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# THE CONSITENCY OF TERM STRUCTURE MODELS: The Short Rate, the Long Rate, and Volatility


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Abstract

The results presented in this paper show how, under certain conditions, the parameters of the short interest rate process are related to one another. Short rate volatility, as a function of the short rate, is shown to be critical in determining the deterministic part of short rate movements. Sufficient conditions are established for the short rate to be mean-reverting. The long rate is investigated. Conditions for it to be constant are stated.

## I. Introduction

Three distinct approaches to modelling the term structure of interest rates have appeared in the literature. Each concentrates on modelling some specific aspect of the term structure, allowing other aspects to be contingent upon this feature. For instance the earliest models, such as that of Vasicek [12], modelled the short rate process. Bond prices are taken to be contingent upon this factor, and the term structure is taken directly from the bond prices. Unfortunately these models possess an arbitrarily determined price of risk. Because the price of risk is exogenous to the model there is no guarantee that mis-specifying it might not lead to arbitrage.

A second approach due to Cox, Ingersoll and Ross [4] models an economy as a whole. This approach overcomes a weakness of simpler models, such as Vasicek, by endogenously determining the price of risk within the model. Both the interest rate process and associated prices of risk arise from the existence of an equilibrium within an economy whose agents have known utility functions. Cox, Ingersoll and Ross presented as a special case of their general economic model an economy where the short rate is the sole determinant of bond prices, but where the price of risk itself is a function of the short rate. The model is guaranteed to be arbitrage free.

The third approach has been to model the term structure as a whole. Due to Heath, Jarrow and Morton [7], who improved and generalized a discrete time model of Ho and Lee [8], and independently due to Babbs [1], these models have theoretical and practical attractions. The Heath, Jarrow and Morton model was originally formulated in terms of the evolution of the term structure of forward rates. However it may equivalently be formulated directly as the evolution of the term structure of spot rates, or indeed of bond prices.

Empirical validation of models of any type has been problematical. For instance, a number of studies have attempted to estimate parameters for the Cox, Ingersoll and Ross model; recent examples are [6] and [11]. Chan, Karolyi, Longstaff and Sanders [2] compare a variety of models of the short term interest rate. Using a Generalized Method of Moments they estimated the parameters for the models and reached conclusions about the success of the models in fitting their data set. They found that the most successful models allowed the volatility of the short rate to be dependent on the level of the short rate.

Set in a context equivalent to a Markovian version of Heath, Jarrow and Morton, this paper presents some results concerning the long rate, the short rate, and the term structure of volatility. Along with Hull and White [9] the volatility function may fluctuate through time. The volatility function is explicitly allowed to be a function of the short rate and of the corresponding spot rate. The results allow the consequences of changing assumptions about the way in which volatility depends upon the short rate to be explored. Because the formulation here is in terms of spot rates, rather than forward rates, it is easy to derive results about specific interest rate processes. In fact we obtain explicit results for the long rate and the short rate process. These expressions are in terms of the volatility function and a price of risk function.

Starting from a general specification of the process followed by spot rates, the spot rate drift is re-expressed in terms of exogenously given volatility and price of risk functions, and other values that depend upon the term structure. A differential equation is obtained that

## II. The Interest Rate Process

In this section we write down a very general form for the spot rate process. It is then recast into a form suitable for future analysis: the drift is expressed as a function of the volatility functions, a price of risk, and the current term structure of interest rates. The initial formulation is appropriate for empirical estimation of the volatility functions. The recast version allows an explicit formulation of the short rate process to be derived.

The existence of a complete set of pure discount bonds is assumed.

The following notation will be used. Current time is denoted by t. T is a time of maturity, and $\tau$ is time to maturity so that $\tau = T - t$.

$B_t(T)$ value at t of a time T maturity pure discount bond.

$r_t(T)$ the yield to maturity at time t of a time T maturity pure discount bond.

$$B_t(T) = e^{-r_t(T).(T-t)}.$$

$r_t(T)$ is the time T maturity spot rate at time t. We shall also write $R_t(\tau) = r_t(t + \tau)$.

$Z_t \equiv r_t(\infty) = \lim_{T \to t} r_t(T)$ is the short rate at time t.

$L_t \equiv r_t(0) = \lim_{T \to \infty} r_t(T)$ is the long rate at time t.

$f^t_s(T)$ is the forward rate at time t for time $s \geq t$ on a T maturity bond.

$f^t_s(T)$ is defined implicitly as:

$$B_t(T) = B_t(s).e^{-f^t_s(T).(T-s)}.$$

$f^t_s$ is the instantaneous forward rate at t for time $s \geq t$, so that

$$f^t_s \equiv f^t_s(0) = \lim_{T \to t} f^t_s(T).$$

$\lambda_t(T)$ is the market price of risk at time t on the time T maturity bond $B_t(T)$.

Writing $dB_t(T) = \mu^B_t(T).B_t(T).dt + \sigma^B_t(T).B_t(T).dw$, $\lambda_t(T)$ is defined to be

$$\lambda_t(T) = \frac{\mu^B_t(T) - Z_t}{\underline{\sigma}_{Bt}(T)}$$

where $dw = (dw_1, dw_2, \ldots, dw_n)$ is a vector of Wiener processes and $\sigma^B_t(T)$ a vector of functions. The correlation matrix $\rho$, where $\rho.dt = \{dw_i.dw_j\}_{i,j = 1,\ldots,n}$ is assumed to be constant. $\underline{\sigma}^2_{Bt}(T) = \sigma^B_t(T)^T.\rho.\sigma^B_t(T)$ is the global volatility function.

$$\mu_t(T) = \frac{d}{ds}( f^t_s(T) )\Big|_{s=t} + \frac{d}{ds}( l^t_s(T) )\Big|_{s=t}.$$

The differential of the forward rate is just

$$\frac{d}{ds}( f^t_s(T) )\Big|_{s=t} = \lim_{s \to t} \frac{f^t_s(T) - f^t_t(T)}{s - t} = \frac{r_t(T) - Z_t}{T - t}.$$

Write $\frac{d}{ds}( l^t_s(T) )\Big|_{s=t} = p_t(T)$, so that we have

$$\mu_t(T) = \frac{r_t(T) - r_t}{T - t} + p_t(T). \tag{3}$$

We now relate $p_t(T)$ to the process followed by a T maturity bond. Because the bond price is a known function, (1), of the T maturity spot price, the process for $B_t(T)$ can be written in terms of the process for $r_t(T)$:

$$dB_t(T) = \mu^B_t(T).B_t(T).dt + \sigma^B_t(T).B_t(T).dw,$$

where

$$\mu^B_t(T) = r_t(T) + \tfrac{1}{2}(T-t)^2.\underline{\sigma}^2_t(T) - (T-t).\mu_t(T),$$

$$\sigma^B_t(T) = -(T-t).\sigma_t(T).$$

The volatility dependency of the drift of $r_t(T)$ comes entirely through $\underline{\sigma}_t(T)$.

The risk premium $\lambda_t(T)$,

$$\lambda_t(T) = \frac{\mu^B_t(T) - Z_t}{\underline{\sigma}_{Bt}(T)}, \tag{4}$$

is a function of $\mu^B_t(T)$ and $\underline{\sigma}_B(T)$, which in turn are functions of $\mu_t(T)$ and $\underline{\sigma}_t(T)$. Substituting for $\mu^B_t(T)$ and $\underline{\sigma}_{Bt}(T)$ into (4) and comparing to (3), we obtain a formula for $p_t(T)$ in terms of $\lambda_t(T)$ and $\underline{\sigma}_t(T)$:

$$p_t(T) = \tfrac{1}{2}(T-t).\underline{\sigma}^2_t(T) - \lambda_t(T).\underline{\sigma}_t(T).$$

This proves (i) and (ii). For (iii) we convert from a rate-for-maturity context, to a rate-to-maturity context. In fact:

$$dR_t(\tau) = dr_t(t+\tau) = dr_t(T) + \frac{\partial r_t(T)}{\partial T}.dT$$

$$= dr_t(T) + \frac{\partial R_t(\tau)}{\partial \tau}.dt$$

since $dT = dt$, and $\frac{\partial r_t(T)}{\partial T} = \frac{\partial R_t(\tau)}{\partial \tau}$. This gives us (iii).

may also be solved directly since in their model both $\lambda_t(\tau)$ and $\underline{\sigma}_t(\tau)$ are proportional to $\sqrt{Z_t}$. $p_t(\tau)$ decomposes into a product of $Z_t$ and $p'_t(\tau)$ where $p'_t(\tau)$ depends only on t and $\tau$, and there is no $R_t(\tau)$ dependence. The solution (7) therefore applies.

## III. The Long Rate

Equation (2) can be simplified if we assume that $\underline{\sigma}$ and $\lambda$ depend upon maturity time T only through time to maturity. This assumption is appropriate to describe an equilibrium situation where there is a degree of time homogeneity. It implies that investors are sensitive to risk only through the time to maturity, and risk is not related to specific future dates. From now on we will write $\underline{\sigma}_t(\tau)$, $\lambda_t(\tau)$, $p_t(\tau)$, etc.

We now seek to describe some of the behaviour of the term structure as it evolves through time. A useful starting point is to search for values of the spot rates for which their drift is zero. This is a set of $\theta(\tau)$ such that $\mu(\theta(\tau),\tau)$ is zero for each $\tau$. In general this set will depend upon the current term structure. These zero drift states are important because if a given spot rate reverts towards some value, then this will be a state with drift zero. In some sense the zero drift states reflect the shape of the term structure by providing a measure of the location of the probability density function of the spot rate at each maturity. In general, however, the zero drift value will not equal the mean of the spot rate distribution for that maturity.

Suppose $\theta(\tau)$ is a set of interest rates such that $\mu(\theta(\tau),\tau)$ is zero for each $\tau$. $\theta(\tau)$ could depend on $\underline{\sigma}_t(\tau)$ and $\lambda_t(\tau)$. $\theta(\tau)$ represents a set of potential reversion states. A zero drift state $\theta \in \theta(\tau)$ is a reversion state if perturbations from $\theta$ tend to revert back to $\theta$. For instance if

$$\left.\frac{\partial \mu(R)}{\partial R}\right|_{\theta} < 0.$$

then small perturbations in R away from $\theta$ will tend to be driven back towards $\theta$. This is sufficient to make $\theta$ a local reversion state. Large perturbations away from $\theta$ may not revert back towards $\theta$. If $\theta$ is also the unique zero of $\mu(R(\tau),\tau)$ then it is a global reversion state, and $R(\tau)$ will always revert towards $\theta(\tau)$.

The differential equation (2) is broadly specified and without further assumptions little progress can be made. However, because the underlying variables are interest rates it is possible to make a number of restrictive assumptions that allow significant simplification. Perhaps the chief assumption we make is that interest rates are Markovian, although this restriction is not required by all the results that follow.

We make the following assumptions:

i) For all feasible term structures $R_t(\tau)$, $R_t(\tau)$ is differentiable in $\tau$.

The limit $R_t(\infty) = \lim_{\tau\to\infty} R_t(\tau)$ exists and is finite, $0 \leq R_t(\infty) < \infty$.

For all pairs $(\tau, R)$, a feasible term structure exists that passes through $(\tau, R)$.

ii) $\underline{\sigma}_t(\tau)$ and $\lambda_t(\tau)$ are Markovian. They are at least twice differentiable in all their arguments.

We have seen that the drift term for $R_t(\tau)$ has three components:

a liquidity term, $\frac{1}{2}\tau \underline{\sigma}_t^2(\tau) - \underline{\sigma}_t(\tau).\lambda_t(\tau)$; a slope term, $\frac{R_t(\tau) - Z_t}{\tau}$; and a tangent term, $\frac{\partial R_t(\tau)}{\partial \tau}$.

Proposition 1.

For every feasible term structure $R_t(\tau)$, the long rate $L_t = R(\infty)$ has zero drift and zero volatility.

If the limiting function $\mu_t(R,\infty)$ has a unique, constant, zero, then the long rate is constant,

$$\mu_t(L_t,\infty) = 0.$$

Along any feasible term structure $R_t(\tau)$, both $\mu_t(R_t(\tau),\tau)$ and $\underline{\sigma}(R_t(\tau),\tau)$ tend to zero at least as fast as $\tau^{-1}$ as $\tau \rightarrow \infty$.

Proof Consider the relationship

$$\mu_t(R_t(\tau),\tau) = \frac{R_t(\tau) - Z_t}{\tau} + \frac{\partial R_t(\tau)}{\partial \tau} + \frac{1}{2}\tau.\underline{\sigma}^2(\tau) - \underline{\sigma}(\tau).\lambda..$$

As $\tau \rightarrow \infty$ the tangent term must go to zero strictly faster than $\tau^{-1}$ or the long rate would be unbounded. The slope term must also go to zero with leading order at least $\tau^{-1}$. Furthermore, $\tau.\underline{\sigma}_t^2(R_t(\tau),\tau)$ must tend to zero for all feasible term structures: by assumption there exists a pair $(\tau_b,R_b)$ such that $\mu_t(R_t(\tau),\tau) < 0$ for all $\tau > \tau_b$ and $R_t(\tau) > R_b$. This implies that as $\tau \rightarrow \infty$, $\mu_t(R_t(\tau),\tau) \le 0$ for $R_t(\tau) > R_b$. But $\frac{\partial R(\tau)}{\partial \tau} + \frac{R(\tau)}{\tau}$ is positive and so is $\frac{1}{2}\tau.\underline{\sigma}^2(\tau)$, so for large R and $\tau$, $\frac{1}{2}\tau.\underline{\sigma}^2(\tau) \le \underline{\sigma}(\tau).\lambda + \frac{Z_t}{\tau}$. If $\underline{\sigma}_t(R_t(\tau),\tau) \rightarrow 0$ slower than $\tau^{-1}$ it is not possible for $\mu_t(R_t(\tau),\tau)$ to become negative for large $R_t(\tau)$. So $\underline{\sigma}_t(R_t(\tau),\tau) \rightarrow 0$ at least as fast as $\tau^{-1}$. All the terms on the right hand side go to zero at least as fast as $\tau^{-1}$, so $\mu_t(R_t(\tau),\tau)$ must also go to zero at least as fast as $\tau^{-1}$. We have shown that the long rate has zero drift and volatility.

Note that the true long rate may not appear within the range of maturities accessible with data from the financial markets. A term structure that appears 'normal' may invert only at maturities of several decades, depending on when $\underline{\sigma}$ settles down to its limiting behaviour. In the absence of bonds of large maturities only limited deductions may be made. Even when perpetual bonds exist only limited inferences may be drawn, since the term structure curve requires the imputation of prices of pure discount bonds.

$$\mu = \frac{1}{P(Z)} \cdot \left[ \int P(Z).\frac{2}{\underline{\sigma}}.\frac{\partial \underline{\sigma}}{\partial \tau}.dZ - \int P(Z).\underline{\sigma}.\frac{\partial \lambda}{\partial R}.dZ - \int P(Z).\frac{\partial}{\partial Z}(\lambda \underline{\sigma}).dZ \right], \tag{8}$$

where P(Z) is the integrating factor

$$P(Z) = e^{\int -\frac{1}{\underline{\sigma}}.\frac{\partial \underline{\sigma}}{\partial R}.dZ}.$$

**Proof** See Appendix.

This proposition defines the short rate process in terms of $\underline{\sigma}(\tau)$ and λ. Equation (8) is of the form $\mu(Z) = \frac{1}{P(Z)}.(c + I(Z))$, where I(Z) is the integral and c is a constant of integration. When $\underline{\sigma}(\tau)$ and λ are specified I(Z) may be computed. This implies that $\underline{\sigma}(\tau)$ and λ and the short rate process may not be independently specified in term structure models. They are related through (9) and (10). Furthermore, under our assumptions, $\underline{\sigma}(\tau)$ and λ are sufficient to determine the short rate process. No other factors, such as forward rates, need be explicitly known.
Note that μ is determined separately at each time t. This means that the constant of integration, c, is determined separately at each t. Therefore c can be a function of time t, $c = c_t$.

We illustrate the proposition with some examples:

<u>Example 1</u> $\underline{\sigma}(\tau)$ has no R or Z dependence, λ is a constant.
(The Vasicek is an example of this sort.)
In this case it is easy to formally integrate the differential equation. We obtain

$$\mu_t = \int \frac{2}{\underline{\sigma}}.\frac{\partial \underline{\sigma}}{\partial \tau}.dZ$$
$$= \kappa.(\theta - Z),$$

where $\kappa = -\frac{2}{\underline{\sigma}}.\frac{\partial \underline{\sigma}}{\partial \tau}$, and κ.θ is the constant of integration.

The process is mean reverting if $\frac{\partial \underline{\sigma}}{\partial \tau} < 0$. In the Vasicek model $\underline{\sigma}(\tau) = \frac{\rho}{\alpha}.\frac{1-e^{-\alpha\tau}}{\tau}$, so $\underline{\sigma}(0) = \rho$, and $\frac{\partial \underline{\sigma}}{\partial \tau} = -\frac{\rho\alpha}{2}$. The reversion rate κ is α.

<u>Example 2</u> $\underline{\sigma}(\tau)$ has no R dependence.
Then $\mu = \int \frac{2}{\underline{\sigma}}.\frac{\partial \underline{\sigma}}{\partial \tau}.dZ - \lambda\underline{\sigma}$
When $\underline{\sigma}(\tau, Z)$ decomposes into a product of a function of time and a function of Z, $\underline{\sigma}(\tau, Z) = \kappa(\tau).f(Z)$, this becomes

$$\mu = c + \frac{2}{\kappa(0)}.\frac{\partial \kappa}{\partial \tau}.Z - \lambda\kappa(0).f(Z).$$

If μ = a.(b - Z), with a and b constant, f must be of the form:

the sharing the same volatility and price of risk.

In these examples, when $\frac{\partial \sigma}{\partial \tau}$ is negative $\mu$ is found to be monotonically decreasing in Z, and to possess a unique zero. These features are certainly not guaranteed to occur. More complex assumptions about the form of the volatility function can result in more complex drift functions.

## Appendix

### Proof of the Proposition

The Ito differential of $\mu$ is

$$d\mu_t(\tau) = \frac{\partial\mu}{\partial t}.dt + \frac{\partial\mu}{\partial R}.dR + \frac{\partial\mu}{\partial D}.dD + \frac{\partial\mu}{\partial Z}.dZ$$

$$+ \tfrac{1}{2}.\frac{\partial^2\mu}{\partial R^2}.dR.dR + \tfrac{1}{2}.\frac{\partial^2\mu}{\partial Z^2}.dZ.dZ + \frac{\partial^2\mu}{\partial Z\,\partial R}.dR.dZ.$$

(Other terms vanish). Under mild regularity conditions [10],

$$dD = \frac{d\mu}{d\tau}.dt + \frac{d\sigma}{d\tau}.dw,$$

and $$dZ = \mu(0).dt + \sigma(0).dw = \left(2.\frac{\partial R_t(\tau)}{\partial\tau}\bigg|_{\tau=0} - \sigma.\lambda\right).dt + \sigma(0).dw.$$

Since $\mu = \frac{R-Z}{\tau} + D + \tfrac{1}{2}\tau.\sigma^2 - \lambda\sigma$, the first order differentials are:

$$\frac{\partial\mu}{\partial R} = \frac{1}{\tau} + \tfrac{1}{2}\tau.\frac{\partial}{\partial R}(\sigma^2) - \frac{\partial}{\partial R}(\lambda\sigma),$$

$$\frac{\partial\mu}{\partial D} = 1,$$

$$\frac{\partial\mu}{\partial Z} = -\frac{1}{\tau} + \tfrac{1}{2}\tau.\frac{\partial}{\partial Z}(\sigma^2) - \frac{\partial}{\partial Z}(\lambda\sigma).$$

The drift is thus

$$d\mu_t(\tau) = \Bigg( \frac{\mu(\tau)-\mu(0)}{\tau} + \mu(\tau).\Big(\tfrac{1}{2}\tau.\frac{\partial}{\partial R}(\sigma^2) - \frac{\partial}{\partial R}(\lambda\sigma)\Big) + \mu(0).\Big(\tfrac{1}{2}\tau.\frac{\partial}{\partial Z}(\sigma^2) - \frac{\partial}{\partial Z}(\lambda\sigma)\Big)$$

$$+ \frac{\partial\mu}{\partial t} + \frac{d\mu}{d\tau} + \tfrac{1}{2}\sigma^2(\tau).\frac{\partial^2\mu}{\partial R^2} + \tfrac{1}{2}\sigma^2(0).\frac{\partial^2\mu}{\partial Z^2} + \sigma(0).\sigma(\tau).\frac{\partial^2\mu}{\partial Z^2}\Bigg).dt$$

$$+ \Bigg( \frac{\sigma(\tau)-\sigma(0)}{\tau} + \sigma(\tau).\Big(\tfrac{1}{2}\tau.\frac{\partial}{\partial R}(\sigma^2) - \frac{\partial}{\partial R}(\lambda\sigma)\Big) + \sigma(0).\Big(\tfrac{1}{2}\tau.\frac{\partial}{\partial Z}(\sigma^2) - \frac{\partial}{\partial Z}(\lambda\sigma)\Big) + \frac{d\sigma}{d\tau}\Bigg).dw.$$

In the limit as $\tau \to 0$, we obtain the process for $d\mu_t(0)$

$$d\mu_t(0) = \Bigg( \frac{\partial\mu}{\partial t} + 2.\frac{d\mu}{d\tau} - \mu(0).\Big(\frac{\partial}{\partial Z}(\lambda\sigma) + \frac{\partial}{\partial R}(\lambda\sigma)\Big) + \sigma^2(0).\Big(\tfrac{1}{2}.\frac{\partial^2\mu}{\partial R^2} + \tfrac{1}{2}.\frac{\partial^2\mu}{\partial Z^2} + \frac{\partial^2\mu}{\partial Z^2}\Big)\Bigg).dt$$

$$+ \Bigg(2.\frac{d\sigma}{d\tau} - \sigma(0).\Big(\frac{\partial}{\partial R}(\lambda\sigma) + \frac{\partial}{\partial Z}(\lambda\sigma)\Big)\Bigg).dw$$

$$= \Bigg( \frac{\partial\mu}{\partial t} + 2.\frac{\partial\mu}{\partial\tau} + \frac{\partial\mu}{\partial R}.(\mu + \lambda\sigma) - \mu\Big(\frac{\partial}{\partial Z}(\lambda\sigma) + \frac{\partial}{\partial R}(\lambda\sigma)\Big)$$

$$+ \sigma^2.\Big(\tfrac{1}{2}.\frac{\partial^2\mu}{\partial R^2} + \tfrac{1}{2}.\frac{\partial^2\mu}{\partial Z^2} + \frac{\partial^2\mu}{\partial Z^2}\Big)\Bigg).dt$$

$$+ \Bigg(2.\frac{\partial\sigma}{\partial\tau} + \frac{\partial\sigma}{\partial R}.(\mu + \lambda\sigma) - \sigma.\Big(\frac{\partial}{\partial R}(\lambda\sigma) + \frac{\partial}{\partial Z}(\lambda\sigma)\Big)\Bigg).dw.$$

There is no explicit $D_t(\tau)$ dependence, this depends only on Z. Therefore the coefficient of dw must equal $\frac{\partial\mu}{\partial Z}.\sigma$ and $\mu(0)$ satisfies the differential equation

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