Efficient Monte Carlo Valuation of Contingent Claims

ABSTRACT

In this paper we discuss the use of antithetic and control variates for reducing the variance or error in a Monte Carlo valuation. We describe a choice of control variates which have an economic interpretation. By combining these two techniques we obtain an increase in the efficiency of the Monte Carlo method of many orders of magnitude. We also consider how the hedge ratios (DELTA, GAMMA, THETA, KAPPA, RHO) can be computed efficiently within our variance reduced Monte Carlo simulation. Finally we describe how recent ideas of Breen (1991) and Ho *et al* (1991) can be used to obtain approximate values for American style options by Monte Carlo simulation.

The Monte Carlo simulation technique for option valuation was first introduced by Boyle (1977). It is a numerical valuation technique which is typically used to value complex European-style options. The basis of the technique is the observation by Cox and Ross (1976) that if a riskless hedge can be formed the option value can be expressed as the discounted risk-neutrally adjusted expectation of its payoff. The technique is thus to simulate many times the risk neutrally adjusted random trajectory of the underlying instrument, and thence its payoff, and to estimate the option price as the discounted mean of these simulated payoffs.

The technique has been used extensively in the literature to generate comparative prices for other valuation techniques and in empirical investigations of option pricing models (most notably in models with stochastic volatility, Johnson and Shanno (1987), Hull and White (1987) and Scott (1987)). But little work has been done on the efficiency of the technique since Boyle (1977). Hull and White (1988) described how one of the techniques we will discuss, control variates, can be applied to a generalised lattice valuation technique. Recently Kemna and Vorst (1990) proposed Monte Carlo simulation as a valuation method for arithmetic average rate (asian) options. Here they employed the analytical formula for the geometric average as a control variate in order to increase the efficiency of the method to an acceptable level.

In section 2 we discuss the use of antithetic and control variates for reducing the variance, i.e. the error, in this estimation. This allows the implementation of the Monte Carlo technique to be more efficient, because not so many simulations are necessary to achieve a given confidence interval for the estimate of the option value. We describe a choice of control variates which have an economic interpretation and we show how antithetic and control variates can be combined to obtain an increase in the efficiency of the Monte Carlo method of approximately 1000 times for a realistic problem such as a lookback call with stochastic volatility.

Section 3 considers how the hedge ratios (DELTA, GAMMA, THETA, KAPPA, RHO) can be computed efficiently within our variance reduced Monte Carlo simulation. By exploiting specific aspects of the model we have for the underlying random processes we can considerably improve the efficiency of the computations.

In section 4 we describe how recent ideas of Breen (1991) and Ho *et al* (1991) can be used to obtain approximate values for American style options efficiently from Monte Carlo simulations.

Finally, Section 5 contains the conclusions.

I Variance Reduction Techniques

A good introduction to this area can be found in Kleijnen (1974) and Ripley (1987). The two techniques which we will consider are antithetic variates and control variates. Firstly we describe the implementation of the simplest, antithetic variates. Suppose that in order to estimate the mean μ of a probability distribution, we generate M independent random samples (or 'trials') $\{y_1, ..., y_M\}$ from the distribution. Then the simple Monte Carlo estimator of μ is given by the sample mean

$$\hat{\mu} = \frac{1}{M} \sum_{i=1}^{M} y_i \tag{1}$$

Now, suppose we also generate $\{y_1^*, ..., y_M^*\}$ which have the same distribution but are negatively correlated with the y_i's. Suppose we estimate $\hat{y}_i = (y_i + y_i^*)/2$, this is unbiased and

$$\mathrm{Var}(\hat{y}) = \frac{1}{2}\mathrm{Var}(y)(1 + \mathrm{Corr}(y, y^*)) \tag{2}$$

Thus we will obtain a more precise estimate from n pairs of (y_i, y_i^*) than from $2n$ simulations of y_i if $\mathrm{Corr}(y_i, y_i^*) < 0$. Furthermore, it may be computationally cheaper to compute (y_i, y_i^*) than to compute y_i twice.

We will now describe the control variate technique in general terms. Suppose that simultaneously with generating the samples $\{y_1, ..., y_M\}$, we also generate the control variates $\{x_1, ..., x_M\}$, where each x_i is an m-vector $(x_j^1, ..., x_j^m)^T$, (the symbol T denoting 'transpose'), whose components have zero mean. Thus, the $m+1$ vectors (y_i, x_i) are uncorrelated for different values of i, but their components are correlated among themselves in a constant way. Then for any m-vector ξ, the quantity

$$\hat{\mu}_c = \frac{1}{M}\sum_{i=1}^{M}(y_i - \sum_{j=1}^{m} x_i^j \xi_j) \tag{3}$$

will also be an estimator of μ, and the variance (i.e. the error) of this will be minimised if we choose ξ to be such that

$$\Sigma_{xx}\xi = \Sigma_{xy} \tag{4}$$

where Σ_{xx} is the covariance matrix of x, and Σ_{xy} is the m-vector of covariances between x and y.

In general, for good choices of the control variates we will not be able to write down Σ_{xx} and Σ_{xy} (if we know the matrix Σ_{xx} and vector Σ_{xy}, then the Equations (3) and (4) allow us to use the control variates x_i to reduce the variance in our simple Monte Carlo estimation (1) of μ). However, we can use the following procedure to obtain the variance-reduced estimate of $\hat{\mu}_c$:

Find $\hat{\beta} \equiv (\hat{\beta}_0, ..., \hat{\beta}_m)^T$ to fit in a least squares sense the system of equations

$$X_i \hat{\beta} = y_i, i = 1, ..., M \tag{5}$$

where $X_i = (1, x_i^1, ..., x_i^m)$. Then $(\hat{\beta}_1, ..., \hat{\beta}_m)$ is the choice of $(\xi_1, ..., \xi_m)$ which will minimise the variance of the estimator (3), and $\hat{\beta}_0$ is the corresponding variance-reduced estimator $\hat{\mu_c}$ of μ. In fact $\hat{\beta}$ can be calculated simply as

$$\hat{\beta} = (X^T X)^{-1} X^T Y \tag{6}$$

provided none of the control variates are redundant. The $m \times m$ matrix $X^T X$ and m-vector $X^T Y$ can be calculated conveniently as the simulations proceed via the equations

$$(X_i^T X_i)_{j,k} = (X_{i-1}^T X_{i-1})_{j,k} + (x_i)_j (x_i)_k \tag{7}$$

$$(X_i^T y_i)_j = (X_{i-1}^T y_{i-1})_j + (x_i)_j (y_i) \tag{8}$$

I . A The Variance Reduction Techniques Applied to the Elementary European Call

In this section we describe how to use these techniques to value a European call with strike price X and time to maturity T, on a non-dividend-paying stock with volatility σ. Our purpose here is to give an elementary presentation of the implementation of the variance reduction techniques we have described. In particular we introduce our specific choices of antithetic and control variates which have a simple economic interpretation. This example also illustrates the power of these techniques. We will then apply the approach to the more realistic problem of a lookback option with stochastic volatility in Section 2.2.

First, divide the lifetime $[0, T]$ of the option into a partition $\{0 \equiv t_0 < t_1 < ... < t_N \equiv T\}$ such that for each $h = 1, ..., N$, we have $t_h - t_{h-1} = \Delta t \equiv \frac{T}{N}$. Then the logarithmic increments $\Delta_t s \equiv \log S_t - \log S_{t-\Delta t}$ of the stock price over each of these time steps are iid normal, with mean $(r - \frac{1}{2}\sigma^2)\Delta t$ and variance $\sigma\sqrt{\Delta t}$, and we have

$$\log S_T = \log S_0 + \sum_{h=1}^{N} \Delta_{t_h} s \tag{9}$$

Thus, from N simulations of the standard normal distribution, we can simulate the final stock price S_T, and the option payoff value $\max\{(S_T - X), 0\}$. This payoff value plays the role of trial value y_i for any of $i = 1, ..., M$. We generate the $\Delta_{t_h}s$ by

$$\Delta_{t_h}s = (r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\tilde{z} \tag{10}$$

where $\tilde{z}$ is a standard normal random variable.

Table I shows the results for a typical European Call.

We see that the simple Monte Carlo estimate with no variance reduction has an unacceptably large error (standard deviation) for the 1000 simulations we have used.

Firstly we apply the antithetic variate technique. The most straightforward way to do this is to generate an underlying stock price path based on $-\tilde{z}$ simultaneously with the price path in Equation 10. We then obtain our estimate of the option value as shown in Section 2. This reduces the variance by a factor of almost 25.

In order to apply the control variate technique we must make a choice of the functional form for the control variates. They should be chosen so as to capture as far as possible the variation in the option value created by the random changes in the underlying stock price. The functions which do this exactly are the partial differentials of the option value with respect to the underlying price. Now in the case of the elementary European call we know these exactly, but normally we will not. However, any reasonable choice of control variates will give good variance reduction, so we can use analytical results for similar or related options. Our control variates should also have zero mean so we should design them to capture the differences from the expected values over each time step.

In fact this approach is equivalent to replicating the option with a hedged portfolio. By including control variates based on all the relevant hedge ratios we can hedge away all risk or account for all variation in the option value generated by changes in the underlying random variables in the continuous time limit. To be strictly correct we must inflate the differences between our hedged portfolio and the option over each time step by the riskless rate to maturity. However this typically has little effect.

In our example we will use DELTA and GAMMA based control variates,

$$x_i^1 = \sum_{h=1}^{N} \left.\frac{\partial \phi}{\partial S}\right|_{t_h} (\Delta_{t_h} S - E[\Delta_{t_h} S]) \tag{11}$$

$$x_i^2 = \sum_{h=1}^{N} \left.\frac{\partial^2 \phi}{\partial S^2}\right|_{t_h} ((\Delta_{t_h} S)^2 - E[(\Delta_{t_h} S)^2]) \tag{12}$$

(Note that we must be able to write down the expectations in these expressions analytically, this will normally be straightforward.)

Application of these control variates alone produces a variance reduction of 300 times.

We can combine the antithetic and control variate techniques. The control variates become the sum of the appropriate control variates for the two price processes generated by the antithetic procedure and the samples are the means of the pay-offs for the two price processes. The combined procedure reduces the variance of the estimate by 12000 times.

We have achieved a reduction in the standard deviation of the estimate such that the estimate is now accurate to two decimal places. To obtain this reduction for the simple Monte Carlo estimate we would have had to perform a factor of 12000 more simulations.

I . B Variance Reduction for a Lookback Call Option

In this section we demonstrate that the same techniques can be applied in straightforward way to more complex and realistic problems and give similarly dramatic results.

Consider the problem of valuing a FX lookback call option with stochastic volatility. The pay-off of a lookback call option is the difference between the terminal value of the underlying asset and its minimum value during the life of the option. There is an analytical formula for the value of a continuous fixing lookback option (Goldman, Sosin and Gatto (1979)). But if the fixings occur at discrete times this has a significant effect on the value of the option and the stochastic volatility will also affect the value of the option.

We assumed the following processes for the underlying FX rate S and the volatility σ of the FX rate are,

$$dS = \mu S dt + \sigma S dz_S \tag{13}$$

$$d\sigma = \alpha(\bar{\sigma} - \sigma)dt + \theta\sigma dz_\sigma \tag{14}$$

We generated control variates corresponding to the first differential with respect to each of the stochastic variables S, and σ (DELTA and LAMBDA) and the second differential with respect to S (GAMMA). That corresponding to S is the same as equation (11), (but note that $\Delta_{t_h}S$ is now different), and that corresponding to σ is also the same, but with S replaced by σ, and finally the third control variate is the same as equation (12). The partial differentials were obtained from the analytical formula for the continuous fixing lookback option (the large reduction in the variance of the Monte Carlo estimate of the elementary European call was partly due to the partial differentials or hedge ratios in the control variates being exact).

Table II shows the results of applying the variance reduction techniques to our lookback call.

For this example we have obtained a total variance reduction of 1200. We could increase the variance reduction by adding further hedge ratios, for example a control variate based on $\frac{\partial^2 \phi}{\partial \sigma^2}$. However, we have reduced the error in the estimate such that the value is accurate to the first two decimal places with only 1000 simulations. Thus the valuation could be performed on a reasonably powerful desktop computer in approximately 30 seconds. Without these variance reduction techniques it would take a factor of 1200 more simulations or 10 hours computation time on the same computer.

II Efficient Computation of Hedge Ratios

The basic technique for obtaining hedge ratios from Monte Carlo valuations is to compute their finite difference approximations. For example, to compute DELTA = $\frac{\partial \phi}{\partial S}$ we compute

$$\frac{\phi(S + \Delta S) - \phi(S - \Delta S)}{2\Delta S} \tag{15}$$

The first potential problem is that if we compute the estimates $\phi(S + \Delta S)$ and $\phi(S - \Delta S)$ using independent random variables then the error in the estimates may be of the same order of magnitude as the change in the option value over ΔS. The second problem is that we immediately double the computational time by having to compute two estimates of the option value. For other hedge ratios we will have to compute more estimates. The solution to both problems is to compute the estimates are far as possible in parallel.

Firstly by computing the estimates using the same standard normal random variables we ensure that the errors in the estimates are similar in magnitude and sign and so will tend to cancel out in the calculation of the derivatives. Secondly by computing the estimates in parallel we can make further computational savings depending on the particular problem.

For example for DELTA and GAMMA ($= \frac{\partial^2 \phi}{\partial S^2}$) for which we compute

$$\frac{\phi(S + \Delta S) - 2\phi(S) + \phi(S - \Delta S)}{\Delta S^2} \tag{16}$$

we need the option value at $S - \Delta S$, S and $S + \Delta S$. Now for a lognormal underlying where r and σ are independent of S the simulated changes in the logarithm of the underlying price are independent of the initial value of the underlying price (see equation (10)). Furthermore, although the control variates do depend on the value of the underlying at each time step, because the initial values are very close we can use the control variates for the central value for all three estimates. So we can simulate a single path for the changes in the logarithm of the underlying and use this to compute the three different payoffs and from that the three option values. Note that it is only because we have considerably reduced the variance of the option value estimates that the GAMMA estimate will be usefully accurate for the number of simulations we use (even so the error is likely to be of the order of a few percent).

For THETA ($= \frac{\partial\phi}{\partial S}$) we simply additionally accumulate the relevant variables before the final time step is made along each path and perform the estimation as normal. This gives us $\phi(t)$ and $\phi(t + \Delta t)$, so we can compute THETA from its finite difference approximation. Note that this will be sensitive to size of the time step, but for typical times to maturity of one year or less and a reasonable number of time steps (of the order of 100) the accuracy is good.

For RHO ($= \frac{\partial\phi}{\partial r}$) and LAMBDA ($= \frac{\partial\phi}{\partial\sigma}$) we need to consider how the changes in the logarithm of the underlying price (Δs_i) along the ith path are formed,

$$\Delta s_i = (r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{\Delta t}\sum_{h=1}^{N} \tilde{z}_h \tag{17}$$

Now if r is not stochastic we can form,

$$\Delta s_i \mid_{r-\Delta r} = ((r - \Delta r) - \frac{1}{2}\sigma^2)T + \sigma\sqrt{\Delta t}\sum_{h=1}^{N} \tilde{z}_h \tag{18}$$

and

$$\Delta s_i \mid_{r+\Delta r} = ((r + \Delta r) - \frac{1}{2}\sigma^2)T + \sigma\sqrt{\Delta t}\sum_{h=1}^{N} \tilde{z}_h \qquad (19)$$

easily by simply accumulating the sum of the standard normal variables $\tilde{z}_h$. Similarly for LAMBDA we can form the relevant changes Δs_i. If r or σ are themselves stochastic then we can first check to see if we can simulate random changes which are independent of the initial value of the variable. If this is not the case then we may have to fully simulate the separate paths.

III American Style Options

As stated in the introduction the basis of the Monte Carlo technique is that any option can be expressed as the discounted risk-neutrally adjusted expectation of its payoff. In order to value American style options we would have to check at each time step along each path whether it was optimal to exercise the option and if so we would take its payoff at that point in time discounted over the time the option had lived. However, to test if the option should be exercised we must know its value if held, so we would have to perform another Monte Carlo valuation at that point. Thus the computational cost explodes for this problem.

Geske and Johnson (1984) and more recently Breen (1991) and Ho *et al* (1991) have shown that the value of an American style option can be obtained accurately by extrapolating from the values of options exercisable at a finite number of times. In particular Breen (1991) extrapolates from options exercisable at one (European), two and three times, valued by the binomial method. Ho *et al* extrapolate from the European option and the maximally valued twice exercisable option. Using this approach we can prevent the computational cost of the Monte Carlo method exploding for American style options.

Consider the valuation of a complex, path dependent option on a single underlying random variable which can be exercised halfway through its life as well as at maturity. In order to value this option we need to know the position of the early exercise boundary or in other words the value of the underlying asset at which it becomes optimal to exercise the option at the halfway point. Figure A depicts how we might value this option by Monte Carlo simulation. Let $t = 0$ be the current time and $t = T$ be the maturity time of the option. At $t = T/2$ we need to know the value of the option if held to maturity as a function of the underlying asset price. We can obtain this approximately by performing Monte Carlo simulations at the points indicated in Figure A. If the computational cost of the simulations is low we would add more points and use a simple linear interpolation between the points. If the computational cost is high then we would use a small number of points and fit a C1 peicewise quadratic function to the points. From this we can easily obtain the position of the early exercise boundary. Valuation of the option is then simply a matter of simulating the underlying from $t = 0$ to $t = T/2$, computing the payoff as the maximum of the option value if held and the value if exercised and then taking the discounted mean of the payoffs.

Figure B illustrates how the value of a twice exercisable put changes with the time of the intermediate exercise point ($X = 100$, $T = 1.0$, $S = 100$, $\sigma = 0.1$). As r increases (and as T increases and as the option becomes more in the money) the value depends more sensitively on the exact time of the intermediate exercise opportunity. But for reasonable values of these parameters an intermediate time of exercise of halfway through the life of the option gives an option value close to the maximum for a twice exercisable option. For certain types of option it may be possible to determine specific points in its life where the early exercise opportunity maximally affects the option value. In this case these points would be used as the limited early exercise points in the Monte Carlo simulation.

Now, consider the case of a lognormal underlying asset where the changes in the log of the asset are independent of its initial value. We simulate paths of the underlying process and its antithetic counterpart as normal together with the relevant control variates from $t = 0$ to $t = T$. At $t = T/2$ we store the changes in the logarithm of the underlyings (which are independent of the initial underlying value) and the control variates. We then compute the value of the European option at $t = 0$ by the normal least squares estimation procedure. We then partition the range of underlying values reached at $t = T/2$ and compute the European option value if held from $T/2$ to T at these discrete values of the underlying. We do this using the previously stored set of simulated changes in the logarithm of the underlyings and the control variates in the normal way. Note that the control variates will not be exactly correct but they will still be good control variates and therefore give good variance reduction. This is much faster than simulating the paths again and we can do this because the changes in the logarithm of the underlying are independent of its initial value. We can now compute the value of the twice exercisable option at $T/2$ (as the maximum of its immediate exercise value and its value if held) for each value of the underlying reached from its initial value via the original simulated changes. We obtain the options held value at $T/2$ by interpolating from our discrete partition values. Using these we can compute the twice exercisable option value at $t = 0$ by the normal least squares estimation again using our stored control variates (see Figure A).

Finally, following Ho *et al* (1991) we can estimate the American option value by extrapolating from the European and twice exercisable option values. Assume an exponential relationship between the American option value $\phi(\infty)$ and the value of an option with a limited number of exercise points $\phi(n)$,

$$\phi(\infty) = \phi(n)\exp(\alpha/n) \tag{20}$$

Setting $n = 1$ and $n = 2$ gives

$$\phi_A = \frac{\phi_2\phi_2}{\phi_E} \tag{21}$$

where ϕ_A is the American option value, ϕ_E is the European option value and ϕ_2 is the twice exercisable option value.

Figure C shows a comparison of the various option values ($T = 0.5$, $S = 100$, $\sigma = 0.2$, $r = 0.1$). The American put option values are computed from the Barone-Adesi and Whaley (1987) approximation.

Now, suppose the option is a function of more than one underlying random variable, for example if we have stochastic volatility or interest rates. The early exercise boundary will be multidimensional. We must therefore obtain the option value if held at the early exercise points by simulating from the range of possible values of all the random variables. However, if we fit a polynomial to early exercise boundary then the number of simulations we must perform will only be roughly an order of magnitude greater. By applying the variance reduction techniques we have discussed this level of computation can be still be achieved in realistic times.

IV Conclusions

Careful use of variance reduction techniques can make Monte Carlo simulation an efficient and powerful method for valuation of complex contingent claims. The number of simulations required to obtain acceptable levels of error can be reduced to a level such that the valuations can be performed in almost real-time on a powerful desktop computer. Furthermore, the hedge ratios can also be computed accurately and efficiently by careful implementation of the simulation. Finally, American style options can be valued approximately by extrapolating from the value of European options and options exercisable at limited points during their life. Often, these computations can be made very efficiently by exploiting the lognormal nature of the underlying asset.

Finally the techniques we have described have implications for the rapidly expanding field of parallel processing. The number of simulations needed to obtain accurate valuations has been reduced to a similar level as the number of processing units which current parallel hardware contains. This greatly simplifies the implementation of Monte Carlo simulations on this type of hardware and simultaneously we obtain a real-time valuation technique.

References

Barone-Adesi G and Whaley R E , Efficient Analytical Approximation of American Option Values, Journal of Finance 42(2), pp 301 - 320, 1987.

Boyle P P , Options: A Monte Carlo Approach, Journal of Financial Economics 4, pp 323 - 338, 1977.

Breen R , The Accelerated Binomial Option Pricing Model, Journal of Financial and Quantitative Analysis 26, pp 153 - 164, 1991.

Cox J C and Ross S A , The valuation of options for alternative stochastic processes, Journal of Financial Economics 3, pp 145 - 166, 1976.

Geske R and Johnson H E , The American Put Valued Analytically, Journal of Finance 39, pp 1511 - 1524, 1984.

Goldman M B, Sosin H and Gatto M , Path dependent options: 'Buy at the low, sell at the high', Journal of Finance 34(5), pp 1111-1127, 1979.

Ho T S, Stapleton R C and Subrahmanyam M G , The Valuation of American Options in Stochastic Interest Rate Economies, Working Paper, The Management School, Lancaster University, 1991.

Hull J and White A , The Pricing of Options on Assets with Stochastic Volatilities, Journal of Finance 42, June, pp 281 - 300, 1987.

Hull J and White A , The Use of the Control Variate Technique in Option Pricing, Journal of Financial and Quantitative Analysis 23(3), September, pp 237 - 251, 1988.

Johnson H E and Shanno D , Option Pricing when the Variance is Changing, Journal of Financial and Quantitative Analysis 22, June, pp 143 - 151, 1987.

Kemna A G Z and Vorst A C F , A Pricing Method for Options on Average Asset Values, Journal of Banking and Finance 14, pp 113 - 129, 1990.

Kleijnen J P C , Statistical Techniques in Simulation, Marcel Dekker, New York, 1974.

Ripley B D , Stochastic Simulation, John Wiley & Sons, New York, 1987.

Scott L O , Option Pricing when the Variance Changes Randomly: Theory, Estimation and an Application, Journal of Financial and Quantitative Analysis 22(4), December, pp 419 - 438, 1987.

Table I

Results of applying variance reduction techniques

to Monte Carlo valuation of an elementary European call

Elementary European call option value	18.58	
Variance reduction method	Variance	
None	0.18	(0.42)
Antithetic variate	0.0073	(0.085)
Control variates	0.0006	(0.024)
Antithetic and Control variates	0.000015	(0.0039)

Exercise price = 100

Time to Maturity = 2 years

Initial Stock Price = 100

Volatility = 10%

Riskless Rate = 10%

Number of time steps (N) = 104

Number of simulations (M) = 1000

(Figures in parentheses are standard deviations)

Table II

Results of applying variance reduction techniques to Monte Carlo valuation of a lookback call

Continuous fixing analytical formula value	2.79	
FX lookback call option value	4.525	
Variance reduction method	Variance	
None	0.0096	(0.098)
Antithetic variate	0.00092	(0.030)
Control variates	0.000046	(0.0068)
Antithetic and Control variates	0.000008	(0.0028)

Time to Maturity = 0.5 years

Initial FX Rate = 100

Volatility = 6%

Domestic Interest Rate = 5%

Foreign Interest Rate = 7%

Rate of Mean Reversion (α) = 1.35

Volatility of the Volatility (θ) = 0.085

Number of time steps (N) = 128

Number of simulations (M) = 1000

(Figures in parentheses are standard deviations)

Figure A: Valuation of a twice exercisable option by Monte Carlo simulation

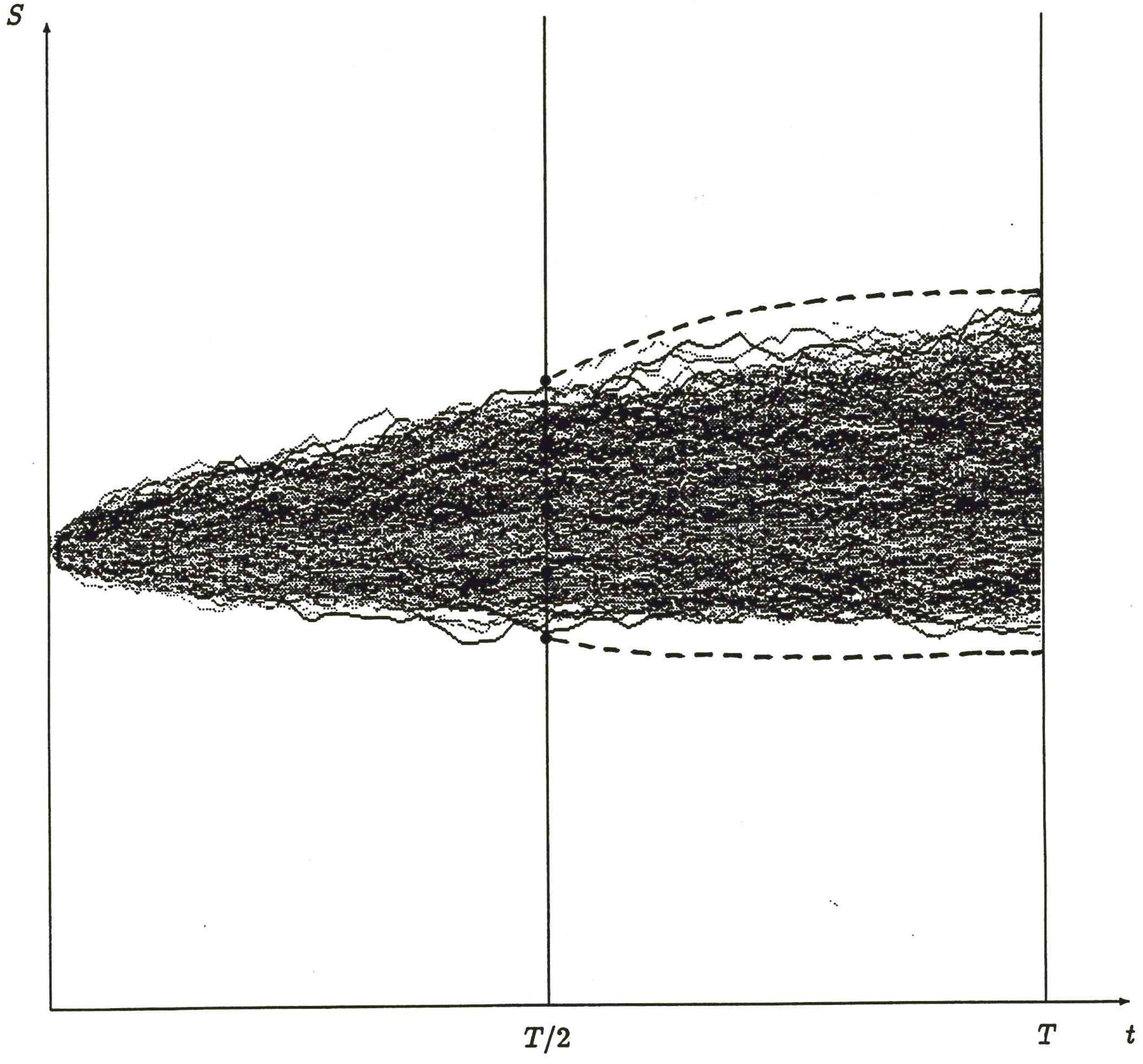

Figure B: Value of a twice exercisable option for various interest rates (r)

(valuation by binomial method with 100 time steps)

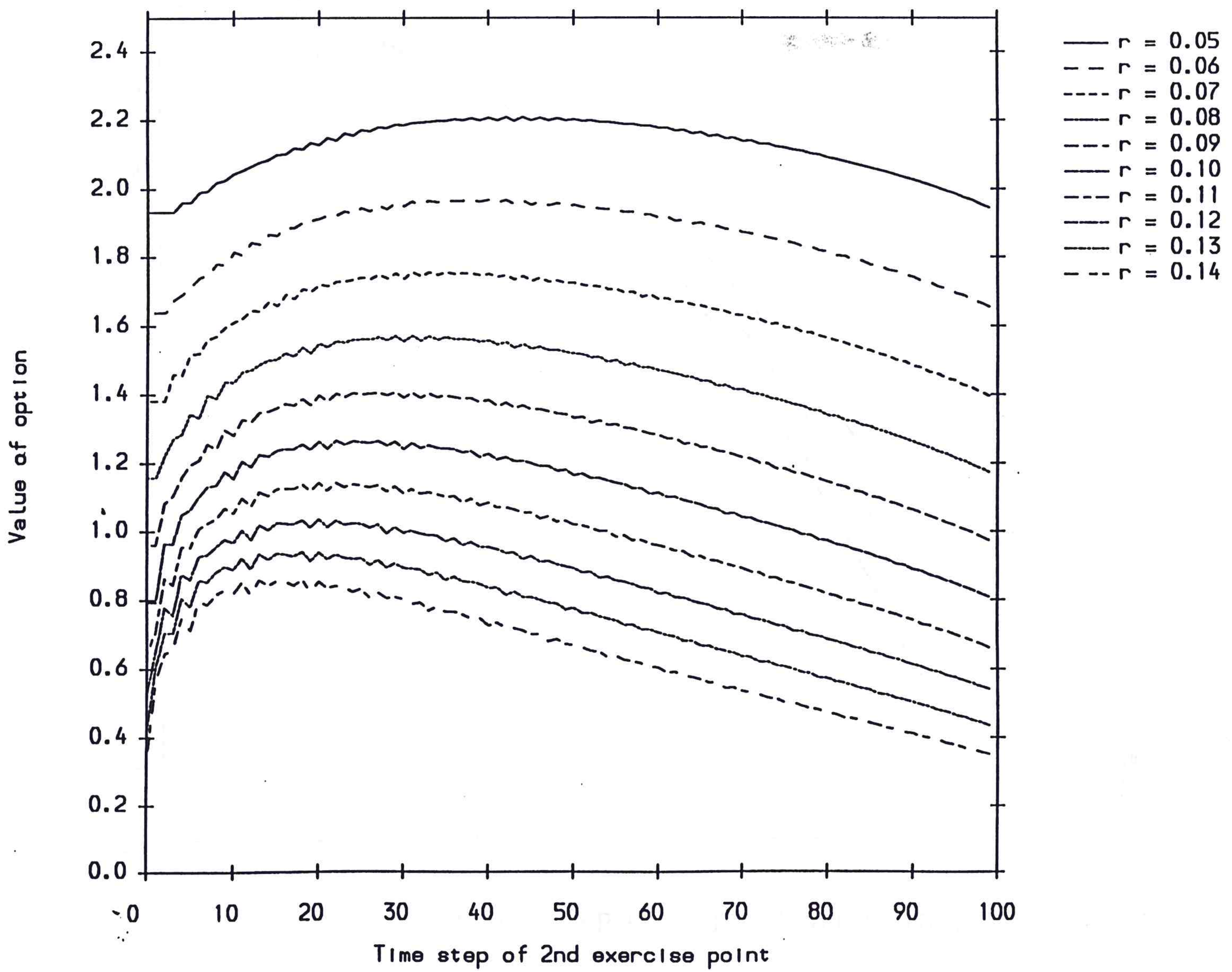

Figure C: Comparison of option values for $\sigma = 0.20$

(American put computed by Barone-Adesi and Whaley (1987) approximation)