Interest Rate Swaps and Default-Free Bonds: A Joint Term Structure Model

Simon Babbs

Midland Montagu
Capital Markets

December 1991

The author is grateful to Stewart Hodges for comments on earlier drafts of this paper.

Financial Options Research Centre
Warwick Business School
University of Warwick
Coventry
CV4 7AL
Phone: 0203 523606

FORC Preprint: 91/27

Abstract: Typical interest rate swaps are contracts contingent on future borrowing rates of banks then of good credit worthiness. Differences between the term structures for default-free bonds and for swaps are more appropriately modelled in terms of "required return differentials" rather than of default risks. We present a theory for the joint evolution of the two term structures, consistent with their initial positions, and for pricing contingent claims on either or both structures. We explore a class of models within our general framework, and exhibit a readily computable valuation formula for european-style contingent claims; we provide a concrete example.

1 INTRODUCTION

The interest rates at which different borrowers can raise funds generate several different term structures within a single currency. In many currencies, quotations are widely available for interest rate swaps[1] based on short term interbank interest rate indices, thereby creating a further term structure. Some indication of the importance of swaps is given by recent estimates by the International Swap Dealers Association that the total value of outstanding contracts exceeds $2,500 billion. In addition to swaps themselves, a variety of contingent claims on the swap term structure are traded; these include interest rate caps and floors, and various kinds of options to enter or terminate swaps.

This paper presents a theory for the joint evolution of the term structures for default-free bonds and for interest rate swaps, consistent with whatever initial term structures are observed, and for pricing contingent claims based on either or both term structures. We explore the properties of a class of models within our general framework, and exhibit a readily computable valuation formula for european-style contingent claims; we provide a concrete example. The applicability of our approach includes the integrated evaluation of trading strategies embracing a very wide range of interest rate instruments, including: spot and forward trading in default-free bills and bonds, and associated futures contracts; interest rate swaps based on government and private sector interest indices, and options thereon; futures contracts on interbank interest rates; caps and floors; and options on all these instruments.[2]

The investigation of term structure is a longstanding topic within financial economics. One substantial body of literature adopts a framework in which all bonds, and any contingent claims with which it may concern itself, are default-free. Its primary application is therefore to own-currency bonds issued by or effectively guaranteed by central government. We have in mind here both the literature seeking to explain the determination of the default-free term structure (eg Vasicek [1977], Cox, Ingersoll and Ross [1985]) and the more recent

---

1 For an account of interest rate swaps see (eg) Antl [1986] or Bicksler and Chen [1986].

2 Our framework embeds the single term structure model independently developed by Heath, Jarrow and Morton [1987],[1989] and Babbs [1990],[1991a]. The closed-form results obtained by Babbs [1990] for: caps and floors; european options on swaps; futures on interbank interest rates, and margined options thereon; and his binomial approximation scheme for american options on swaps and futures, therefore carry over.

literature which aims to model the dynamics of that term structure consistently with whatever initial structure is actually observed (eg Ho and Lee [1986], Babbs [1990],[1991a], Heath, Jarrow and Morton [1987],[1989]).

A second substantial literature addresses the term structures corresponding to the debt obligations, at various maturities, and one or more degrees of subordination in the event of default, of an individual default-risky borrower (ie one who may default).

The most common approach in this second body of literature (eg Merton [1974],[1977], Black and Cox [1976], Ho and Singer [1982],[1984] and Selby [1983]) is based on the capital structure of a corporate borrower. It builds on an insight due to Black and Scholes [1973] that all corporate liabilities, whether debt or equity, can be viewed as contingent claims on the total assets of the firm. To simplify the analysis it is often assumed that the default-free interest rate is constant; the total assets of the firm then become the sole stochastic variable. Default occurs when the assets of the firm are inadequate to meet payments on debt, or fail some criterion specified in covenants that form part of the debt contracts. Cooper and Mello [1991] - to be discussed later - have recently applied this approach to examine default risks on swaps between a default-risky and a default-free counterparty.

An alternative approach to the modelling of default has recently been proposed by Jarrow and Turnbull [1991]. They model both the default-free term structure, and the term structure representing the *relative* prices of different maturities of default-risky debt, consistently with both of the observed initial term structures, using an extension of the technology independently developed by Heath, Jarrow and Morton [1987],[1989] and Babbs [1990],[1991a]. Default risk enters the model via the *absolute* price of maturing default-risky debt. This price is unity until an exogenously specified jump process, representing cumulative credit downgradings, reaches the "bankruptcy" state; the price then jumps downwards and subsequently follows a diffusion process reflecting the course of corporate restructuring. Under this approach, as under the capital structure approach discussed in the previous paragraph, it is solely the direct losses arising from potential bankruptcy that distinguish corporate bonds from default-free debt and justify the existence of separate term structures.

An altogether different approach is contained in the later sections of Ramaswamy and Sundaresan [1985] (RS). RS assume that the default-free term structure evolves according to the "square root" model of Cox, Ingersoll and Ross [1985], coupled with the Local Expectations Hypothesis - according to which expected total returns from all default-free bonds equal the instantaneous default-free interest rate:[3]

$$E_t[dF(t)] + x(t)dt = F(t)r(t)dt \qquad (1)$$

where F is the price of the bond, x its (continuously paid) coupon rate, and r the instantaneous default-free rate. Turning to default-risky debt, RS do not model explicitly events of default; instead they assume that the expected total returns from default-risky bonds must exceed those on otherwise identical default-free bonds by a maturity-independent differential, p:[4]

$$E_t[dF(t)] + x(t)dt = F(t)\{r(t)+p(t)\}dt \qquad (2)$$

RS do not make explicit which of two possible interpretations they would place on this approach. Under one interpretation, the benefits of holding a default-risky bond to maturity consist precisely of the promised cashflows. Since RS' approach does not include explicit default, these benefits are identical to those arising from an otherwise identical default-free bond. The Law of One Price then contradicts the simultaneous validity of (1) and (2). Jarrow and Turnbull [1991] interpret RS this way and criticize their approach accordingly. Under the alternative interpretation, to which we subscribe, (2) states that the returns from holding default-risky bonds which, *ex post*, do not promptly default, include additional capital gains which accrue at a rate represented by a required return differential p providing compensation for, *inter alia*[5], losses on similar bonds which (also *ex post*) do default.[6] Specifically, equation (2) revises the expected change in price as given by (1) in order precisely to reflect the differential. No anomaly is involved.

We believe that the required return differential approach to spreads between term structures, pioneered by RS, may prove more successful than approaches which attribute the spreads solely to prospective direct losses arising from possible default. Prudent investment in default-risky bonds requires the costs of at least rudimentary credit assessment, and of the implementation of policies to

---

3 see RS equation (5) p260

4 see RS equation (9) p268; we have generalized the coupon for ease of comparison with the equation for default-free bonds

5 See the next paragraph for a discussion of other possible influences on p.

6 Technically, this interpretation involves an additional element of conditioning in the expectation in (2).

ensure diversification by economic sector of the borrowers whose bonds are held. Institutional arrangements may restrict the types of bond in which some agents can invest. Holdings of different classes of bonds may have different taxation implications for some investors and, in the case of bank investors, may be subject to different weightings for the purposes of computing capital adequacy ratios (see Bank for International Settlements [1988]). The markets in different types of bonds may exhibit varying liquidity and other characteristics. All these factors may cause equilibrium yield to differ significantly from those which might plausibly be seen as attributable to the direct costs of possible defaults. A required return differential approach can cover the influences of all manner of factors, essentially regardless of their source.

The *a priori* arguments just advanced in favour of a required return differential approach as opposed to a solely default-oriented approach find empirical encouragement in both Jones, Mason and Rosenfeld [1984] and in RS. Using the capital structure approach of Merton [1974],[1977] and others, Jones *et al.* find evidence (p619) that "[default] risk is not playing a significant role in explaining investment grade bond prices. This also suggests that a stochastic interest rate model could be a better predictor". RS indicate (pp267-8) that they found the capital structure approach "simply unable to account for the magnitude of the discounts [between market prices of floating rate notes and values obtained from the default-free term structure]".

A commonplace assumption of contingent claims analysis is that the agents in the economy can both lend and borrow freely at the same rate of interest. In the literature on default-risky debt, the issuers of the debt play a purely passive role, all securities trading being carried out by investors who are assumed to be default-free. The natural choice of the rate for freely available borrowing is then held to be the default-free rate. In the context of the interest rate swaps market, however, the most prominent participants are banks and other default-risky corporations; these same agents also trade extensively in default-free ("government") bonds. It would therefore be problematic to take default-free interest rates as applicable to freely available borrowing[7]. In contrast, the central authorities, as the issuers of default-free bonds, do not trade actively in swaps, or

[7] Private sector agents' ability to borrow at rates close to default-free rates via mechanisms such as repo markets in government bonds does not permit *net* raising of funds, since such borrowings must be secured by pledging government bonds of greater value. We apply our model to repo markets in Section 7.

even in their own debt[8]. We therefore deem it natural to take the borrowing rates applicable to agents such as banks (ie interbank borrowing rates) as the rates for freely available borrowing. This feature of our work appears to be an innovation.

Of course, having recognised that the prominent participants in the swaps markets are themselves default-risky agents introduces a new problem. The existing literature on the valuation of default-risky claims is devoted to claims held by default-free investors *on* default-risky counterparties[9], rather than claims and obligations held *by* default-risky agents. In the present paper, we sidestep this issue by treating agents in our trading model as default-free. One could argue from this that the required return differential in our model therefore covers factors *other than* default; we would rather intend it as approximating the full reality. It should also be noted that there are strong reasons for regarding default-risk on swaps as far less significant than on bonds: swaps consist of *exchanges* of fixed and floating payments initially perceived as of equal value, and exclude the exchange of the notional principal amount; the value of a swap, unlike that of a bond, is therefore unlikely to exceed a small percentage of the principal amount. The low credit risk on swaps is reflected in the approach of banking regulators expressed in Bank for International Settlements [1988], under which the risk weighting attaching to a new swap is only a tiny fraction of that attaching to a bond or loan of the same principal amount[10].

At first glance, Cooper and Mello [1991] (CM) might be interpreted as indicating that default-risk might play a larger part in the swap term structure than suggested by our previous paragraph. CM adopt the capital structure approach, of Merton [1974],[1977] and others, to value a swap claim by a default-free investor upon a default-risky corporation. After a general analysis in terms of a stylized

---

8 We feel it reasonable to regard the central authorities' open market operations as part of their debt issuance/retirement, rather than as securities trading.

9 We have in mind the term structure literature already cited and, outside such a context, Johnson and Stulz [1987].

10 Under the "original exposure" method the fraction does not exceed $L\%$ where L is the maturity of the swap in years. Under the alternative "current exposure" method, widely viewed as superior, the fraction for a new swap at quoted market rates does not exceed $\frac{1}{2}L\%$.

swap, CM consider the equilibrium spreads[11] on 5-year fixed interest and floating interest debt and on a swap from floating to fixed interest. CM provide illustrative estimates of these spreads; the equilibrium swap spread equals a modest but significant fraction[12] of those on fixed or floating debt. We would conjecture, however, that the swap spread has been substantially increased by CM's assumption - which they accept is of uncertain validity - that swap claims are subordinated to debt in the event of default.

The impact of default-risk on the swap term structure is lessened, not just by the small scale of the exposure relative to that on debt, but by the fact that the typical swap is between two default-risky counterparties - whereas in CM one counterparty is default-free. We believe that the equilibrium spread for a swap between two default-risky counterparties of comparable credit standing - two banks for instance - would be near zero.

The most common variety of interest rate swaps - and those regarded by practitioners as defining the swap term structure - are contracts which provide for periodic "fixed rate" payments by one counterparty, against periodic "floating rate" payments by the other, based on short-term interbank interest rates - rather than the default-free rate used in Sundaresan [1989] (see below) and in Cooper and Mello [1991]. Such swaps should be seen as contingent contracts on future interbank rates. Those rates do not correspond to the future borrowings of an individual corporation, but to borrowings by members of a perpetually renewed pool of corporations who are of good credit standing at the time the funds are raised. Such a pool cannot readily be seen as a single borrower, and indeed cannot collectively default - short of a general banking collapse. In consequence, neither the capital structure approach of Merton [1974] and others, nor that of Jarrow and Turnbull [1991], can be applied to the evolution of interbank rates, since, under both those approaches, only the possibility of explicit default justifies the existence of separate term structures. By extension, neither approach can readily applied to the swap term structure.

We draw the conclusion from the above discussions that a required return differential approach is more appropriate to the joint modelling of the default-free and interest rate swap term structures than one which concentrates heavily on the possibility of default.

---

11 ie the additional *per annum* interest rates required to be paid by the default-risky corporation over and above those payable in comparable default-free contracts.

12 lying between one ninth and one half, depending on the parameter values used

Sundaresan [1989] applies the required return differential approach of Ramaswamy and Sundaresan [1985] to determine the swap term structure, given stochastic processes for the default-free term structure and the required return differential. This entails accepting the RS assumption of the Local Expectations Hypothesis, and the inconsistency of the Cox, Ingersoll and Ross [1985] term structure model with the initial term structure actually observed. While we prefer to avoid such assumptions, we do not thereby imply any theoretical criticism. However, Sundaresan's approach also assumes that agents can borrow freely at default-free interest rates; we have argued above that this is inappropriate to the swap markets.

We believe that we have now given sufficient justification to motivate the modelling approach to be adopted in this paper. Namely, we shall not include the occurrence of default explicitly; instead we shall adopt a generalized version of the required return differential approach of Ramaswamy and Sundaresan both to model, and to justify the existence of, distinct default-free - hereafter "government" - and swap term structures. Since major banks are prominent participants in the swap market, we shall use interbank rates as representing the rates for freely available borrowing.

In Section 2 we rehearse the structure of a simple interest rate swap. This leads us to the fundamental relationship, (3), between swap rates and the swap term structure.

In Section 3 we set out the formal structure of our model. We consider a finite horizon continuous time pure trading economy akin to that used in Harrison and Kreps [1979]. In modelling both the "government" and swap term structures we exploit the technology recently independently developed by Babbs [1990],[1991a] (Babbs) and Heath, Jarrow and Morton [1987],[1989] to model the dynamics of each by means of Ito processes driven by a finite number of Brownian motions, consistently with whatever initial term structures are actually observed. We feel that the last mentioned feature of the technology is of significant practical importance; moreover, we are able in this framework to avoid making strong assumptions on preferences (such as the Local Expectations Hypothesis assumed by Ramaswamy and Sundaresan).

In Section 4 we show that our model of securities trading is viable, ie capable of being supported in a general equilibrium, and that all contingent payoffs are priced by arbitrage, in the senses defined by Harrison and Kreps [1979]; our demonstration builds on extensions to Harrison and Kreps' results, and applications to term structure models, recently obtained by Babbs [1990],[1991a].

Our results hinge on the existence of a unique equivalent martingale measure; we are also able to deduce from this that the model is complete in the sense that all claims can be attained by admissible trading strategies.

We noted above a number of factors underlying a yield differential between the "government" and swap term structures. These factors may bear differently on different agents. How would allowing for this affect the viability of our model? Will agents who apply different required return differentials assign the same values to contingent claims? We show in Section 5 that viability is preserved, and that contingent claim values are assigned unanimously.

Up to the end of Section 5, our framework is quite general, and the valuation formula for contingent claims correspondingly abstract. In Section 6, we illustrate our framework by imposing a particular family of term structure models. We explore briefly the behaviour of the term structure dynamics, and of the spread between short term "government" and interbank/swap interest rates. We also derive a easily computable valuation formula for european-style contingent claims expiring at any date in the economy. Specializing still further, and supplying particular parameter values, we take a reasonably complex example of some interest to market participants - an option to enter a swap, whose fixed rate will be set at the expiry date of the option at a fixed spread above whatever yield is then ruling on a reference government bond - and show how to obtain a numerical valuation.

Section 7 shows how a repo market in "government" securities can be analyzed in our model, and provides secured funding at a cost close to default-free rates of interest.

Section 8 provides concluding remarks.

2 INTEREST RATE SWAPS

The most common variety of interest rate swap is a contract requiring one party, the "fixed payer", to make periodic payments to the other party, over some overall maturity, calculated by applying over each period a fixed rate of interest, the "swap rate", to a constant notional principal amount; in return the other party, the "floating payer", makes periodic payments, over the same overall maturity and with the same notional principal amount, calculated by applying over each period a floating rate of interest equal to the interbank interest rate for the period - as observed at[13] the start of each period.

If we supplement both the "fixed payer's" and the "floating payer's" obligations with a requirement to pay the notional principal amount at the maturity date of the swap, clearly no net payment is required. The supplemented obligations, however, now equate respectively to those on a fixed interest bond with coupon equal to the "swap rate" and to those on a floating rate note paying the periodic interbank rate each period. If we suppose that agents may lend or borrow freely at the interbank rate[14], then the value of this floating rate note must equal its face value: the notional principal amount of the swap. (This argument is essentially identical to that due to Cox, Ingersoll and Ross [1980] p391, which shows that what Ramaswamy and Sundaresan [1985] p256 call a "perfectly-indexed floater" has a value equal to par. It sidesteps the need to value the floater by approaches such as that used by Bicksler and Chen [1986].) Since no premium changes hands when a swap is entered[15], the value of the fixed interest bond must also equal its face value. Applying therefore the well-known decomposition of a coupon bond into a bundle of pure discount bonds[16], we obtain the relationship between swap rates and the swap term structure:

$$c_J \sum_{j=1}^{J} \delta_j B_p(M_j, 0) + B_p(M_J, 0) = 1 \tag{3}$$

where c_J is the swap rate applicable to a maturity of J fixed interest periods, δ_j is the interest accrual factor for the jth period, and $B_p(M_j, 0)$ is the price at the initial time of a "private" sector pure discount bond maturing at the jth payment date, M_j . This relationship resembles that between the coupon on a bond valued at par and the corresponding term structure.

---

13 This observation will, in many currencies, take place two business days prior to the start of the period. Moreover, since the successive interest dates in the swap are calculated from the inception of the swap, the interbank rate observed at the start of a period may refer to an interval of time ending on a date slightly different date from the swap payment date. The analysis we will present generalizes readily to accommodate such practical minutiae.

14 This analysis also sets aside the occurrence of default.

15 Strictly this applies only to the "par" or "at-the-market" swaps most commonly quoted and traded.

16 see eg the start of Section VI of Merton [1974], or of Section I of Cox, Ingersoll and Ross [1981]

By considering the case $J = 1$, it is evident from arbitrage considerations that the one period swap rate must equal the interbank rate for that period.

3 FORMAL STRUCTURE OF THE MODEL

We consider a two-sector continuous trading economy with a finite horizon. One sector (the "government") is not modelled and plays a purely passive role, but is exogenously assumed to issue a range of fixed interest securities. Agents in the other, active "private" sector[17] may trade these securities[18], but may not take short spot[19] positions[20] in them. They also trade fixed interest claims on each other and arbitrary contingent claims.

3.1 THE CONSUMPTION SPACE; AGENTS; INFORMATION

We suppose that consumption occurs only at the outset of the economy, and at the terminal horizon date. Agents' preferences are defined over initial consumption and state-contingent terminal consumption. Formally we construct a consumption space $\Re \times X$ where $X = L^2(\Omega, \mathcal{F}, P)$, and a set of agents, as in Harrison and Kreps [1979] (HK). The technical assumption that state-contingent terminal consumption must have finite second moment affords considerable mathematical convenience.

---

[17] In Section 5 we will subdivide the private sector, in order to discuss the effects of differing levels of required return.

[18] We assume that the aggregate endowments of these agents are sufficient for them to hold all the "government" securities outstanding.

[19] We will assume that spot transactions are for immediate delivery.

[20] We emphasize that this restriction applies only to spot positions; it does not restrict the ability to take short forward positions (which we treat as a particular type of contingent claim). We show in Section 7 how this framework accommodates a repo market such as that operating in US Federal Government securities. In that market, the security provided by the government securities permits long positions to be financed at close to default-free interest rates, even by investors of indifferent credit standing.

The information available to agents at time $t \in [0,T]$ is represented by $\mathcal{F}_t \subseteq \mathcal{F}$ where $\{\mathcal{F}_t : t \in [0,T]\}$ is an increasing sequence of sub-sigma-algebras satisfying the "usual conditions"[21] and where, without loss of generality, $\mathcal{F}_0$ is almost trivial, and $\mathcal{F}_T = \mathcal{F}$.

Following HK, we assume that the (relevant) information available to agents is confined to the past histories of the two term structures:[22]

Assumption *The information filtration $\{\mathcal{F}_t : t \in [0,T]\}$ is that generated by the Brownian motions $Z_g(\ )$, $Z_p(\ )$, introduced below, which drive the term structure dynamics.*

3.2 TERM STRUCTURE DYNAMICS

Let $B_g(M,t)$ and $B_p(M,t)$ denote the prices at t of unit nominal of a pure discount bond maturing at $M \geq t$, issued by "government" and "private" sector borrowers respectively.

Assumption *The term structure dynamics can be described by the following Ito processes*[23]*, consistent with any*[24] *initial term structures:*

$$\frac{dB_g(M,t)}{B_g(M,t)} = \{r_g(t) + s(M,t) + \theta_g(t)\sigma_g(M,t)\}dt + \sigma_g(M,t)dZ_g(t) \qquad (4)$$

and

$$\frac{dB_p(M,t)}{B_p(M,t)} = \{r_p(t) + \theta_p(t)\sigma_p(M,t)\}dt + \sigma_p(M,t)dZ_p(t) \qquad (5)$$

---

21 see eg Karatzas and Shreve [1987] p10

22 The findings of Babbs and Selby [1991] may be used to establish that the key results in this paper - viability; pricing by arbitrage of all contingent claims on either or both term structures; completeness with respect to term structure risks - carry over to a more general information filtration, provided that the volatility, correlation, and required return differential processes are all adapted to the filtration generated by the Brownian motions driving the term structure dynamics.

23 The equations are given in single factor form. The model could be made multi-factor by the simple expedient of adding further terms of similar form; details are left to the reader.

24 Strictly, any initial structures $B_g(M,0)$ and $B_p(M,0)$ whose logarithms are differentiable with respect to *M*.

In these equations $r_g(\ )$, $r_p(\ )$ *are the respective instantaneous spot interest rate processes*[25]. $s(\ )$ *is a deterministic function which we suppose can be written in the form*

$$s(M,t) = \int_t^M \beta(m,t)dm \tag{6}$$

for some function $\beta(\ )$, *and whose role will be discussed further below.* $\theta_g(\ )$, $\theta_p(\ )$ *are essentially bounded predictable processes relating, in some way*[26], *to the "price of interest rate risk" on bonds issued by the two sectors.* $\sigma_g(\ )$, $\sigma_p(\ )$ *are also predictable processes, representing bond price volatilities; and* $Z_g(\ )$, $Z_p(\ )$ *are standard Brownian motions whose innovations have the predictable correlation process given by:*

$$dZ_g(t).dZ_p(t) = \rho(t)\ dt \tag{7}$$

Babbs [1990],[1991a] has shown that such Ito processes can be constructed in a well-defined manner, and that bond prices converge to par at maturity, with probability one.[27]

The role of *s( )* relates to the required return differentials which we will define formally in the next subsection. A portion of these is, of course, reflected in the spread between the parts of expected returns represented by the instantaneous spot rates $r_g(\ )$, $r_p(\ )$. However, this applies to all bonds independently of maturity, and thus cannot reflect any maturity-dependence of required return differentials. Maturity-dependence therefore gives rise to the additional return term *s( )*.[28]

---

25 The instantaneous spot interest rates are of course equal to the yields on immediately maturing bonds:

$$r_k(t) = -\lim_{M \downarrow t} \frac{\ln B_k(M,t)}{M-t}; \quad k = g, p$$

26 We elucidate this relation fully in 4 Remark II below.

27 The presence of *s( )* in (4) requires a slight extension of the construction in Babbs [1990],[1991a]. The key is to regard $B_g(M,t)$ as composed of two factors, one of them being:

$$\exp\left\{-\int_t^M \int_t^m \beta(m,u)dudm\right\}$$

and the other "underlying" process being constructed as set out in Babbs. The rest of the extension is trivial. The presence of a similar term in (5) would give rise to an arbitrage anomaly; in (4) however the effect is offset by the required return differential *D( )* - see (16).

28 The precise relationship is determined in 4 Theorem I as equation (16).

A significant implication of the presence of *s( )* can best be seen by considering the deterministic case, $\sigma_g(\ ) \equiv \sigma_p(\ ) \equiv 0$. The evolution of the "private" sector term structure is then, of course, determined by its initial state. In particular, the instantaneous spot interest rate at each date, $r_p(\ )$, equals the initial instantaneous forward rate for that date:

$$r_p(M) = -\frac{\partial}{\partial M}\ln B_p(M,0) \tag{8}$$

Due to *s( )*, the corresponding equation for the "government" instantaneous spot rate is:

$$r_g(M) = -\int_0^M \beta(M,u)du - \frac{\partial}{\partial M}\ln B_g(M,0) \tag{9}$$

where the relation between *s( )* and $\beta(\ )$ is given by (6). Without *s( )* therefore, both instantaneous spot rates would be given by the initial forward rates, with possibly implausible consequences for the spread between them. Thus the presence of *s( )* implies that, even in the deterministic case, the initial term structures alone do not dictate the future course of the spread between the instantaneous spot rates.

Equation (6) ensures that the instantaneous spot rate matches the yield on immediately maturing bonds; its particular form ensures that the government term structure possesses instantaneous forward rates for all dates and at all times.

3.3 REQUIRED RETURN DIFFERENTIALS

Due to a variety of factors discussed in the introduction, agents require return differentials between the "government" and swap term structures. Since these differentials may be maturity specific we model them by means of the family - indexed by *M* - of predictable processes $D(M,t)$ which we treat formally as attaching to returns at time *t* on the "government"[29] pure discount bond maturing at *M*. The total returns achieved from holdings of "government" bonds are obtained by adding the required return differential to the bond returns. Thus, the

---

[29] It turns out that we must attach the differential to the "government" rather than the "private" sector term structure in order to ensure that a contingent claim between two agents, which happens to promise a fixed payment from one to the other, is valued by discounting the payment using "private" sector rather than "government" discount factors.

total return over the period from t to $t + dt$ from holding one unit nominal of the "government" bond maturing at M is:

$$dB_g(M,t) \; + \; B_g(M,t) \;\; D(M,t) \;\; dt \qquad (10)$$

We assume that the risks associated purely with the price volatility of government bonds are rewarded through the price dynamics of those bonds rather than being reflected in the required return differential. More formally:

Assumption *$D(\;)$ has no direct dependence on the anticipated volatility $\sigma_g(\;)$*

3.4 TRADING STRATEGIES

We adopt what is essentially the most general possible formulation of the concept of a trading strategy, adapted slightly to cope with the required return differential.

As HK were the first to point out, some restriction must be placed on trading strategies to eliminate arbitrage and - a stronger condition - to ensure viability; HK restricted agents to "simple" strategies, involving a finite set of trading dates, which ensured *inter alia* that the corresponding normalized value processes were martingales under a risk adjusted change of probability measure - an "equivalent martingale measure" (EMM). Restrictions sufficient to give this latter property are commonplace in the subsequent literature to prevent arbitrage opportunities. Babbs [1990] (chapter 5, pp129-40), [1991b] showed that the martingale property suffices also for viability with the more general class of trading strategy allowed here; following him therefore, we commence with a provisional definition, adding the EMM-related restriction later once the existence of a (unique) EMM has been established in Section 4.

We allow a trading strategy to involve trading any finite number of securities in an arbitrary fashion, subject only to: some obviously necessary exclusions; budget constraints; and some modest technical regularity conditions. To formalize this, let I be an index to the collection of securities traded in the markets, excluding "government" debt. We initially restrict I to cover only pure discount bonds issued by the "private" sector, but we will word our definition to assist our later generalizations to cover trading in contingent claims:

Provisional definition *A trading strategy is: some finite selection of maturity dates $M_1, \ldots, M_k$; a finite selection of securities, indexed as $a_1, \ldots, a_n \in I$; together with k-dimensional and n-dimensional real valued stochastic processes*

$\gamma : [0,T]\times\Omega \rightarrow \Re^k$ *and* $\psi : [0,T]\times\Omega \rightarrow \Re^n$. *The ith component,* γ_i, *of the former process represents time-state-dependent nominal holdings of the "government" pure discount bond maturing at* M_i; *the jth component,* ψ_j, *of the latter process fulfils the same role for security* a_j. *The processes must satisfy the following requirements:*

(i) γ, ψ *are predictable*[30];

(ii) *the stochastic integrals in (12) are well-defined;*[31]

(iii) *no short spot positions in government bonds:*
$\gamma_i(t) \geq 0, \quad \forall i,t$

(iv) $\gamma_i(t) = 0 \quad \forall t > M_i$. *Similarly,* $\psi_j(t) = 0$ *if security* a_j *matures strictly before* t;

(v) *"self-financing":* $\forall t,\omega$:

$$V(t) \equiv \sum_{i=1}^{k} \gamma_i(t) B_g(M_i,t) + \sum_{j=1}^{n} \psi_j(t) S_j(t) \tag{11}$$

$$= V(0) + \sum_{j=1}^{n} \int_0^t \psi_j(u) dS_j(u)$$

$$+ \sum_{i=1}^{k} \int_0^t \gamma_i(u) \{dB_g(M_i,u) + B_g(M_i,u) D(M_i,u) dt\} \tag{12}$$

where $S_j : [0,T]\times\Omega \rightarrow \Re$ *is the price process of security* a_j

(vi) *terminal value in the terminal consumption space:*

$$V(T,\omega) \in X \tag{13}$$

---

30 A process is said to be predictable if it is measurable with respect to the sub-sigma-algebras generated by left-continuous adapted processes. Making this requirement prevents arbitrage opportunities arising from exploiting price jumps (see Harrison and Pliska [1981] p237).

31 Babbs [1990] achieves this condition by means of the stronger requirement that holdings of securities are locally bounded (see eg Elliott [1982] Definition 6.37 p58). The benefit of the present approach is that, as Harrison and Pliska [1981] pointed out, the replication of even some bounded contingent claims in eg the Black-Scholes economy requires strategies which are not locally bounded.

Remark Requirement (v) above adapts the usual definition of "self-financing" strategy by including the required return differential in the returns on "government" bonds in defining the value process of a trading strategy. This implicitly suggests that the differential returns are immediately realised in cash. We must think in terms of funds being transferred between the trading strategy's portfolio and some kind of reserve. We feel that our formulation is clear and reasonable, and that formalizing a transfer process would add to the technical complexity of our model without benefit to its substance.

4 VIABILITY AND PRICING BY ARBITRAGE; COMPLETENESS

We normalize security prices and trading strategy wealth processes by dividing through by the price of a numeraire security. We select for this purpose the "private" pure discount bond maturing at *T*. The key technical device in HK is the "equivalent martingale measure", which we now define in our context:

Definition I *A probability measure P^* on $(\Omega, \mathcal{F})$ is said to be an equivalent martingale measure (EMM) if and only if the following three conditions hold:*

(i) *P^* and P are equivalent, ie have the same null sets. A necessary and sufficient condition for this is that the Radon-Nikodym derivative*

$$\frac{dP^*}{dP}$$

be strictly positive[32].

(ii) *after adjusting for required return differentials*[33], *normalized security price processes are martingales under P^**

(iii) $$\frac{dP^*}{dP} \in L^2(\Omega, \mathcal{F}, P) \tag{14}$$

---

[32] Note that since P, P^* have the same null sets, we may use the term "almost surely" without qualification as to which probability measure is intended.

[33] No adjustment is required for holdings of "private" sector bonds; for the "government" pure discount bond maturing at *M*, the adjusted price process (before normalization) is

$$B_g(M,t) \ \exp\left\{\int_0^t D(M,u)du\right\}$$

obtained by "re-investing" the required return differential in the bond.

It is convenient at this stage to re-express $Z_g(\ )$, $Z_p(\ )$ in terms of two independent standard Brownian motions $W_1(\ )$, $W_2(\ )$ where

$$dZ_p(t) = dW_1(t) \tag{15a}$$

$$dZ_g(t) = \rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}\,dW_2(t) \tag{15b}$$

Theorem I *The model specified in Section 3 has at most one EMM.*

If this EMM, P^*, *exists, then*

$$D(M,t) = r_p(t) - r_g(t) - s(M,t) \tag{16}$$

and

$$\xi(t) \equiv \exp\left\{-\int_0^t \{\theta_p(u)-\sigma_p(T,u)\}dW_1(u) - \int_0^t \frac{\theta_g(u)-\rho(u)\theta_p(u)}{\sqrt{1-\rho^2(u)}}dW_2(u) - \frac{1}{2}\int_0^t \{\theta_p(u)-\sigma_p(T,u)\}^2 + \frac{\{\theta_g(u)-\rho(u)\theta_p(u)\}^2}{1-\rho^2(u)}du\right\}$$

$$\text{is a } P\text{-martingale, with } \xi(T) \in L^2(P) \tag{17}$$

and P^* *has Radon-Nikodym derivative*

$$\frac{dP^*}{dP} = \xi(T) \tag{18}$$

Conversely, if (16) and (17) hold, define a probability measure, P^*, *by (18). Then a sufficient condition for* P^* *to be an EMM is that:*

$$\exp\left\{\int_0^t \{\sigma_p(M,u)-\sigma_p(T,u)\}dW_1^*(u) - \frac{1}{2}\int_0^t \{\sigma_p(M,u)-\sigma_p(T,u)\}^2du\right\}$$

$$\text{is a } P^*\text{-martingale } \forall M\in[0,T] \tag{19}$$

and that

$$\exp\left\{\int_0^t \{\sigma_g(M,u)\rho(u)-\sigma_p(T,u)\}dW_1^*(u) + \int_0^t \sigma_g(M,u)\sqrt{1-\rho^2(u)}\,dW_2^*(u) - \frac{1}{2}\int_0^t \{\sigma_g(M,u)\rho(u)-\sigma_p(T,u)\}^2+\sigma_g^2(M,u)\{1-\rho^2(u)\}du\right\}$$

$$\text{is a } P^*\text{-martingale } \forall M\in[0,T] \tag{20}$$

where $W_1^*(\ )$, $W_2^*(\ )$ *are standard Brownian motions under* P^*, *given by (A15a)-(A15b).*

Proof see Appendix.

Remark I By the Novikov condition[34], (19) and (20) will always be fulfilled if volatility is deterministic. Thus (16) and (17) are sufficient for the existence of a unique EMM for that case.

Remark II We can now clarify the roles of $\theta_g(\ )$, $\theta_p(\ )$ as prices of risk. Using (15a)-(15b) and (16), we can re-express (4) and (5) as:

$$\frac{dB_g(M,t)}{B_g(M,t)} = \mu(M,t)dt + \sigma_g(M,t)\rho(t)dW_1(t)$$

$$+ \sigma_g(M,t)\sqrt{1-\rho^2(t)}dW_2(t) \tag{21}$$

where

$$\mu(M,t) = r_p(t) - D(M,t) - s(M,t) + \theta_p(t)\sigma_g(M,t)\rho(t)$$

$$+ \frac{\theta_g(t)-\rho(t)\theta_p(t)}{\sqrt{1-\rho^2(t)}}\sigma_g(M,t)\sqrt{1-\rho^2(t)} \tag{22}$$

and

$$\frac{dB_p(M,t)}{B_p(M,t)} = \{r_p(t)+\theta_p(t)\sigma_p(M,t)\}dt + \sigma_p(M,t)dW_1(t) \tag{23}$$

Thus $\theta_g(\ )$, $\theta_p(\ )$ jointly specify the price of the (statistically independent) risks corresponding to innovations in $W_1(\ )$ and $W_2(\ )$.

If we now impose the following:

Regularity Condition *The conditions (16), (17), (19) and (20) of Theorem I are fulfilled.*

and impose on trading strategies an additional requirement:

Definition II *Define a trading strategy by the Provisional Definition in Section 3.4 above, supplemented by the following additional requirement:*

(vii) $V(t)\ /\ B_p(T,t)$ *is a* P^* *-martingale*

then we can derive a further important result:

---

[34] see eg Karatzas and Shreve [1987] Corollary 3.5.13 p199

Theorem II *Our model is viable. Arbitrage opportunities and "suicide" strategies are precluded. Moreover, every contingent claim is uniquely priced by arbitrage at a value:*

$$S_x = B_p(T,0)\, E^*[\ x\ ] \tag{24}$$

where $x \in X$ *is the payoff of the claim at the terminal date of the economy*[35], *and* E^* *denotes expectations under* P^*. *Indeed, our model is complete in the sense that every contingent claim can be attained by an admissible trading strategy.*

Proof By the above Regularity Condition, Theorem I holds, and so there exists a unique EMM.

As we noted in Section 3.4 when introducing our Provisional Definition of trading strategy, HK established a link between the existence of an EMM and viability, under the restriction that agents pursue only "simple" trading strategies. Babbs, however, ([1990] pp129-40; [1991b]) has strengthened this result by showing that it suffices merely that the normalized value processes of trading strategies be martingales under the EMM. The requirement (vii) applied to trading strategies by Definition I of the present section therefore now gives us all that we require to establish viability.

Requirement (vii) from Definition I is also readily shown to preclude arbitrage and suicide strategies.

The pricing formula is simply the pricing system corresponding to the (unique) EMM.

Completeness follows from the fact that any square-integrable random variable, measurable with respect to a Brownian filtration, can be represented as the terminal value of a stochastic integral with a predictable integrand (see Elliott [1982] Theorem 12.33 p144)[36]. ▮

Finally, we can now relax our initial restriction that the index I of traded securities other than "government" bonds covers only "private" bonds, to admit the trading of contingent claims:

---

35 Claims whose payoffs occur before T are included by assuming that any such payoffs are reinvested in the numeraire security: the private sector pure discount bond maturing at T.

36 We can ensure that our prohibition of short spot positions in government bonds is met by appropriate use of forward contracts.

<u>Theorem III</u> *Let the Regularity Condition above hold. Define* P^* *by (17) and (18). Let all contingent claims have measurable and adapted price processes. Expand the index* I *of traded securities to admit such claims.*

Then the model is viable if and only if the price process of the contingent claim giving a payoff x *at* T *is of the form:*[37]

$$S_x(t) = B_p(T,t)\ E_t^*[\ x\ ] \qquad \forall t \in [0,T] \qquad (25)$$

<u>Proof</u> By Theorem I, P^* is the sole candidate EMM. Equation (25) is precisely what is required to ensure that the discounted price processes of contingent claims are P^* -martingales. The result now follows by the existence of a unique EMM, just as in Theorem II above. ▮

5 <u>HETEROGENEOUS REQUIRED RETURN DIFFERENTIALS</u>

We noted earlier that the factors, which lead "private" agents to require a return differential between the "government" and swap term structures, may bear more heavily on some "private" agents than on others. We now proceed to show that allowing for this possibility does not disrupt our model; moreover, all "private" agents assign the same values to contingent claims.

We note from 4 Theorem I that, even in the present extended context, there remains just a single candidate EMM, since the Radon-Nikodym derivative of P^*, given by (17)-(18), is independent of required return differentials. Thus the sole candidate price system for all contingent claims is that given by (24) in 4 Theorem II. Certainly therefore, if the model remains viable, all agents assign the same values to contingent claims.

The "threats" to the viability of that pricing system are twofold. Firstly, agents applying large required return differentials derive correspondingly large total returns (ie after adjusting for the differential) from the government bonds concerned. This suggests that they might obtain excess returns (corresponding to trading strategies whose normalized value processes are submartingales under the EMM) by holding relevant government bonds. Secondly, agents applying small

---

37 Like the corresponding result in HK, this result covers a claim expiring before T, by defining x to be the time T result of reinvesting the expiry proceeds of the claim in the numeraire security.

differentials would derive inferior returns from holding the government bonds concerned, suggesting they might obtain excess returns by shorting them. We proceed to show both these avenues to be closed.

Suppose that there are K classes of "private" agents, applying different required return differentials $D_i(\ )$, $i = 1, \ldots, K$. To avoid the possibility of trading strategies, involving long positions in government bonds, having normalized value processes which are submartingales under the EMM, it is easy to show that we must expand 4 Regularity Condition by assuming that (16) holds with:

$$D(M,t) \equiv \max_{i=1,\ldots,K} D_i(M,t) \qquad (26)$$

representing, for each bond and at each date, the largest differential applied by any of the K classes of agents.

The remaining "threat" to the viability of the constructed price system is, as indicated above, the possibility of obtaining excess returns by shorting bonds. We have prohibited short spot positions in these securities (3.4 Provisional Definition (iii)), but it might appear that the effect of this restriction is undermined by the unfettered scope to take short forward positions via trading in contingent claims. This avenue also proves to be closed, since under the candidate price system, all contingent claims have martingale normalized price processes. Viability follows[38], provided that in equilibrium, all government bonds in issue are held at all times by agents then applying to them the largest required return differentials, ie for whom

$$D_i(M,t) = D(M,t) \qquad (27)$$

This is easily established in the form of the following proposition.

<u>Proposition</u> *Under the constructed price system, the* i*th agent will not hold at time* t *the "government" bond maturing at* M *unless (27) holds. Conversely, all "government" bonds in issue will be held at all times by agents then applying the largest required return differential.*

---

38 To make this step fully rigorous, in terms of the analysis of Harrison and Kreps [1979], we suppose that agents prefer claims to one another on the basis of their costs under the candidate price system, and we "endow" agents with trading strategies which distribute the government bonds in issue among the agents so that they are always held by those applying the greatest required return differential. No agent then has any incentive for further trade.

Proof Inclusive of the required return differentials *D( )*, the discounted value process of any trading strategy is a P^*-martingale. Inclusive of any smaller differential therefore, the discounted value process, $N(t)/B_p(T,t)$ of any strategy involving holding "government" bonds is a P^*-supermartingale[39]; in particular, the initial cost/proceeds *N(0)* of the strategy satisfies

$$N(0) \; > \; B_p(T,t)\; \mathrm{E}^*[N(T)] \tag{28}$$

But the RHS of this equation represents the price of the terminal payoff *N(T)*, as constructed. Thus it would be cheaper to purchase the payoff as a contingent claim than to generate it by the trading strategy. The result follows.

For the converse, we recall that we assumed above (footnote 18) that the aggregate endowments of the "private" sector suffice to hold all government bonds in issue, and that holdings can be funded by borrowings. ▮

6 ILLUSTRATION

To illustrate our general framework, we consider the special case where the volatility and correlation processes are deterministic and the former can take the form:

$$\sigma_g(M,t) \;=\; \{G_g(M)-G_g(t)\}\;\lambda_g(t) \tag{29a}$$

$$\sigma_p(M,t) \;=\; \{G_p(M)-G_p(t)\}\;\lambda_p(t) \tag{29b}$$

where the functions on the RHS are essentially arbitrary. Babbs [1990] (chapters 4 and 6-7, especially pp126-8, pp158-60, and pp175-81) has established the significance of this form by showing that it is precisely the deterministic form required if the future behaviour of each individual term structure is to be described under the EMM in terms of a single Gaussian state variable. We will suppose here that $G_g(\;)$, $G_p(\;)$ are twice differentiable and that the first derivatives are bounded below away from zero.

Of particular note are the implications for the processes for the instantaneous spot rate of interest:

$$dr_p(t) \;=\; \mu_p'(t)dt \;-\; \frac{G_p''(t)}{G_p'(t)}\{\mu_p(t)-r_p(t)\}dt \;-\; G_p'(t)dY_p(t) \tag{30}$$

where

$$\mu_p(t) \;=\; f_p(t) \;+\; G_p'(t)\int_0^t \{G_p(t)-G_p(u)\}\lambda_p^2(u)du \tag{31a}$$

---

[39] We use, of course, here the restriction that holdings of "government" bonds are non-negative.

$$dY_p(t) = \theta_p(t)\lambda_p(t)dt + \lambda_p(t)dZ_p(t) \tag{31b}$$

and $f_p(\ )$ is the initial instantaneous forward rate curve:

$$f_p(M) = -\frac{\partial}{\partial M}\ln B_p(M,0) \tag{31c}$$

The process for the "government" spot rate is similar except that (49c) becomes:

$$f_g(M) = -\int_0^M \beta(M,u)du - \frac{\partial}{\partial M}\ln B_g(M,0) \tag{31c'}$$

These equations look complicated, but on closer inspection can be seen to represent generalized mean-reverting random walks, where the means are adjusted versions of the initial forward rate curve. They may therefore be regarded as making the mean-reversion model of Vasicek [1977] consistent with whatever initial term structures are actually observed, while simultaneously liberalizing the mean-reversion, volatility, and risk parameters.

The implied behaviour of the spread $r_{spread} \equiv r_p - r_g$ between the government and private instantaneous spot rates is somewhat complex, depending in general on the level of interest rates as well as on the current spread. However, in the special - but not at all implausible - case that the government and swap term structures exhibit identical mean reversion: $G_g(\ ) \equiv G_p(\ ) \equiv G(\ )$ say, then the spread exhibits generalized mean-reverting behaviour similar to that of the two spot rates themselves:

$$dr_{spread}(t) = \mu_{spread}'(t)dt - \frac{G''(t)}{G'(t)}\{\mu_{spread}(t) - r_{spread}(t)\}dt - G'(t)dY_{spread}(t) \tag{32}$$

where

$$\mu_{spread}(t) = f_p(t) - f_g(t) + G'(t)\int_0^t \{G(t)-G(u)\}\{\lambda_p^2(u)-\lambda_g^2(u)\}du \tag{33a}$$

$$dY_{spread}(t) = \{\theta_p(t)\lambda_p(t) - \theta_g(t)\lambda_g(t)\}dt + \lambda_p(t)dZ_p(t) - \lambda_g(t)dZ_g(t) \tag{33b}$$

but with instantaneous standard variance:

$$\{G'(t)\}^2 \ \{\lambda_p^2(t) - 2\lambda_p(t)\rho(t)\lambda_g(t) + \lambda_g^2(t)\} \tag{34}$$

which, for realistic parameter values[40], will be very much lower.

---

[40] Informal empirical investigation on the correlation of US dollars swap rates with yields on government bonds, for example, suggests that $\rho(\ )$ exceeds 90%; volatilities in the two markets are comparable.

We will now derive a readily computable valuation formula for european-style contingent claims - on either or both term structures - defined as follows:

Definition *A european contingent claim produces a single payoff at some fixed date, which is a function of the prices, at that date, of some fixed finite collections of "government" and "private" pure discount bonds. Thus, formally, it consists of: some fixed* $M \in [0,T]$ *; collections* $M_{g1} < \ldots < M_{gJ},\, M_{p1} < \ldots < M_{pK} \subset [M,T]$ *; and a payoff function:*

$x(\ B_g(M_{g1},M),\ldots,B_g(M_{gJ},M),\ \ B_p(M_{p1},M),\ldots,B_p(M_{pK},M)\ )$

satisfying the regularity condition:

$$\frac{x}{B_p(T,M)} \in X \tag{35}$$

Theorem *A european contingent claim is priced by arbitrage at:*

$$S_x(0) \;=\; B_p(T,0)\ \mathrm{E}^*\left[\frac{x}{B_p(T,M)}\right] \tag{36}$$

which can conveniently (for many purposes) be re-expressed as

$$S_x(0) \;=\; B_p(M,0)\ \mathrm{E}^{(M)}[x] \tag{37}$$

where $\mathrm{E}^{(M)}$ *denotes expectations under a probability measure* $P^{(M)}$ *on* $(\Omega,\mathcal{F}_M)$ *equivalent to P, and having Radon-Nikodym derivative*

$$\begin{aligned}\frac{dP^{(M)}}{dP} \;=\; \exp\Bigg\{ &-\int_0^M \{\theta_p(u)-\sigma_p(M,u)\}dW_1(u) \\ &- \int_0^M \frac{\theta_g(u)-\rho(u)\theta_p(u)}{\sqrt{1-\rho^2(u)}}dW_2(u) \\ &-\frac{1}{2}\int_0^M \{\theta_p(u)-\sigma_p(M,u)\}^2 + \frac{\{\theta_g(u)-\rho(u)\theta_p(u)\}^2}{1-\rho^2(u)}du\Bigg\}\end{aligned} \tag{38}$$

Proof The theorem follows from 4 Theorem II by straightforward manipulations. For details of these in an analogous result see Babbs [1990] (6.3.1 Theorem, pp154-6). ▮

Remark The above Theorem holds in general, not just for the volatility and correlation structures assumed in this example. It can be thought of in terms of working in terms of an equivalent probability measure under which expected spot prices at *M* equal current forward prices for settlement at that date. In a stochastic interest rate setting, the concept of risk neutrality requires qualification as to whether risk is measured in current value terms or in terms of forward values at some horizon; Babbs and Salkin [1989] have therefore argued that this "current forward equals expected spot" idea can be regarded as more fundamental than the celebrated "risk neutrality" insight into option pricing, due to Cox and Ross [1976].

Proposition *For european contingent claims, and with the deterministic specifications of volatility and correlation assumed in this Section, the general formula (25), given by 4 Theorem III, for the price process* $S_x(\ )$ *of a contingent claim giving a payoff* x *at* T *, reduces to:*

$$S_x(t) \;=\; B_p(M,t)\ \mathrm{E}_t^{(M)}[\ x(\vec{h}_g,\vec{h}_p)\ ] \tag{39}$$

where $\vec{h}_g$, $\vec{h}_p$ *are respectively J- and K-dimensional vectors with representative ith elements:*

$$h_{gi} \;=\; \frac{B_g(M_{gi},t)}{B_g(M,t)} \exp\left\{ \{G_g(M_{gi})-G_g(M)\}X_g + \int_t^M a(M_{gi},u)-a(M,u)du \right\} \tag{40a}$$

$$h_{pi} \;=\; \frac{B_p(M_{pi},t)}{B_p(M,t)} \exp\{\ \{G_p(M_{pi})-G_p(M)\}X_p - \frac{1}{2}\{G_p(M_{pi})-G_p(M)\}^2\ \mathrm{E}_t^{(M)}[X_p^2]\ \} \tag{40b}$$

representing $B_g(M_{gi},M)$ *and* $B_g(M_{pi},M)$ *respectively, where*

$$a(L,u) \;\equiv\; s(L,u) \;+\; \rho(u)\sigma_p(M,u)\sigma_g(L,u) \;-\; \frac{1}{2}\sigma_g^2(L,u) \tag{41}$$

and where X_g, X_p *are bivariate Normally distributed under* $P^{(M)}$ *with zero means and the following (co-)variances:*

$$\mathrm{E}^{(M)}[X_g^2] \;=\; \int_t^M \lambda_g^2(u)du \tag{42a}$$

$$\mathrm{E}^{(M)}[X_p^2] \;=\; \int_t^M \lambda_p^2(u)du \tag{42b}$$

$$\mathrm{E}^{(M)}[X_g X_p] \;=\; \int_t^M \lambda_g(u)\rho(u)\lambda_p(u)du \tag{42c}$$

Proof see Appendix.

6.1 WORKED EXAMPLE

We take as our example a 6-month european-style option to enter a 2-year interest rate swap, with a notional principal amount of $50mn, under which the option-holder would receive semi-annual fixed interest at a rate 0.5% higher than the yield, at the expiry of the option, on a 7.75% government bond maturing 1 year and 11 months later. A theoretically coherent valuation model for such an instrument, consistent with the observed initial term structures, has hitherto been lacking; we will demonstrate that it can be valued quite easily under our model. In our model, the value of the option is given by equation is (39) in 6 Proposition; it is this equation we need to compute.

The first step is to formulate the payoff function x precisely. Assuming that the bond pays coupons semi-annually and that yields are expressed in like manner, the yield y just specified equates the value of the outstanding cashflows on the bond with the sum of their present values discounted at y. Thus, measuring time in years,[41] y is defined by:

$$\frac{17.75}{2\ 100}\sum_{i=1}^{4}\left(1+\frac{1}{2}y\right)^{-\left(i-\frac{1}{6}\right)} + \left(1+\frac{1}{2}y\right)^{-3\frac{5}{6}}$$

$$= \frac{17.75}{2\ 100}\sum_{i=1}^{4} B_g\left(\frac{i}{2}+\frac{5}{12},\frac{1}{2}\right) + B_g\left(2\frac{5}{12},\frac{1}{2}\right) \qquad (43)$$

and, using (3), the payoff to the option holder is, per unit nominal:

$$x = \max\left\{\frac{1}{2}(y+0.5\%)\sum_{i=1}^{4} B_p\left(\frac{1}{2}(i+1),\frac{1}{2}\right) + B_p\left(2\frac{1}{2},\frac{1}{2}\right),\ 0\right\} \qquad (44)$$

Our valuation result (ie 6 Proposition) directs our attention next to the joint distribution, under $P^{(M)}$, of the various $B_g\left(\frac{i}{2}+\frac{5}{12},\frac{1}{2}\right)$ and $B_p\left(\frac{1}{2}(i+1),\frac{1}{2}\right)$; this is given by (40a)-(42c). One method of computing the option value now readily presents itself:

(i) Once the volatility functions $G_g(\ )$, $\lambda_g(\ )$, $G_p(\ )$, $\lambda_p(\ )$, the correlation function $\rho(\ )$, and the addition return function $s(\ )$ have been numerically specified, the various integrals of combinations of these functions, needed in equations (41)-(42c), can be obtained either analytically or, failing that, numerically.

---

[41] Purely for ease of exposition, we treat every month as being exactly a twelfth of a year. We make similar simplifying assumptions below in respect of swaps.

(ii) Replace the correlated variables X_g, X_p with independent standard Normal variates Y_1, Y_2 where:

$$X_g = \sqrt{\int_0^{\frac{1}{2}} \lambda_g^2(u)du} \; Y_1 \tag{45a}$$

$$X_p = \frac{\int_0^{\frac{1}{2}} \lambda_g(u)\rho(u)\lambda_p(u)du}{\sqrt{\int_0^{\frac{1}{2}} \lambda_g^2(u)du}} Y_1$$

$$+ \sqrt{\int_0^{\frac{1}{2}} \lambda_p^2(u)du - \frac{\left(\int_0^{\frac{1}{2}} \lambda_g(u)\rho(u)\lambda_p(u)du\right)^2}{\int_0^{\frac{1}{2}} \lambda_g^2(u)du}} \; Y_2 \tag{45b}$$

(iii) For any value of Y_1, we may obtain *y* from (45a), (40a) and (43). Then, for any value of Y_2, we can use this value of *y* to derive the option payoff *x* from (45b), (40b) and (44).

These steps can be incorporated into a simple quadrature scheme to evaluate $E^{(M)}[x]$, and hence, via (39), the option value $S_x(0)$. We now illustrate this procedure by means of our example. As that example is essentially only illustrative, we will adopt one of the simplest specifications consonant with plausibility, and supply stylized numerical values. We suppose that

$$G_g(x) = G_p(x) = \frac{1 - e^{-\xi x}}{\xi} \tag{46a}$$

$$\lambda_g(x) = \lambda_p(x) = e^{\xi x} \; \kappa \tag{46b}$$

with

$$\xi = 0.1531; \quad \kappa = 0.01449 \tag{46c}$$

We also suppose that the instantaneous correlation between the term structures is constant:

$$\rho \equiv 0.9 \tag{47}$$

The significance of (46a)-(46b) is that the instantaneous spot interest rates both follow "Vasicek-type" dynamics, consistent with whatever the initial term structures. By this we mean here that they exhibit constant variability, with a constant rate of reversion towards deterministic means. To see this, we substitute (46a)-(46b) into (30) to obtain:

$$dr_l(t) \;=\; \mu_l'(t)dt \;-\; \xi\{\mu_l(t)-r_l(t)\}dt$$

$$-\;\theta_l(t)\kappa dt \;-\; \kappa dZ_l(t); \qquad l=g,p \tag{48}$$

The values chosen for ξ, κ are annualized versions of the values reported in a study of US government bond data by Dybvig [1989][42]. We further suppose that the initial term structures are flat, with:

$$B_g(m,0) \;=\; e^{-8.0\%\, m}, \qquad \forall m \tag{49a}$$

$$B_p(m,0) \;=\; e^{-8.5\%\, m}, \qquad \forall m \tag{49b}$$

Finally, we suppose that the required return differential between government and private sector bonds increases gradually with maturity. (16) indicates that this is one way of viewing the role of *s( )*. In turn, (6) suggests we specify *s( )* via $\beta(\;)$; we choose to set

$$\beta \;\equiv\; -0.03\% \tag{50}$$

so that the required return differential on zero coupon bonds, for example, rises by 0.03% for every year of their tenor.

We have now assembled a detailed specification of all the ingredients we require. All that remains is to carry out the steps (i)-(iii) above, and apply a numerical quadrature scheme to evaluate the expectation on the RHS of our valuation equation, (39). As these computations are now purely mechanical, we shall simply state the result: the value of the option described above is $132,500; equivalent to 0.265% of the notional principal amount.

7 REPOS

We have assumed that the interest rates for freely available borrowing are those on the "private" sector term structure. Where a repo market exists, such as that in US Federal Government securities, it might appear that an alternative - and cheaper - means of funding exists, since short-term quoted repo rates are usually significantly lower than interbank rates, being more akin to Treasury bill yields.

That this is not the case can be seen by considering a purchase of bonds, using repo funding. As soon as the bonds are purchased, they are pledged as collateral for a loan from the repo counterparty. The proceeds of the loan are used to

---

[42] In Dybvig's notation the parameters κ, η for which he reports estimates on p13.

supply the purchase price of the bonds.[43] The initial flows of both bonds and cash therefore cancel out. At the maturity of the repo, unencumbered title to the bonds is recovered by repaying the repo loan with interest. The net effect is therefore essentially identical to that of entering a contingent claim in the form of a forward contract to purchase the bonds, on the maturity date of the repo, and for a sum equal to the repayment cost of the repo loan. By our results on the viability of our model - specifically 4 Theorem III - therefore, no theoretical anomaly arises. Nevertheless, the practical issue remains, of whether models within our framework suggest short-term repo rates closer to government than private sector rates.

Under the family of models discussed in Section 6, the formula in 6 Proposition readily gives the fair forward price at time zero to purchase, at time *M*, one unit nominal of the "government" pure discount bond[44] maturing at *M'* as:

$$\frac{B_g(M',0)}{B_g(M,0)} \exp\left\{\int_0^M s(M',u)-s(M,u)\,du + \int_0^M \{\sigma_g(M',u)-\sigma_g(M,u)\}\{\rho(u)\sigma_p(M,u)-\sigma_g(M,u)\}du\right\} \quad (51)$$

Dividing this forward price by the spot price of the bond, and expressing the ratio as a continuously compounding funding cost, y, over the period $[0,M]$, we obtain:

$$y = -\frac{1}{M}\ln B_g(M,0) + \frac{1}{M}\int_0^M s(M',u)-s(M,u) + \{\sigma_g(M',u)-\sigma_g(M,u)\}\{\rho(u)\sigma_p(M,u)-\sigma_g(M,u)\}\;du \quad (52)$$

which, since the yields on immediately maturing bonds equal the instantaneous interest rate (see footnote 25), by the continuity of the integrand[45], and by (6),

$$\to r_g(0) + s(M',0), \quad \text{as} \quad M \to 0 \quad (53)$$

---

[43] In practice, the repo loan would be slightly less than the value of the bonds, in order to provide a greater margin of security for the loan. The outcome of the argument is not, however, affected by such details.

[44] The analysis for coupon bonds follows readily from the observation that a coupon bond is a linear bundle of pure discount bonds.

[45] Strictly, the integrand will only be continuous if we exclude essentially pathological behaviour of $\beta(\;)$, $\sigma_g(\;)$ and $\sigma_p(\;)$.

Thus, provided the maturity-dependence of required return differentials - which we recall from (16) is precisely what *s( )* represents - is small, this family of models within our general framework gives short-term repo rates close to government rates.

An interesting converse of (53) is that, at least for the family of models discussed in Section 6, the dependence of short-term repo rates upon the maturity of the government bonds in question is a possible source of evidence on the maturity-dependence of required return differentials.

8 CONCLUDING REMARKS

In this paper, we have argued that interest rate swaps are contracts contingent on future borrowing rates of banks then of good credit worthiness, and that differences between the term structures for default-free bonds and for swaps are therefore more appropriately modelled in terms of "required return differentials" rather than of default risks. We then constructed a theory for the joint evolution of the two term structures, consistent with their initial positions, and for pricing contingent claims on either or both structures. It was shown that our models are compatible with a complete markets general equilibrium. We explored a class of models within our general framework, and exhibited a readily computable valuation formula for european-style contingent claims; and provided a concrete example.

This paper has not discussed the valuation of frequently resettled claims, such as futures and options margined like futures, or offered proposals for the concrete evaluation of american-style contingent claims. We conjecture that these could be handled satisfactorily by appropriate extensions of the single term structure results in Babbs [1990].

We have employed only a single stochastic factor for each term structure, implicitly supposing perfect instantaneous correlation between movements in different parts of each individual term structure, while the correlation between movements in the two structures are independent of the maturities concerned. This could be avoided - at a computational price - by introducing additional stochastic factors, whose influence - as specified by the corresponding volatility function - varies with tenor in a different fashion from that of the existing factors. It would be theoretically interesting, though mathematically considerably harder, to make the function *s( )* stochastic, so that the dependence of the required return differential upon maturity could be subject to random variation.

Notwithstanding the previous two paragraphs, we believe we have developed a theory of the joint evolution of the default-free and interest rate swap term structures, and for the valuation of contingent claims, which is of considerable generality. The most pressing task is to investigate empirically whether a model encompassed by our framework provides a useful description of the reality!

APPENDIX

Proof of 4 Theorem I Suppose that there exists an EMM, P^*. By definition of an EMM,

$$\frac{dP^*}{dP} \in L^2(P) \tag{A1}$$

Hence

$$\eta(t) \equiv E\left[\left. \frac{dP^*}{dP} \right| \mathcal{F}_t \right] \tag{A2}$$

is a square-integrable P-martingale with

$$\eta(0) = E\left[\frac{dP^*}{dP} \right] = 1 \tag{A3}$$

Using 3.1 Assumption, we can apply martingale representation theory[46] to obtain that

$$\eta(t) = 1 + \int_0^t \alpha_1(u)dW_1(u) + \int_0^t \alpha_2(u)dW_2(u) \tag{A4}$$

for some predictable P-square-integrable processes $\alpha_1(\ ), \alpha_2(\ )$.

By (5),

$$\frac{d\{B_p(M,t)/B_p(T,t)\}}{B_p(M,t)/B_p(T,t)} = \{\sigma_p(M,t) - \sigma_p(T,t)\} \times [\{\theta_p(t) - \sigma_p(T,t)\}dt + dW_1(t)] \tag{A5}$$

46 see eg Liptser and Shiryayev [1977] Theorem 5.5 p162 - essentially the Kunita-Watanabe Representation Theorem.

Since P^* is an EMM, $B_p(M,t)/B_p(T,t)$ is a P^*-martingale; implying[47] that $B_p(M,t)\eta(t)/B_p(T,t)$ is a *P*-martingale. Applying Ito's lemma,

$$\frac{d\{B_p(M,t)\eta(t)/B_p(T,t)\}}{B_p(M,t)/B_p(T,t)}$$

$$= \{\sigma_p(M,t)-\sigma_p(T,t)\}\ [\alpha_1(t)+\{\theta_p(t)-\sigma_p(T,t)\}\eta(t)]dt$$

$$+\ [\alpha_1(t)+\{\sigma_p(M,t)-\sigma_p(T,t)\}\eta(t)]dW_1(t)+\alpha_2(t)dW_2(t) \qquad (A6)$$

Now an Ito process is a martingale only if[48] it has zero drift. Thus we require

$$\alpha_1(t)\ +\ \{\theta_p(t)-\sigma_p(T,t)\}\eta(t)\ =\ 0 \qquad (A7)$$

Now consider a "buy-and-hold" strategy in the "government" bond maturing at *M*, modified to reflect the required return differential arising from the existing holding by "re-investing" it in the bond. The value process of this strategy is the price process of the bond, adjusted for required return differentials:

$$V(t)\ =\ \exp\left\{\int_0^t D(M,u)du\right\}\ B_g(M,t) \qquad (A8)$$

Hence, following normalisation, applying Ito's lemma and substituting (A7) gives us:

$$\frac{d\{V(t)\eta(t)/B_p(T,t)\}}{V(t)/B_p(T,t)}\ =\ \nu(t)dt\ -\ \eta(t)\{\theta_p(t)-\sigma_g(M,t)\rho(t)\}dW_1(t)$$

$$+\left\{\eta(t)\sigma_g(M,t)\sqrt{1-\rho^2(t)}+\alpha_2(t)\right\}dW_2(t) \qquad (A9)$$

which cannot be a *P*-martingale unless

$$\nu(t)\ \equiv\ \alpha_2(t)\sigma_g(M,t)\sqrt{1-\rho^2(t)}$$

$$+\,\eta(t)[r_g(t)+s(M,t)+D(M,t)-r_p(t)+\sigma_g(M,t)\{\theta_g(t)-\rho(t)\theta_p(t)\}]$$

$$=\ 0 \qquad (A10)$$

Now, $\eta(\ )$, $\alpha_2(\ )$ are independent of *M*, and by 3.3 Assumption *D()* is independent of $\sigma_g(\ )$; so we deduce that (16) must hold, and that

$$\eta(t)\{\theta_g(t)-\rho(t)\theta_p(t)\}\ +\ \alpha_2(t)\sqrt{1-\rho^2(t)}\ =\ 0 \qquad (A11)$$

---

47 see eg Liptser and Shiryayev [1977] Lemma 6.6 p226.

48 It is well known that this condition is necessary but not sufficient. Its necessity flows from the result that, on a finite interval, a continuous local martingale of finite variation is constant (see eg Elliott [1982] Lemma 11.39 p121, bearing in mind that continuity implies predictability - *op. cit.* Theorem 6.32 p56).

Combining (A7) and (A11),

$$\frac{d\eta(t)}{\eta(t)} = -\{\theta_p(t)-\sigma_p(T,t)\}dW_1(t) - \frac{\theta_g(t)-\rho(t)\theta_p(t)}{\sqrt{1-\rho^2(t)}}dW_2(t) \qquad (A12)$$

whose solution is

$$\eta(\ ) = \xi(\ ) \qquad (A13)$$

whence (17) holds.

Thus, if P^* is an EMM, then $\xi(\ )$ is a *P*-martingale with

$$\xi(T) = \eta(T) = \frac{dP^*}{dP} \in L^2(P) \qquad (A14)$$

Conversely, if the condition (17) in the theorem is fulfilled, then, by the Girsanov Theorem[49], (A1) defines a probability measure P^* equivalent to *P*, under which

$$W_1^*(t) \equiv W_1(t) + \int_0^t \{\theta_p(u)-\sigma_p(T,u)\}\,du \qquad (A15a)$$

and

$$W_2^*(t) \equiv W_2(t) + \int_0^t \frac{\theta_g(u)-\rho(u)\theta_p(u)}{\sqrt{1-\rho^2(u)}}\,du \qquad (A15b)$$

are standard Brownian motions, and we may re-express the discounted process (A5) as

$$\frac{d\{B_p(M,t)/B_p(T,t)\}}{B_p(M,t)/B_p(T,t)} = \{\sigma_p(M,t)-\sigma_p(T,t)\}\,dW_1^*(t) \qquad (A16a)$$

Similarly, the price process for the government bond maturing at *M*, after adjusting for required return differentials, and normalisation, can be re-expressed as

$$\frac{d\{V(t)/B_p(T,t)\}}{V(t)/B_p(T,t)} = \{\sigma_g(M,t)\rho(t)-\sigma_p(T,t)\}\ dW_1^*(t) + \sigma_g(M,t)\sqrt{1-\rho^2(t)}\ dW_2^*(t) \qquad (A16b)$$

49 see eg Karatzas and Shreve [1987] Theorem 3.5.1 p191

The unique solutions of (A16a) and (A16b) are the respective P^*-supermartingales:

$$\frac{B_p(M,t)}{B_p(T,t)} = \frac{B_p(M,0)}{B_p(T,0)} \exp\left\{\int_0^t \{\sigma_p(M,u)-\sigma_p(T,u)\}dW_1^*(u) - \frac{1}{2}\int_0^t \{\sigma_p(M,u)-\sigma_p(T,u)\}^2 du\right\} \qquad (A17a)$$

and

$$\frac{V(t)}{B_p(T,t)} = \frac{V(0)}{B_p(T,0)} \exp\left\{\int_0^t \{\sigma_g(M,u)\rho(u)-\sigma_p(T,u)\}dW_1^*(u) + \int_0^t \sigma_g(M,u)\sqrt{1-\rho^2(u)}\,dW_2^*(u) - \frac{1}{2}\int_0^t \{\sigma_g(M,u)\rho(u)-\sigma_p(T,u)\}^2+\sigma_g^2(M,u)\{1-\rho^2(u)\}du\right\} \qquad (A17b)$$

If (19) and (20) hold, the RHS of (A17a) and (A17b) are, in fact, P^*-martingales; thus P^* is an EMM as required. ▮

Proof of 6 Proposition By Girsanov's theorem, we can re-express (4) and (5), via (16a)-(16b) as:

$$\frac{dB_g(L,u)}{B_g(L,u)} = \{r_g(u)+s(L,u)+\sigma_p(M,u)\rho(u)\sigma_g(L,u)\}\ du + \rho(u)\sigma_g(L,u)\ dW_1^{(M)}(u) + \sqrt{1-\rho^2(u)}\,\sigma_g(L,u)dW_2^{(M)}(u) \qquad (A18a)$$

and

$$\frac{dB_p(L,u)}{B_p(L,u)} = \{r_p(u)+\sigma_p(M,u)\sigma_p(L,u)\}\ du + \sigma_p(L,u)\ dW_1^{(M)}(u) \qquad (A18b)$$

where $W_1^{(M)}, W_2^{(M)}$ are standard Brownian motions under $P^{(M)}$, defined by:

$$W_1^{(M)}(s) = W_1(s) + \int_0^s \{\theta_p(u)-\sigma_p(M,u)\}du \qquad (A19a)$$

$$W_2^{(M)}(s) = W_2(s) + \int_0^s \frac{\theta_g(u)-\rho(u)\theta_p(u)}{\sqrt{1-\rho^2(u)}}du \qquad (A19b)$$

By applying Ito's lemma, we can now derive, by means of extensive but entirely straightforward manipulations, that, for each cashflow date on which the payoff function x depends:

$$B_k(M_{ki}, M) = h_{ki}, \quad k = g, p \qquad (A20)$$

The result follows. ▮

BIBLIOGRAPHY

B Antl (editor) "Swap Finance" Euromoney publications, Vols I and II (1986)

S H Babbs "The Term Structure of Interest Rates: Stochastic Processes and Contingent Claims" PhD Thesis, London University (1990)

S H Babbs "A Family of Ito Process Models for the Term Structure of Interest Rates", FORC Preprint 90/24, University of Warwick (February 1991)

S H Babbs "The Viability of Continuous Time Securities Market Models" Working paper (last revised July 1991)

S H Babbs and G R Salkin "The Valuation of Multivariate Contingent Claims in Continuous Time Models" Working paper (September 1987; revised September 1989)

S H Babbs and M J P Selby "Pricing by Arbitrage in Incomplete Markets" Working paper (latest draft May 1991)

Bank for International Settlements, Committee on Banking Regulations and Supervisory Practices "International Convergence of Capital Measurement and Capital Standards" (July 1988)

J Bicksler and A H Chen "An Economic Analysis of Interest Rate Swaps" *Journal of Finance* 41 (July 1986), 645-55

F Black and J C Cox "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions" *Journal of Finance* 31 (May 1976), 351-67

F Black and M S Scholes "The Pricing of Options and Corporate Liabilities" *Journal of Political Economy* 81 (May/June 1973), 637-59

I A Cooper and A S Mello "The Default Risk of Swaps" *Journal of Finance* 46 (June 1991), 597-620

J C Cox, J E Ingersoll and S A Ross "An Analysis of Variable Rate Loan Contracts" *Journal of Finance* 35 (May 1980), 389-403

J C Cox, J E Ingersoll and S A Ross "A Re-examination of Traditional Hypotheses about the Term Structure of Interest Rates" *Journal of Finance* 36 (September 1981), 769-99

J C Cox, J E Ingersoll and S A Ross "A Theory of the Term Structure of Interest Rates" *Econometrica* 53 (March 1985), 385-407

J C Cox and S A Ross "A Survey of Some New Results in Financial Option Pricing Theory" *Journal of Finance* 31 (May 1976), 383-402

P H Dybvig "Bond and Option Pricing Based on the Current Term Structure" Working paper (February 1989)

R J Elliott "Stochastic Calculus and Applications" Applications of Mathematics 18, Springer Verlag (1982)

J M Harrison and D M Kreps "Martingales and Arbitrage in Multiperiod Securities Markets" *Journal of Economic Theory* 20 (1979), 381-408

J M Harrison and S R Pliska "Martingales and Stochastic Integrals in the Theory of Continuous Trading" *Stochastic Processes and their Applications* 11 (1981), 215-60

D C Heath, R A Jarrow and A Morton "Bond Pricing and the Term Structure of Interest Rates: A New Methodology" Working paper, Cornell University (1987)

D C Heath, R A Jarrow and A Morton "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation" Working paper, Cornell University (1989)

T S Y Ho and S-B Lee "Term Structure Movements and Pricing Interest Rate Contingent Claims" *Journal of Finance* 41 (December 1986), 1011-29

T S Y Ho and R Singer "Bond Indenture Provisions and the Risk of Corporate Debt" *Journal of Financial Economics* 10 (1982), 375-406

T S Y Ho and R Singer "The Value of Corporate Debt With a Sinking Fund Provision" *Journal of Business* 57 (1984), 315-36

R A Jarrow and S M Turnbull "Pricing Options on Financial Securities Subject to Credit Risk" Working paper (May 1990, revised June 1991)

H E Johnson and R M Stulz "The Pricing of Options with Default Risk" *Journal of Finance* 42 (June 1987), 267-79

E P Jones, S P Mason and E Rosenfeld "Contingent Claims Analysis of Corporate Capital Structures: an Empirical Investigation" *Journal of Finance* 39 (July 1984), 611-27

I Karatzas and S E Shreve "Brownian Motion and Stochastic Calculus" Graduate Texts in Mathematics 113, Springer Verlag (1987)

R S Liptser and A N Shiryayev "Statistics of Random Processes I: General Theory" Springer-Verlag (1977)

R C Merton "On the Pricing of Corporate Debt: the Risk Structure of Interest Rates" *Journal of Finance* 29 (May 1974), 449-70

R C Merton "On the Pricing of Contingent Claims and the Modigliani-Miller Theorem" *Journal of Financial Economics* 5 (1977), 241-9

K Ramaswamy and S M Sundaresan "The Valuation of Options on Futures Contracts" *Journal of Finance* 40 (December 1985), 1319-40

M J P Selby "The Application of Option Theory to the Evaluation of Risky Debt" PhD Thesis, London Business School (May 1983)

S M Sundaresan "Valuation of Swaps" First Boston Working Paper Series FB-89-15 (May 1988, revised June 1989)

O A Vasicek "An Equilibrium Characterization of the Term Structure" *Journal of Financial Economics* 5 (November 1977), 177-88