Expected Turnover in a Binomial Tree

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Introduction

The amount of turnover involved in hedging options positions by means of a delta hedge on the underlying asset can be very important when transactions costs are involved. Practitioners often use hedges based on either a Black Scholes (1973) model or a Binomial model, (eg. Cox, Ross, Rubinstein (1979)), and make revisions to the hedge at discrete time intervals. The purpose of this paper is to examine how the expected level of transactions turnover implied by delta hedging using a simple (additive) binomial tree depends on the number of time intervals used to characterize the tree. We have found that the relationship between the frequency of revisions of a delta hedging strategy, and the expected level of turnover is often poorly understood by even quite sophisticated participants in options markets. The paper shows that the volume of turnover depends on the square root of the number of intervals, and is therefore unbounded as we approach the continuous time case. Leland (1985) contains a similar result. The direct approach adopted here provides some additional insights.

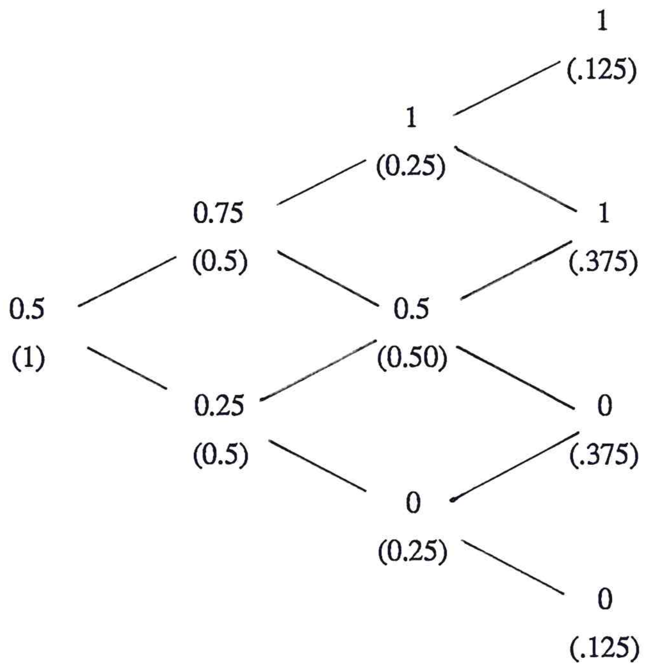
The Analysis

We consider the simplest possible case of a call option under an additive binomial process with no drift. Figure 1 shows how we would calculate expected turnover for a diagram with 3 time intervals. We shall first discuss this case before establishing the general case and examining its properties.

Figure 1

Expected Turnover in a Binomial Tree

Each node shows the delta value and beneath it in brackets the probability of reaching that node.



t =	0		1		2		3	
Expected turnover each period:		0.25		0.25		0.25		Total: 0.75

We assume equal increments, and equal probabilities of up movements and down movements. Under this tree structure the hedge ratio x at each node is a simple average of the two later stage values it is connected to.

The weights attached to the binomial probabilities are also shown on the figure. These are proportional to 'Pascal triangle' weights, and the probability of each node is a simple average of the two adjacent earlier stages (eg. 0.375 is an average of 0.25 and 0.50, 0.125 is an average of 0.25 and 0.0 (outside tree)).

Each node shows the delta value and beneath it in brackets the probability of reaching that node. The weights attached to the binomial probabilities are also shown on the figure. These are proportional to 'Pascal triangle' wieghts, and the probability of each node is a simple average of the two adjacent <u>earlier</u> stages (eg, 0.375 is an average of 0.25 and 0.50, 0.125 is an average of of 0.25 and 0.0 (outside tree)).

Expected Turnover is Constant Through Time

In the example, we see that the expected turnover is 0.25 in each of the three periods, making a total of 0.75. In the first period we shall either revise upwards by 0.25 (from 0.5 to 1.0), or downwards by 0.25. The same is true in the second period (whether we start from 0.75 or 0.25 we always move by 0.25). In the third period we either move by 0.5 with 50% probability, or not at all with 50% probability. The expectation is that we move by 0.25.

It is no coincidence that expected turnover is constant in this way. We show next that it must be constant for any contingent claim where the boundary condition dictates hedge ratios (deltas) that are monotonic in the asset price. This enables us to immediately write down a general formula for total turnover and to then obtain an accurate and simple approximation to it.

Suppose that at any time t, the price of the underlying asset has prices defined on a grid i = 1, ..., n.

For simplicity of exposition we will suppose that the same subscript k refers to all prices running along the top of the binomial tree. Thus a primitive triangle would be referenced by i at time t, and i and i-1 at the next time t+1.

The corresponding deltas at any time t will be denoted by $x_{1t}, \ldots, x_{nt}$ or $\underline{x}_t$ for short. We shall use $p_{1t}, \ldots, p_{nt}$ or $\underline{p}_t$ for short to denote the probabilities of these prices occuring at this date. Finally we shall use the notation B to denote the backward shift operation in the domain of the price grid, so that $Bx_{it} = x_{i-1,t}$.

In a going from date t to date t+1, the turnover depends on the differences between adjacent x's at t+1, and on the probability of reaching the corresponding node at t.

Thus the expected turnover from date t to t+1 is given by

$$T_t = \underline{p}'_t (I-B)\, \underline{x}_{t+1}. \qquad (1)$$

Both p and x follow simple recursion relations in time, but in opposite directions:

$$\underline{x}_t = {}^1/_2\,(I + B)\underline{x}_{t+1} \qquad (2)$$

$$\underline{p}'_t = {}^1/_2\,(I + B)\underline{p}_{t-1} \qquad (3)$$

The way $\underline{p}'_t$ propagates from a unit impulse $\underline{p}_o$ ensures that our attention is solely confined to the triangular region, despite both $\underline{x}$ and $\underline{p}$ being continued beyond it.

Substituting equations (2) and (3) into (1), we can obtain an expression for the expected turnover, at the earlier date, t-1.

$$
\begin{aligned}
T_{t-1} &= \underline{p}'_{t-1}(I - B)\,\underline{x}_t \\
&= \underline{p}'_t(1/2(I+B))^{-1}(I-B)\ 1/2(I+B)\underline{x}_{t+1} \\
&= \underline{p}'_t(1/2(I+B))^{-1}\ 1/2(I+B)(I-B)\underline{x}_{t+1} \\
&= \underline{p}'_t(I-B)\underline{x}_{t+1} \\
&= T_t
\end{aligned}
$$

This completes the proof that, conditional on a given starting point, the expected turnover is equal at all dates.

The Level of Turnover

Consider the situation with $N = 2n + 1$ periods, and an option which is issued exactly at the money. Figure 1 is a special case of this situation. In the last period a transaction only occurs if we arrive at the middle node in the penultimate date. ie. after 2n periods. In this case the turnover is 0.5.

The probability of such occurence is given by the binomial probability

$$\frac{\binom{2n}{n}}{2^{2n}}$$

or

$$\frac{2n!}{n!\ n!\ 2^{2n}}$$

(NB. For our earlier example we get for n = 1, 2/4 = 0.50)

Expected turnover in the last period is therefore:

$$\frac{2n!}{n!\ n!\ 2^{2n+1}}$$

and total expected turnover is given by:

$$\text{Total Turnover} = \frac{2n!\ (2n+1)}{n!\ n!\ 2^{2n+1}} \qquad (4)$$

In order to understand the properties of this turnover figure, we need to use Stirling's formula to obtain a good approximation that is much simpler.

Stirling's formula gives

$$n! \approx n^{n+1/2}\ e^{-n}\ \sqrt{2\pi} \qquad (5)$$

Substituting this into (4) and rearranging, we obtain

$$\text{Total turnover} \approx \frac{2n+1}{\sqrt{4\pi n}}$$

$$= \sqrt{\frac{4n^2+4n+1}{4\pi n}}$$

$$\approx \sqrt{\frac{2n+2}{2\pi}} = \sqrt{\frac{N+1}{2\pi}}$$

where N is the number of stages in the tree.

(For N=3, this gives 0.80 as against the exact 0.75 figure. The approproximation rapidly becomes very accurate for larger values of N). For even N, it appears that the formula $\sqrt{N/2\pi}$ holds.

Conclusions

Table 1 shows the levels of turnover for various grid sizes:

Table 1

Expected Turnover and Grid Size: At the Money Option

N	Formula Turnover	Round trip Turnover (revisions)	Total Round Trip Turnover
10	1.26	0.63	1.13
50	2.82	1.41	1.91
100	3.99	1.99	2.49
200	5.64	2.82	3.32
400	7.98	3.99	4.49
	Unbounded	Unbounded	Unbounded

Note that the formulae refer to the turnover required for intermediate revisions (not including the initial purchase costs or final liquidation ones). Turnover was also measured as a one-way volume rather than as a round trip one.

The final columns adjust for this by dividing by two, and adding 0.50 to adjust for the round trip cost of establishing the original delta hedge of 0.50, and the expected liquidation turnover of 0.50.

Finally, there are three other points worth noting

1. The at-the-money option is the least favourable case. We can easily compute figures for options starting in or out of the money by using the appropriate probability for hitting the expiry date note closest to the excercise price.

2. The analysis has been confined to an additive process, and a zero interest rate. However, the results generalise in a fairly straight forward way to the usual log-normal Black-Scholes assumptions, and the results are very similar.

3. The realised turnover is highly path dependant, with paths which approach near the exercise price close to expiry showing much higher turnover levels than others.

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