Estimating the Gilt-Edged Term Structure:

Basis Splines and Confidence Intervals

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ABSTRACT

Studies estimating the term structure using a linear approximation to the discount function frequently use spline functions. Many of these functions can be shown to generate a regressors' matrix which is nearly perfectly collinear. This paper presents a form of spline function, "B-splines", which avoid this problem. Procedures to aid application to term structure estimation are given, and are used to estimate the term structure for British Government securities. Rarely are estimation errors reported, which casts doubt upon the results of many studies. Simple, yet robust, formulae are derived and used to demonstrate the strengths of the B-spline methodology.

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ESTIMATING THE GILT-EDGED TERM STRUCTURE: BASIS SPLINES AND CONFIDENCE INTERVALS

JAMES M. STEELEY*

INTRODUCTION

The term structure of interest rates defines the array of discount factors on a collection of default-free pure discount (zero-coupon) bonds that differ only in their time to maturity. There are a number of reasons why we might wish to observe these rates, for example, to value other certain cash flow streams, or to test the theories of the stochastic evolution of the term structure (e.g. Ho and Lee, 1986). As with many other countries, most bonds in the UK, other than the very short maturity Treasury Bills, pay coupons, and so such rates are not directly observable. Consequently, studies to estimate the term structure have used various methods of fitting the following standard discounting equation to bond prices and cash flows

$$P = \frac{C_1}{(1+R_1)} + \frac{C_2}{(1+R_2)^2} + \ldots + \frac{C_n}{(1+R_n)^n} \quad (1)$$

where the market price of the bond is P, and the payments to be made at the ends of periods $1,2,\ldots n$ are $C_1,C_2,\ldots C_n$. The spot interest rates applicable to these payments are $R_1,R_2, R_3,\ldots,R_n$, and we may regard the term structure as being the series of spot rates.

The particular method chosen is largely determined by the intended use of the interest rate estimates, however, two techniques seem to prevail. Both estimate a linear approximation to the discount function, but differ in their choice of approximation function. McCulloch (1971) used polynomial spline functions, whereas Schaefer (1973 and 1981) used a set of Bernstein polynomials.

Unless spline functions are carefully chosen, certain matrices formulated for use in the estimation are likely to be ill-conditioned. Shea (1982) demonstrates this to be the case in the McCulloch formulation. In this paper, a form of spline function that is not subject to this problem is presented. Associated procedures which aid application to term structure estimation are provided, and then used to estimate the term structure for British Government fixed interest securities ('gilt-edged' stock) at several dates over a twenty-month

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period. Many previous studies have not considered the sizes of the error due to estimation. Appropriate formulae are suggested and it is shown how they can provide useful information both during and after estimation.

METHODOLOGY AND PREVIOUS STUDIES

Conceptually, term structure estimation is reasonably straightforward. Let us assume that we have a set of m default free bonds, where the i^{th} bond has a price P_i, pays an amount C_{ij} at time t_j, and where there are n different periods. If the number of bonds with linearly independent vectors of cash flows exceeds the number of payment dates, then because future cash flows are known and prices are observable, the discount function can be estimated by ordinary least squares from the following equation

$$P_i = \sum_{j=1}^{n} C_{ij} d_{t_j} \tag{2}$$

where d_{t_j} is the discount factor for date t_j. This discount factor is equivalently the price of a pure discount bond paying £1 at time t_j in every state of nature. Each discount factor defines a spot rate, the rate at which the discount factor must be compounded until t_j to reach £1, and the set of these rates may be regarded as the term structure of interest rates.

Carleton and Cooper (1976) successfully employed this method by selecting a sample of bonds which only incurred four different payment dates per year. However, to ensure the cash flow matrix had sufficient rank, it was also necessary to restrict the maximum maturity under consideration to seven years. Consequently, spot rates for gilts of medium and long maturities could not be obtained in this manner. Further difficulties are caused by only obtaining discrete point estimates. Discrete spot rate estimates will generate forward rates that do not lie on a smooth curve.[1]

Approximation Functions

An alternative approach which avoids these difficulties consists of estimating a linear approximation to the (continuous) discount function. In other words, instead of estimating each d_{t_j} value directly we substitute an approximation of the form

$$d(t) = \sum_{l=1}^{L} \alpha_l f_l(t) \tag{3}$$

and estimate the α_l coefficients which are applied to the L approximating functions chosen. On substitution of this function into our price equation (2), we obtain

$$P_i = \sum_{l=1}^{L} \alpha_l \sum_{j=1}^{n} C_{ij} f_l(t). \qquad (4)$$

We still have a linear regression equation but now we can choose how many coefficients we wish to estimate.

The Weierstrass Theorem has been used to justify polynomial approximation. This theorem says that we can approximate arbitrarily closely over a given interval any continuously differentiable function by a polynomial. Depending on the required degree of accuracy, a higher order polynomial may be necessary. There are dangers, however, in using higher order polynomials. Although these may provide a greater degree of accuracy, if they are fitted through limited data, then it is possible for the approximation to fluctuate wildly over its range.

Two methodologies have been developed that avoid the problems with simple polynomials. McCulloch (1971 and 1975) used various piecewise polynomials, or spline functions, and Schaefer (1973 and 1981) used a set of Bernstein polynomials. In using the former technique, extreme care is required when choosing the form of the component, or basis, functions of the piecewise polynomial. Not all bases are equally capable of defining spline regressors useful for reliable estimation. Indeed, Powell (1981, p. 227–8) shows that it is extremely bad practice to work with a spline function composed of power functions and truncated power functions (equivalent to that used by McCulloch) as inaccuracies arise from the subtraction of large numbers. The inaccuracies arise because this and certain othe[illegible] generate a regressors' matrix which is nearly perfectly collinear.

APPROPRIATE SPLINE FUNCTIONS: B-SPLINES

Instead, it is recommended that a basis of 'B-splines', which are identically zero over a large portion of the approximation space, is used. These prevent the loss of accuracy due to cancellation and they also have good convergence properties.

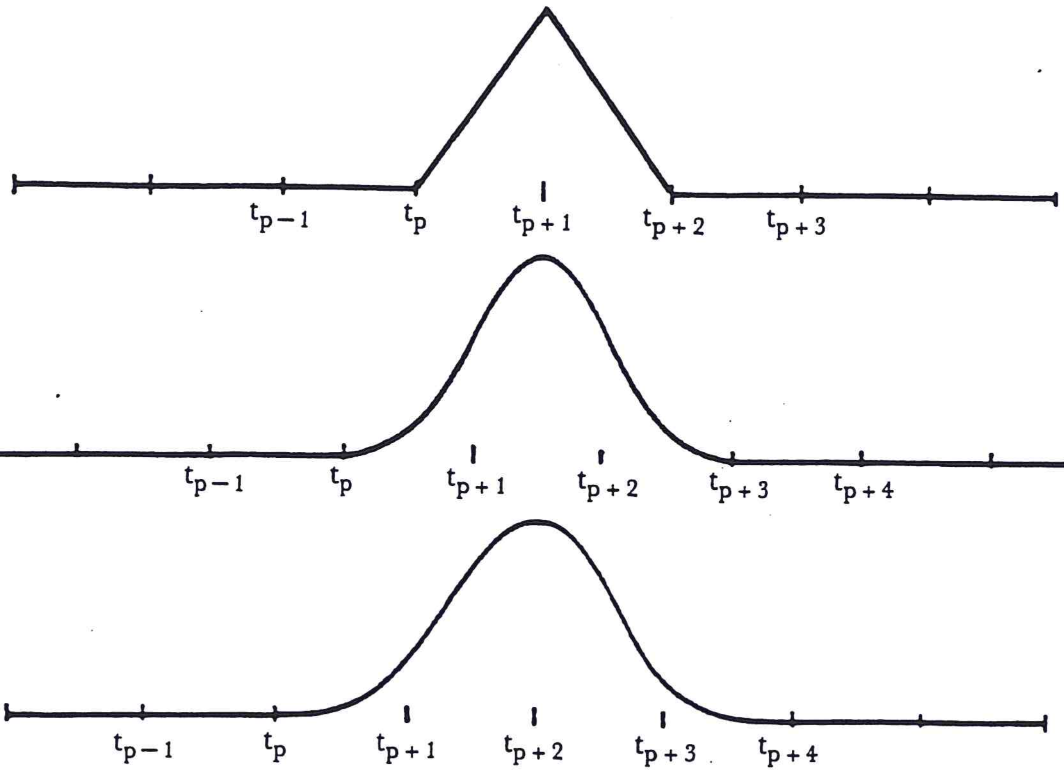
The function

$$B_p^k(t) = \sum_{l=p}^{p+k+1} \left[\prod_{h=p, h\neq l}^{p+k+1} \frac{1}{(t_h - t_l)} \right] (t-t_l)_+^k \qquad -\infty < t < \infty \qquad (5)$$

is known as a k-order B-spline. The subscript $+$ has the meaning $(t-t_l)_+ = \max\,[0,(t-t_l)]$, and the subscript p denotes that $B_p^k(t)$ is only non-zero if t is in the interval $[t_p, t_{p+k+1}]$. Example graphs of first, second and third order B-splines are given in Figure 1. The borders between the n sections of the approximation space are known as knots, and any approximation space will be spanned by $n+k$ basis functions.

Figure 1

B-splines of Degrees One, Two and Three



Source: Powell (1981)

At this point, for clarity of exposition, we shall begin references to a simple example. A linear B-spline function ($k=1$) would be given by

$$B_p(t) = \sum_{l=p}^{p+2}\left[\prod_{h=p,h\neq l}^{p+2}\frac{1}{(t_h-t_l)}\right](t-t_l)_+ \qquad -\infty<t<\infty. \tag{6}$$

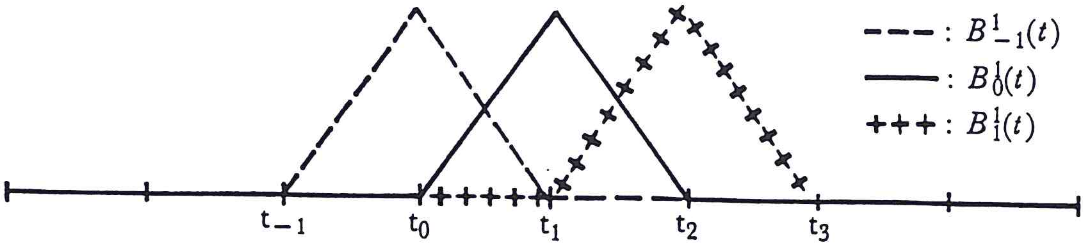
It is non-zero over the interval $[t_p, t_{p+2}]$ and takes the following values

$$B_p(t) = \begin{cases} 0 & \text{for } t\leq t_p \\ (t-t_p)/(t_{p+1}-t_p)(t_{p+2}-t_p) & \text{for } t_p\leq t\leq t_{p+1} \\ (t_{p+2}-t)/(t_{p+2}-t_p)(t_{p+2}-t_{p+1}) & \text{for } t_{p+1}\leq t\leq t_{p+2} \\ 0 & \text{for } t_p\leq t. \end{cases}$$

The first natural step to construct our basis of $n+k$ B-splines is to include the $n-k$ functions $\{B_p : p=0,1,\ldots,n-k-1\}$ because they are linearly independent. Assuming that the approximation space has two segments ($n=2$ and, say, $p=0$), we first include the function $B_0^{\,1}(t)$ which can be evaluated in the manner described above, and appears as the solid line in Figure 2. It is zero until t_0 at which point it has a positive linear slope until t_1, whereupon it takes a negative linear slope until becoming zero once more at t_2. As we require a total of $n+k$ basis functions, another $2k$ basis functions are needed.

Figure 2

B-splines of Degree One



A convenient way of choosing them so that they are also B-splines is to introduce some extra knots outside the interval $[a,b]$.[2] Then we construct a total basis of B-splines, that is, $\{B_p^k(t): p=-k, -k+1, \ldots, n-1\}$. In our example, we add the knots t_{-1} and t_3 and span the approximation interval with the $n+k=3$ functions $B^1_{-1}(t)$, $B_0{}^1(t)$ and $B_1{}^1(t)$ (Figure 2). For spline functions of higher degree the extra $2k$ functions have the effect of including the right-hand 'tails' of those functions which were first non-zero in regions to the left of t_0. In our linear example, over the interval t_0 to t_1, the right hand portion of $B^1_{-1}(t)$ is added to the existing left hand portion of $B_0{}^1(t)$. Hence we note the general result that each segment has present within it non zero portions of $k+1$ functions, in our example this numbers two.

Calculation of Higher Degree B-splines

In order to obtain a smooth forward rate curve, we must have a spline function of at least order three. The calculation, by the earlier formula (5), is inconveniently complicated; manually or computer assisted. However, Powell (1981, p. 234—5) has shown that the following recurrence relation holds for k-order B-splines for all real values of t

$$B_p^k(t) = \frac{(t-t_p)B_p^{k-1}(t) + (t_{p+k+1}-t)B_{p+1}^{k-1}(t)}{(t_{p+k+1}-t_p)} \qquad (7)$$

and it is recommended that they are calculated from the tableau, Figure 3, computing columns in sequence from the left. If t is in the interval $[t_p, t_{p+1}]$, then the numbers in the first column have the values

$$B_q^1(t) = 0 \qquad q \neq p-1, \qquad q \neq p$$
$$B_{p-1}^1(t) = (t_{p+1}-t)/[(t_{p+1}-t_{p-1})(t_{p+1}-t_p)]$$
$$B_p^1(t) = (t-t_p)/[(t_{p+1}-t_p)(t_{p+2}-t_p)].$$

We already know that we need $k+1$ functions for each portion of the approximation interval, and for the segment $t_p \leq t \leq t_{p+1}$ these begin with $B_{p-k}^k(t)$ and end with $B_p^k(t)$. It is straightforward to see that the recurrence relation will deliver the first of the $k+1$ required functions, i.e. $B_{p-k}^k(t)$, from initial calculation of the k functions $B^1_{p-k}(t)$ to $B^1_{p-1}(t)$.[3] To get the first two of the required functions i.e. $B^k_{p-k}(t)$ and $B^k_{p-k+1}(t)$, the $k+1$ functions $B^1_{p-k}(t)$ to $B^1_p(t)$ are needed. Similarly, and in general, to get the $k+1$ functions $B^k_{p-k}(t)$ to $B^k_p(t)$, the calculation of the $2k$ initial functions $B^1_{p-k}(t)$ to $B^1_{p+k-1}(t)$ is required.[4]

ESTIMATION PROCEDURE

In matrix notation, the linear regression problem described by equation (2) may be written as

$$P = D\alpha, \tag{8}$$

where P is a vector of gross price observations on m different bonds, α is a vector of $L = n + k$ approximation coefficients, and D is the $(m \times L)$ matrix of the summed products of the cash flows and evaluated basis splines.[5] In particular, if t_{ij} is the point in time when bond i receives cash flow j, then the general element of the matrix D is given by

$$D_{il} = \sum_{j=1}^{n} C_{ij} f_l(t_{ij}) \quad i = 1,2,\ldots,M., \quad l = 1,2,\ldots,L. \tag{9}$$

where

$$f_l(t_{ij}) = B^k_p(t_{ij}) \quad \text{and} \quad t_p \leq t_{ij} < t_{p+1}.$$

The least squares estimate of the vector of coefficients on the basis functions, $\hat{\alpha}$, may be substituted into equation (3) to obtain an estimate of the discount function.

Constraining the Discount Function

Several authors such as Rose and Schworm (1980) and Shea (1984) have noted that term structure estimates using McCulloch spline functions would often generate forward rates that were unstable and fluctuated widely, often drifting off to negative values. Vasicek and Fong (1982) have pointed out that discount functions are principally exponential decays and since polynomials have different curvature, a polynomial spline function will tend to oscillate around an exponential discount function, and that this explained earlier results. However, Shea (1985) argued that the difficulties of modelling exponential decays with polynomial functions do not extend to local polynomial approximations to exponential functions. He further demonstrated that the functions suggested by Vasicek and Fong are equally likely to give unstable forward rates. Chambers, Carleton and Waldman (1984) have incorporated the exponential characteristic in a different manner. They suggested that the spot rate curve rather than the discount function should be approximated, using an exponential function. The need for non-linear estimation makes it computationally more difficult, particularly if estimation errors are to be calculated.

If it is desirable, we may introduce constraints on the forward rates as done by Schaefer (1973 and 1981) and examined by Shea (1984). Schaefer constrained the slope of the discount function to be everywhere negative, while Shea reported that simple restrictions of fixed proportions between first derivatives were often sufficient. For example, he found that constraining the slope at the last knot to be one half of the slope near to the next-to-last knot removed negative forward rates at near maximum sample maturity. Within B-spline technology, these derivatives may be obtained by differentiating, with respect to time t, both the recurrence relation, equation (7), and the first order B-spline formulae, below equation (6), and then constructing a tableau similar to Figure 3 to guide the calculation of the derivatives in the same manner as the levels.

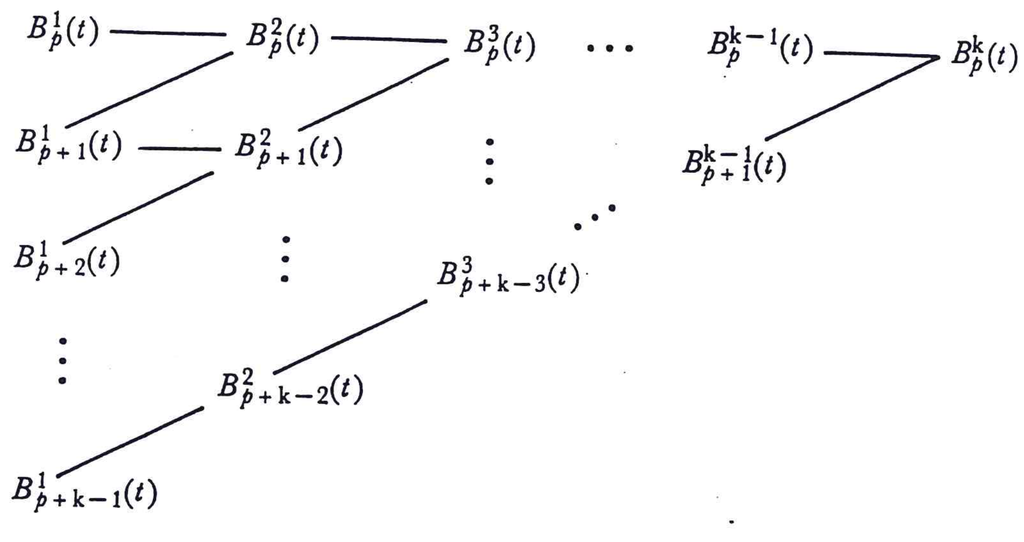
However, the addition of constraints to spline approximation functions is necessarily ad hoc, and the results of Shea (1984) demonstrate that non-negativity constraints can dramatically alter the structure of the forward rate curve in places other than where negative rates are constrained away. Furthermore, there may be some explanation which makes it undesirable to impose constraints upon estimation. In fact, the only natural constraint to impose on the function is that it should take a value of unity at time zero, that is

$$d(0) = \sum_{l=p-k}^{l=p+n} \alpha_l B_l^k(0) = 1.0 \qquad (10)$$

where $t = 0$ is in the interval $(t_p \leq t < t_{p+1})$, and, by construction, only the first

Figure 3

Tableau for Calculating Higher Order B-splines



Source: Powell (1981)

$k+1$ of the above summed functions will be non-zero, that is, those from $p-k$ to p. The effects of this constraint will be discussed later.

The linear restriction on the constraints may be written as $W\alpha = w$, where W is an L element row vector of basis functions evaluated at $t=0$, and w is the scalar, unity. The restricted least squares estimator of α is obtained in the standard manner (see e.g. Johnston, 1984, p. 205). Thus, we may solve for the estimated discount function using equation (3), where

$$f_l(t) = B_p^k(t) \qquad t_p \leq t < t_{p+1}. \tag{11}$$

For each discount factor d_t, there is a spot rate of interest R_t, and the set of these is the term structure of interest rates. The formula to calculate the spot rates is simply

$$R_t = \left[\frac{1}{d_t}\right]^{1/t} - 1 \tag{12}$$

and the implied forward rates are given by

$$r_t = \left[\frac{d_{t-1}}{d_t} - 1\right]. \tag{13}$$

Confidence Intervals

A feature conspicuous by its absence in the majority of papers on term structure estimation is a report upon the accuracy of the estimated discount function and interest rates. The formulae suggested by McCulloch (1971) and also used in his later paper (1975) have been largely ignored in all subsequent papers. However, Schaefer (1981) does indicate those interest rates which are 'unreliable' due to being estimated beyond the maturity of the longest available bond.

That errors have for so long gone unreported is perhaps not surprising when it is remembered that, as a theoretical model, equation (1) leaves no room for residuals. However, as indicated in note 5, the residuals can measure the accuracy of approximation in practical situations such as this. A further reason for their absence is the fact that the procedure to retrieve the relevant variance-covariance matrices is, relative to the rest of the procedure, statistically complex. In this paper, we apply some established statistical results to general sufficiently robust standard errors for all the products derived from the least squares procedure.

The variance-covariance matrix of the vector of discount factors may be obtained by applying the result that the variance of a linear combination $x'y$ is given by the quadratic form $x'\text{var}(y)x$, and hence

$$\text{var}(\hat{d}) = FAF' \tag{14}$$

where F is the $(T \times L)$ matrix whose general element is given by equation (11), and A is the variance-covariance matrix of the restricted least squares coefficients.

Since forward rates are essentially a ratio of discount factors, the variance of the forward rate may be estimated by var(d_{t-1}/d_t). On applying the formula for the variance of a ratio (see e.g. Bulmer 1979, p. 79), we obtain the following approximation for the variance of the forward rate

$$\operatorname{var}(\hat{f}_t) \approx \frac{E(d_{t-1}^2)}{E(d_t^2)}\left\{\frac{\operatorname{var}(d_{t-1})}{E(d_{t-1}^2)} + \frac{\operatorname{var}(d_t)}{E(d_t^2)} - \frac{2\operatorname{cov}(d_{t-1},d_t)}{E(d_{t-1})E(d_t)}\right\}. \tag{15}$$

This formula can be viewed as an alternative to that of McCulloch (1971) which is derived from the above formula for use with instantaneous rates. The McCulloch formula pre and post multiplies the matrix A by a matrix representing the difference between the ratio of matrix F and vector $\hat{d}$, and the derivatives of the same. The formula suggested above is computationally less difficult. It is a satisfactory approximation provided that the variances of the discount factors used are substantially smaller than the square of their expected values (the square of discount factors themselves, for unbiased estimates), and provided that the distribution of the denominator discount function is positive.

The McCulloch (1971) formula for spot rates relies on an approximation to a logarithmic function which may not be very accurate given the range of variables involved. When estimated it was systematically smaller and hence less robust than the alternative formula suggested below. We already have standard errors and hence confidence bounds upon the discount factors which can be used to define, through application of equation (12), upper and lower bounds upon the spot rates. These may be simply reinterpreted as standard errors by appropriately differencing the confidence bounds. In particular, the following formula is suggested

$$\text{s.e.}(\hat{R}_t) \approx \left[\frac{\left[\dfrac{1}{d_t - k.\ \text{s.e.}(d_t)}\right]^{1/t} - \left[\dfrac{1}{d_t + k.\ \text{s.e.}(d_t)}\right]^{1/t}}{2k}\right]. \tag{16}$$

Because of the non-linear relationship between the discount factor and spot rate, there must be an asymmetric confidence region about the spot rate. Although, the arithmetic averaging in the above formula suppresses this, the discrepancy from linearizing is generally small. Furthermore, this formula is intuitively consistent with the forward rate formula, inasmuch as empirically s.e.$(\hat{f}_1)$ = s.e.$(\hat{R}_1)$ for all k in $0.01 < k < 30$. McCulloch's (1971) formulae do not possess this property.

RESULTS

Data and Parameter Settings

The data used in this study were obtained from Datastream, and consisted of closing mid-market prices on forty nine high coupon, fixed interest, redeemable gilts.[6] Twenty two such observations were drawn at four weekly intervals over the period from 6th March, 1986 until 15th October, 1987, that is between the change to quoting all market prices net of accrued interest, and the stock market crash. During that period six gilts in the sample matured and three, with the required characteristics, were issued and subsequently quoted fully-paid. Sample sizes thus range between forty two and forty four.

There are several parameters which have to be chosen. To minimize the losses in degrees of freedom yet still imply smooth forward rate curves, cubic ($k = 3$) B-splines were used. Due to the relatively limited supply of bonds with a maturity of greater than eighteen years, this was the maximum term ($T = 18$) over which discount factors were estimated. In the sample chosen, the maximum bond maturity encountered was twenty four years and so the edges of the approximation space can be set at say $a = 0$ and $b = 25$. The flexible nature of the approximation process means that the only constraint on the setting of the b knot is that it is greater than the maximum maturity of any bond present in the sample. The further out the knot the further out can the discount function be estimated, however once past the maximum maturity bond (and often before, with a limited supply of data), the estimates become very unreliable. This phenomenon is clearly shown in Schaefer (1981, Table 3, p. 430–1), though he employs a different technique of estimation.

There is essentially only one a priori guideline for setting the within sample knots, that is, dividing the bonds into short, medium and long maturity as classified by the market. There are two difficulties with this approach. Firstly, the market definition of 'short' and some participants' definitions are different: under five years and under seven years respectively. Secondly, such a definition causes a strong clustering of bonds, leaving the long end of the market poorly represented. Conversely, the clustering at the short end might imply that a further division is useful. In order to avoid biases and inefficiencies from accepting an under parameterized function while still preserving consistency, we commence with a likely over-parameterized model and change the position and reduce the number of knots until the standard errors of the interest rates are minimized.[7] As the process continued, it was found that the interest rate estimates tended to stay within a relatively narrow set of values long before the standard errors were finally minimized. This is a strength of this methodology, as it allows reasonable confidence about the interest rate estimates even if standard errors are not completely minimized. The outcome of this process for the samples studied here suggested the following division into $n = 3$ portions; up to five years, up to ten years and over ten years. This division placed approximately equal quantities of bonds in each segment. Such a starting point is recommended for future studies, although the number of segments has still to be chosen. This strategy maximizes degrees of freedom over individual portions of the discount function.[8]

Estimated Coefficients and Interest Rates

There were $n = 3$ plus $k = 3$, that is, $L = 6$ approximation function coefficients to estimate for each date. The vectors of restricted least squares estimates a are provided (Table 1) as a databank for other researchers who need only use equations (3) and (11) to generate interest rates of varying terms as required.[9]

Figures 4 and 5 show the evolution of the estimated spot and forward rate curves for maturities of between one and eighteen years over the twenty month sample period. The spot rate curve (Figure 4) appears humped in shape and to have moved extensively in a parallel fashion. The forward rate curve (Figure 5) exaggerates the shape of the spot rate curve as would be expected. The short rate $R_1 = r_1$ is in the foreground in both diagrams and fluctuates more than the long rate. We note also that the forward rate curve is never negative, and

Table 1

Date	Sample Size	*Approximation Function Coefficients* a_1	a_2	a_3	a_4	a_5	a_6
06-03-86	44	8.9494	10.0127	24.4558	−3.5955	26.4838	−135.8266
03-04-86	44	8.8771	10.1582	26.2281	−1.7565	22.1166	−95.6332*
01-05-86	44	8.7397	10.4710	26.4010	−1.4472*	18.2164	−62.4611*
29-05-86	43	8.7496	10.4473	26.4933	−3.3381	23.7268	−102.8525*
26-06-86	43	8.7595	10.4316	25.8893	−3.5968	24.4475	−104.4071*
24-07-86	44	8.8014	10.3438	25.1638	−2.9285	20.0201	−59.2118
21-08-86	44	8.7700	10.4124	25.4618	−2.9927	22.0531	−75.2774
18-09-86	44	8.8211	10.3130	23.8958	−3.6029	21.1760	−78.0470
16-10-86	44	8.9050	10.1334	22.7766	−3.8561	22.8458	−85.9986
13-11-86	44	8.8981	10.1554	22.2320	−4.2376	23.0039	−92.2683
11-12-86	44	8.9173	10.1061	22.7165	−4.8246	24.9539	−98.3064
08-01-87	43	8.8736	10.1947	23.7305	−2.9116	22.1254	−89.1255
05-02-87	43	8.8411	10.2651	24.0780	−3.4023	22.6534	−86.6559
05-03-87	43	8.7945	10.3569	25.4065	−2.4153	20.2634	−71.6676
02-04-87	44	8.7470	10.4647	25.4821	−2.3971	22.5476	−93.5764
30-04-87	44	8.7093	10.5456	25.9762	−1.9789	23.5021	−98.6961
28-05-87	43	8.7157	10.5329	25.8075	−1.5656	23.4407	−96.3109
25-06-87	43	8.7237	10.5168	25.6018	−2.5597	26.0113	−112.3366
23-07-87	43	8.7626	10.4329	25.1455	−2.9624	25.6546	−110.3432
20-08-87	43	8.8653	10.2140	23.6945	−3.6420	24.5855	−103.7808
17-09-87	43	8.8087	10.3412	23.9178	−3.5880	24.7398	−102.5023
15-10-87	42	8.8314	10.2908	23.7782	−4.7753	27.1726	−119.9067

*Not significant at a 5 percent level.

Figure 4

Time Path of the Spot Rate Curve

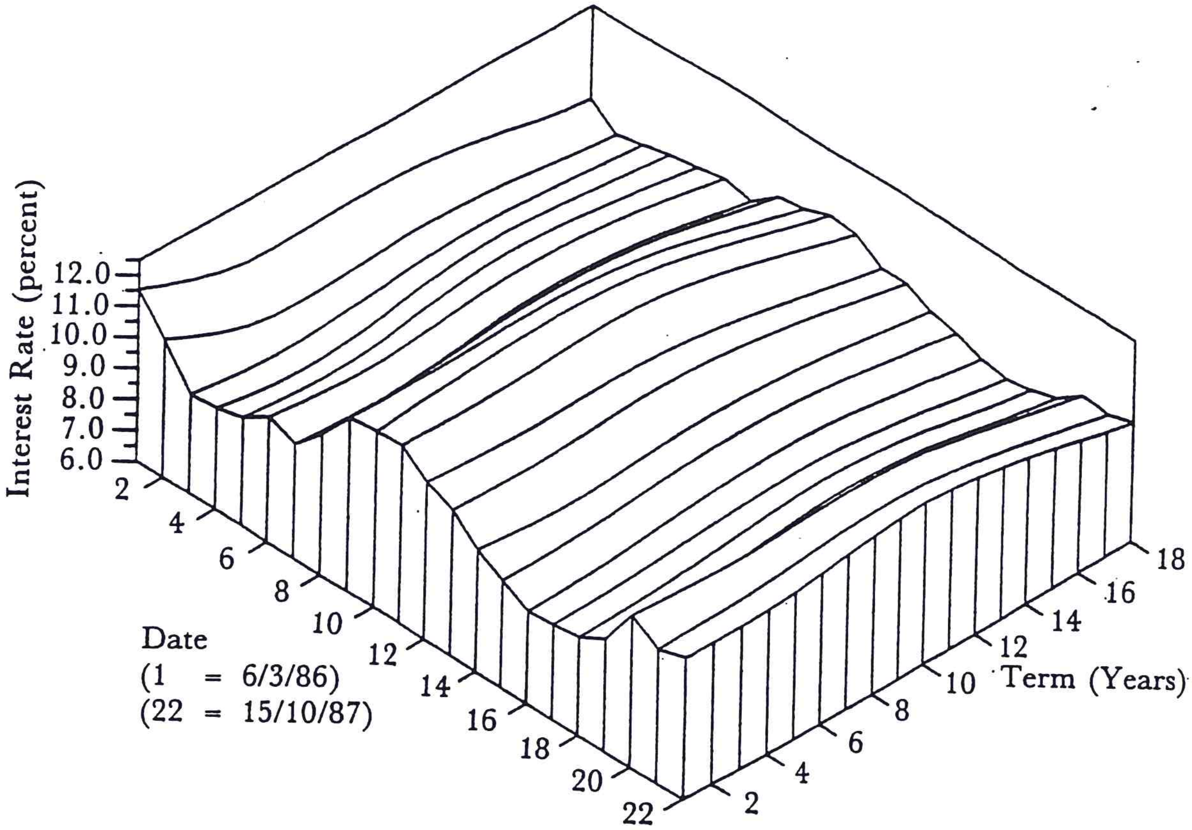
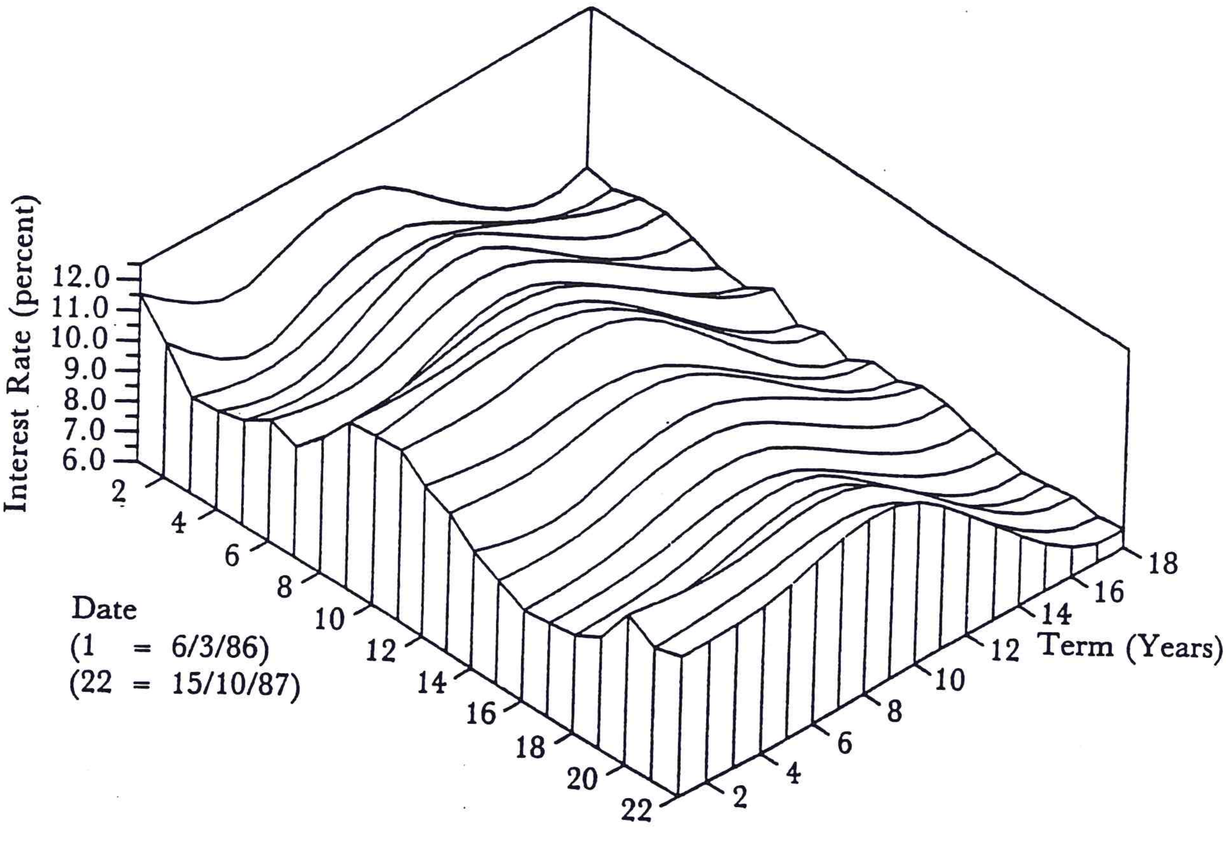


Figure 5

Time Path of the Forward Rate Curve



confirm that even for unreliable estimates beyond eighteen years there is no tendency for rates to become negative. Furthermore, results for a selection of dates outside this sample do not produce negative forward rates.

The accuracy of these estimates can be gauged by calculating the standard errors in the manner described earlier. Figures 6 and 7 show two examples of spot rate curves and forward rate curves respectively with bands at one standard error drawn either side.[10] For the spot rate curves, the standard error is typically 0.04 percent and this rises with maturity to around 0.07 percent at eighteen years. Similarly, the standard error of the forward rate curves rises with maturity to a value of 0.35 percent at year eighteen from a typical value of 0.20 percent. The size of the standard errors for the two chosen examples are echoed among the figures for the intervening periods and suggest a high degree of reliability in our estimates. The effect of sparse data can be demonstrated by comparing the standard errors on the twenty year spot and forward rates before and after July 1986, when the maximum maturity increased from eighteen years to twenty four years. At dates after July 1986, the twenty year figures are little different from the eighteen year figures, whereas before July 1986, the figure is typically 0.25 percent for spot rates and 2.50 percent for forward rates.

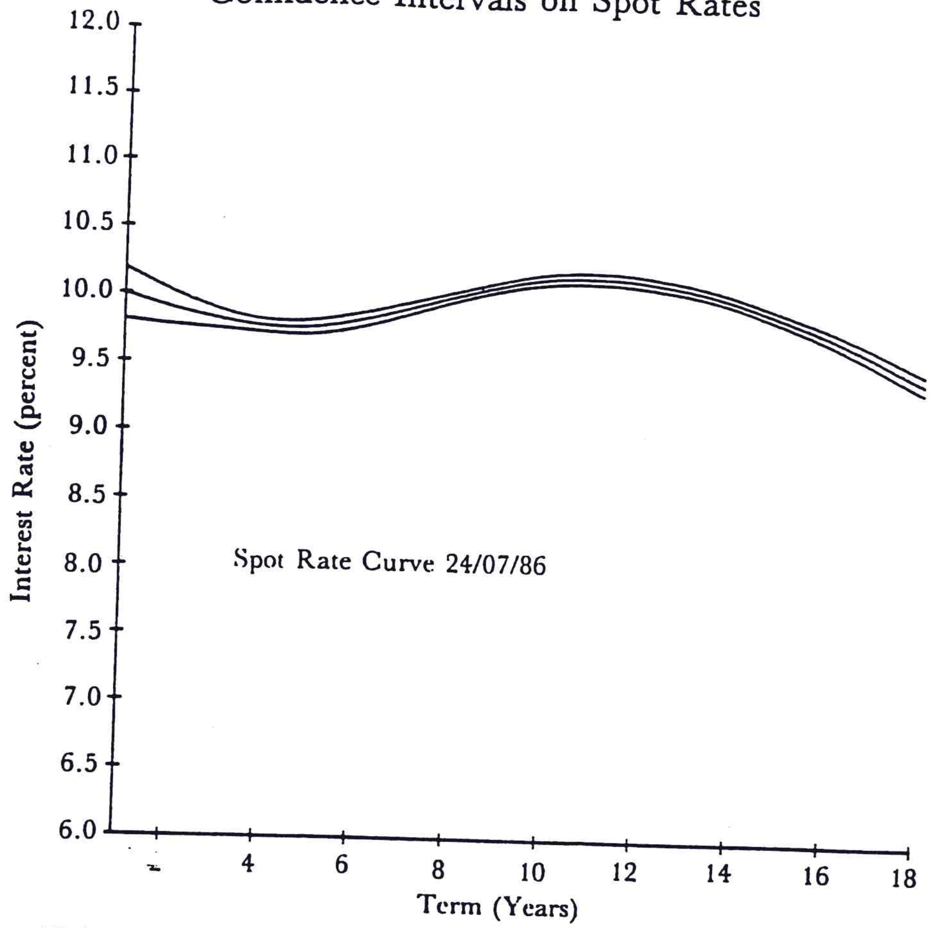
Figures 6 and 7 also show that the confidence bands tend to expand at very short maturities. This means that constraining the discount function to pass through unity at time zero may not identify the short maturity end of the term structure with the required level of accuracy. This suggests that in markets where an appropriate short rate of interest exists, it may be preferable to constrain the short end of the term structure to take this particular value rather than constrain the discount function.

If the term structure estimates are valid, then when bond prices are recreated using equation (4), there should be a negligible error and no systematic effects present. Most recreated prices are less than fifty pence away from the market price (quoted in pounds), that is, most prices are less than one half of one percent away from the market. Significantly, this is less than the average bid-offer spread. Furthermore, no estimated price is greater than one percent different from the market price.

We have deliberately chosen our sample to avoid the type of tax effects examined by Schaefer (1981). It is possible nevertheless, that the choice rule applied has not completely removed them. Any remaining effects will be manifested by a strong relationship between coupon size and residual. This is because the market value of a stock which is predominantly held by high rate tax payers is likely to be mispriced by a model assuming a zero tax rate. Given that gilt preferences are governed extensively by coupon size (see note 6), residuals strongly related to coupon size indicate the presence of tax effects. To assess the extent of remaining effects the correlations between coupon and residual were calculated. They were all small (less than twenty five percent), with the majority between five and fifteen percent. This is insufficient to

Figure 6

Confidence Intervals on Spot Rates



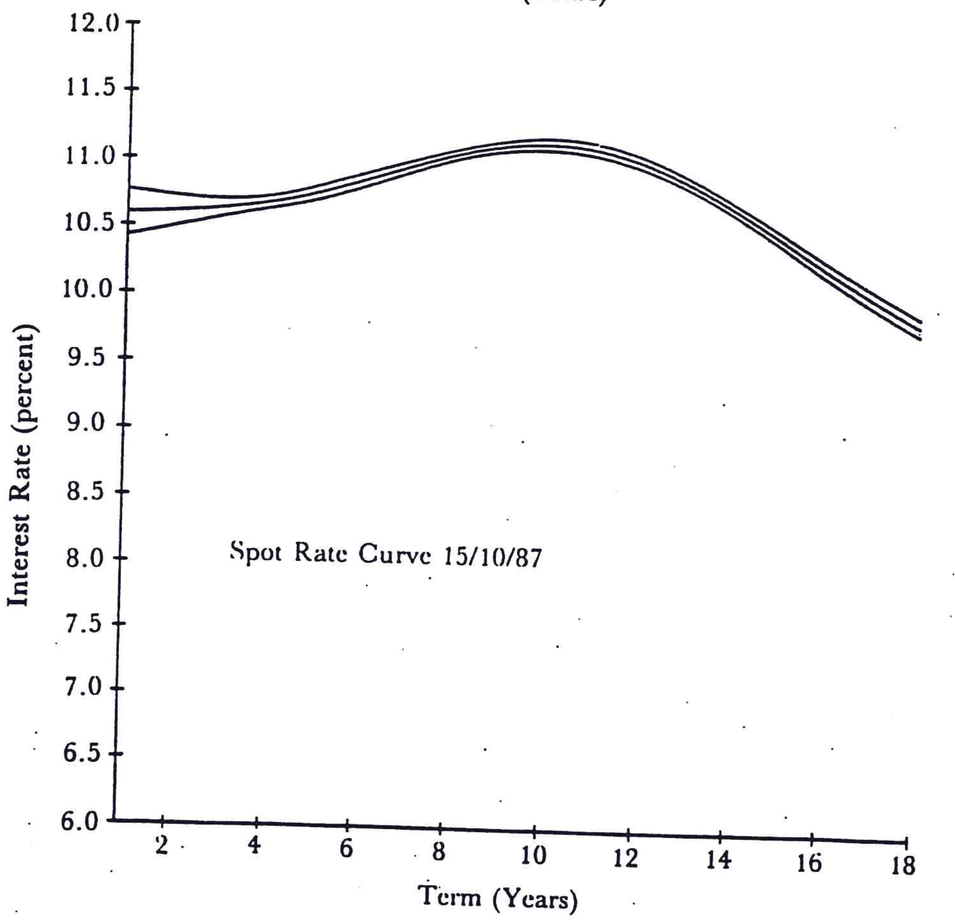
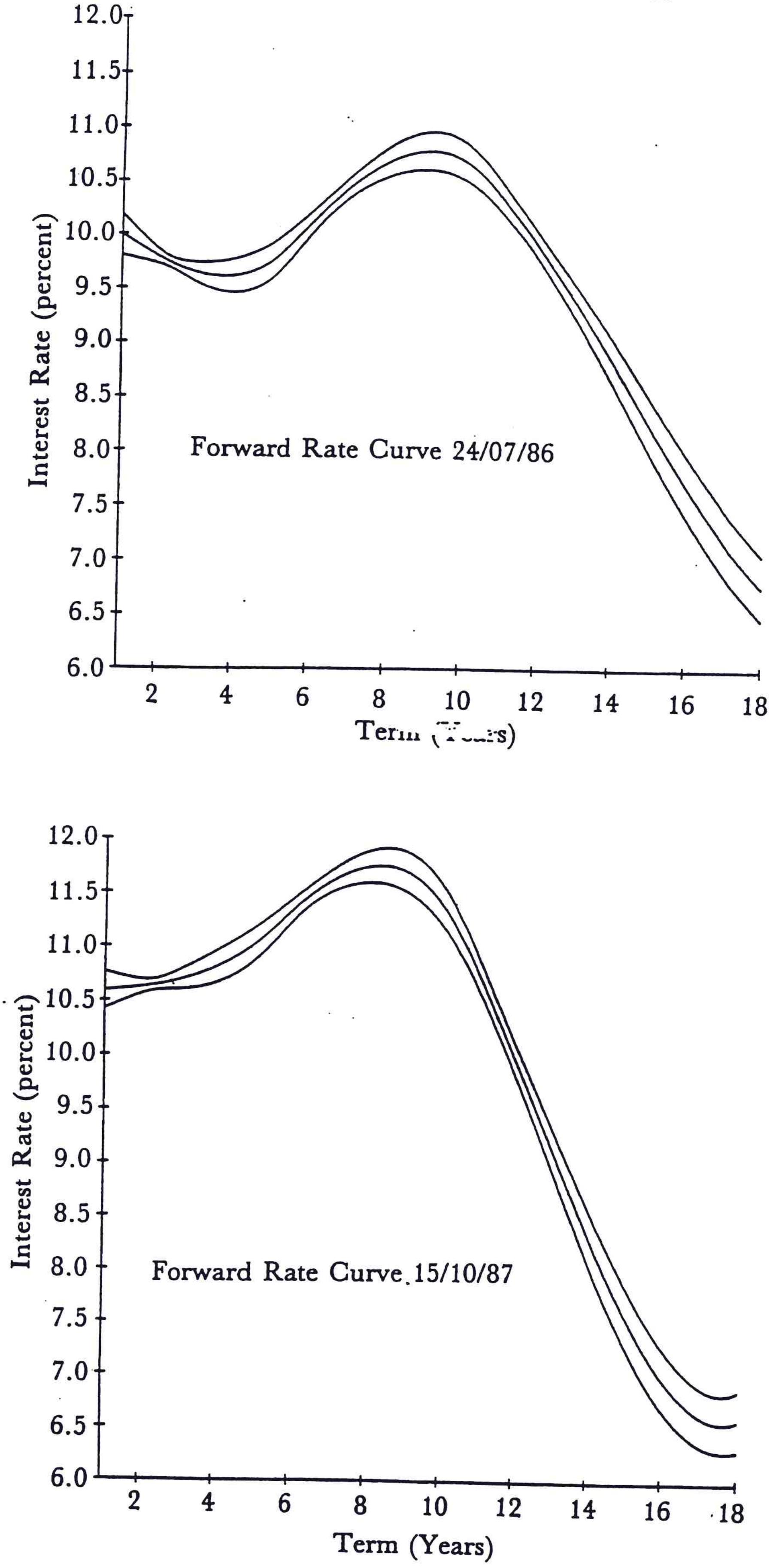


Figure 7

Confidence Intervals on Forward Rates



seriously bias the term structure estimates. Finally, we note the finding that these estimates are robust to minor changes in constituent securities.

SUMMARY

This paper sought to present a method of estimating discount functions using spline functions which will not incur problems of multicollinearity in the regresssors' matrix. Such a solution is to use a form of spline functions defined by Powell (1981) and known as B-splines. We outline the approximation function technique and appraise previous studies. Comprehensive details of the form of B-splines and the estimation processes involved are given in order to establish a definitive procedure for this form of term structure estimation.

It is argued that the calculation of appropriate standard errors provides both useful information during the estimation stages and a necessary guide to performance at the end. The path of the term structure estimates for several dates is provided, and the estimated approximation function coefficients are provided as a database for other researchers. By criteria discussed above, the estimated interest rates fit well and thus the spline procedure is re-established as a robust alternative to the Bernstein polynomial approach.

NOTES

1 However this methodology more closely reflects economic reality than the approximation methodology in the sense that no clear meaning can be attached to a spot rate estimated for a date at which no bond actually makes a payment.

2 Specifically, we let $\{t_j : j = -k, -k+1, \ldots, -1\}$ and $\{t_j : j = n+1, n+2, \ldots, n+k\}$ be any points on the real line that satisfy the conditions

$$t_{-k} < t_{-k+1} < \ldots < t_{-1} < t_0 = a$$

$$b = t_n < t_{n+1} < t_{n+2} < \ldots < t_{n+k}.$$

3 To see this, exchange the subscript p in the tableau for $p-k$ (Figure 3).

4 The paper by Shea (1985) recommends a form of basis spline derived by DeBoor (1978) which also avoid the ill-conditioning problem. It is simple to show that these functions are non-trivially different from those presented here. A comparison of the implied zero-order splines, which are special cases of those considered here, and the results of DeBoor's recurrence program are adequate proof. Furthermore, the Powell presentation is more accessible and consequently the potential of the B-spline technology is more easily appreciated and certainly more simple to use.

5 As a theoretical model, equation (8) leaves no room for residuals (see Schaefer 1981, p.416). However, practical considerations permit residuals to represent, *inter alia*, estimation error (see Steeley 1990, p.142–144). The residuals are assumed to possess the classical regression properties.

6 High coupon stocks are used in an attempt to estimate the term structure without distortions due to the low coupon/short maturity preference of high rate tax payers, which could invalidate the assumed properties of the regression residuals.

7 This 'general to specific' modelling approach was popularized by Hendry (1979). The absence of a theoretical economic model in this circumstance allows more freedom of action: we eliminate variables (the space between two knots) on grounds of statistical insignificance alone.

8 Including those outside the sample, the knot settings used were −3, −2, −1,0,5,10,40,45, 50,60.

9 It should be noted that until July 1986, the maximum bond maturity in the samples was 18 years, and hence interest rates calculated beyond this maturity are likely to be unreliable. Although the maximum maturity thereafter increased to twenty four years, the paucity of data means that estimation much beyond eighteen years is likely to give unreliable estimates.

10 The spot rate confidence bounds depicted have not been linearized by equation (16). They are simply

$$\left[\frac{1}{d_t \pm \text{s.e.}(d_t)}\right]^{1/t}.$$

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