The Economic Journal, **103** (*March*), 395–405. Published by Blackwell Publishers, 108 Cowley Road, Oxford OX4 1JF, UK and 238 Main Street, Cambridge, MA 02142, USA.

QUASI MEAN REVERSION IN AN EFFICIENT STOCK MARKET: THE CHARACTERISATION OF ECONOMIC EQUILIBRIA WHICH SUPPORT BLACK–SCHOLES OPTION PRICING*

Stewart Hodges and Andrew Carverhill

This paper is concerned with the behaviour of the risk premium on the market portfolio of risky assets. It provides a characterisation for the evolution of the market risk premium in a simple economy where the variance of the equity market return and the risk free interest rate are both constant. These are precisely the assumptions which enable index options to be valued using the Black and Scholes (1973) option valuation model.

This analysis is motivated by recent empirical studies (for example, Poterba and Summers (1988), Fama and French (1988), and Lo and MacKinlay (1988)) which suggest some degree of predictability related to mean reversion in equity returns. Apparent observed mean reversion might be due either to market inefficiencies, or to systematic variation in the equilibrium risk premium.[1] Until now it has been impossible to construct tests to distinguish between these two alternatives, as we have had no theory to explain what kinds of variation in risk premia might be experienced within an equilibrium model. That is what this paper provides.

We assume that the market asset has price S which follows the process:

$$\frac{dS}{S} = \mu(S,t)\,dt + \sigma\,dz. \tag{1}$$

For option pricing, the rate of drift $\mu(S,t)$ is essentially arbitrary: it is eliminated in the derivation of the Black–Scholes formula. The purpose of this paper is to characterise what behaviour for $\mu(S,t)$ is required for consistency with equilibrium. For convenience we shall prefer to work with

$$\alpha(S,t) = \frac{\mu(S,t)-r}{\sigma} \tag{2}$$

which measures the instantaneous reward per unit of risk, where r is the risk free interest rate. The kind of equilibrium which we are concerned with here is that characterised by a single representative investor. Thus we assume agents maximise expected utility over utility functions which are not state dependent. Their demands aggregate to the demands of a single representative agent who holds the market portfolio at all times. The main result of the paper is that we derive a non-linear partial differential equation which equilibrium alphas must

* We are grateful to Dr Michael Selby, FORC, for introducing us to the literature on Burgers' equation, and to Dr Les Clewlow, FORC, for further assistance with this equation.

[1] See, for example, Cechetti *et al.* (1990).

satisfy. The equation is known as Burgers' equation. It occurs in a number of other contexts such as models of the flow of viscous fluid, and the flow of traffic, and closed form solutions are possible depending on the boundary conditions imposed. The question we address has previously been studied by Bick (1990). Parts of our analysis are very similar to his.[2] Our assumption of the constancy of the variance rate enables us to obtain stronger results.

The development of the paper is as follows. We first provide a formal description of the type of equilibrium that we assume. We assume an equilibrium of a conventional kind which can be characterised by the utility function of a representative agent (see, for example, Huang and Litzenberger (1988)). Together with our other assumptions, and following the related literature of Cox and Leland (1982), Dybvig (1988*a*, *b*), and Bick (1990), we demonstrate particular path independence and monotonicity properties. The following sections show the strength of those properties. We illustrate them first within a binomial context, and show that they imply that alpha must satisfy an unusual difference equation. Taking the limit for small time increments results in the partial differential equation known as Burgers' equation. A more satisfactory, but less intuitive, derivation is possible in a continuous-time framework using the Girsanov formula for the change of probability measure associated with moving between the objective probabilities and the risk neutral ones. The following section describes the nature of solutions to Burgers' equation, and their implications for our understanding of the dynamics of the risk premium in capital markets. Finally, the paper concludes by discussing possible alternative kinds of equilibrium which would not have the properties described in this paper, and makes some suggestions for future research.

I. PATH INDEPENDENCE AND MONOTONICITY RESULTS

We begin by considering an economy with a single asset whose price S_t follows the process

$$\frac{dS}{S} = [r + \sigma\alpha(S, t)]\, dt + \sigma\, dz. \tag{3}$$

For simplicity we shall also assume that within the time horizon H of interest to us no dividends are paid and that the instantaneous riskfree rate r is constant. The question posed is how must $\alpha(S, t)$ behave if we require this economy to correspond to an equilibrium characterised by individuals maximising the expected utility of their wealth at dates greater than or equal to H. Following Harrison and Kreps (1979), we note that any arbitrage-free price system can be sustained as a competitive equilibrium characterised by a single representative agent. For any economy which does not permit arbitrage, there exists a risk neutral probability measure under which the rate of drift of all assets is the risk free rate, r. We shall define P as the objective probability measure, and Q as the risk neutral one. The ratio of the risk neutral density to

[2] Since completing this paper, our attention has been drawn to the related important work completed independently by He and Leland (1991).

the objective one (i.e. the Radon–Nikodym derivative dQ/dP) gives the state-price density which defines the marginal utilities of this agent.

Dybvig's Probability Distribution Pricing Model (1988*a*, *b*) pushes this analysis one step further. One of Dybvig's main results is to demonstrate that a state-dependent pattern of terminal wealth is efficient if and only if it is decreasing (non-increasing) in the terminal state-price density. The meaning of efficiency in this context is that of first degree stochastic dominance. In other words a probability distribution of wealth is said to be efficient if it maximises expected utility for some (non-state dependent) and strictly increasing von Neumann–Morgenstern utility function. No assumption of risk aversion is required. Within the simple assumptions we have already made, Dybvig's results imply that for equilibrium, the state-price density function must be monotonic decreasing in S at every $t \leqslant H$. This also means (loosely stated) that any two paths with equal probabilities under P also have equal probabilities under Q. This is a path independence property which seems to have been first noted in the work of Cox and Leland (1982), and is also utilised in Bick's (1990) analysis. More generally and precisely, dQ/dP itself must be a path independent function of S and t. We show in the next section how this characterisation leads directly to Burgers' equation for the evolution of alpha.

We should also note that while efficient wealth payoffs for an investor must always be monotonic decreasing in the state-price density, within more generally specified economies (e.g. with stochastic interest rates and/or volatility) the value of the market portfolio need not be monotonic in the state-price density.[3]

II. BINOMIAL EXAMPLE AND DERIVATION

Fig. 1 shows the basic binomial tree of our analysis. The single state at time zero is labelled state 0. The two states at the next time instant δt are labelled 1 and 2. At each of those states, s, the objective probability of S increasing to uS is denoted by p_s. The probability of a down move to dS is $p'_s = 1 - p_s$.

The nature of the restrictions imposed by the path independence property of equilibrium is best illustrated by a numerical example. We suppose $S =$ £100, $u = 1{\cdot}10$, $d = 0{\cdot}90$ and that the risk free interest rate r is zero. Thus at the state labelled 1, the index value is £110 and at that labelled 2 it is £90. Our choice of u and d ensure that the state prices are equal, i.e. $q_1 = q_2 = \frac{1}{2}$ etc, as it costs £0·5 in state 0 to construct a £1 payoff in state 1. The equilibrium market clearing condition therefore implies that the objective probability of u followed by d exactly equals that of d followed by u.

In other words

$$p_0 p'_1 = p'_0 p_2.$$

As a result, if the risk premium is known for states 1 and 2, this condition tells us what it must be for state 0. At any state s the percentage expected return is just $20p_s - 10$, which is also the risk premium (since $r = 0$). Suppose $p_1 = 0{\cdot}6$ and $p_2 = 0{\cdot}8$, so after one period if the index is at £110 the risk premium is 2%,

[3] See, for example, Dybvig (1988*a*) footnote 6.

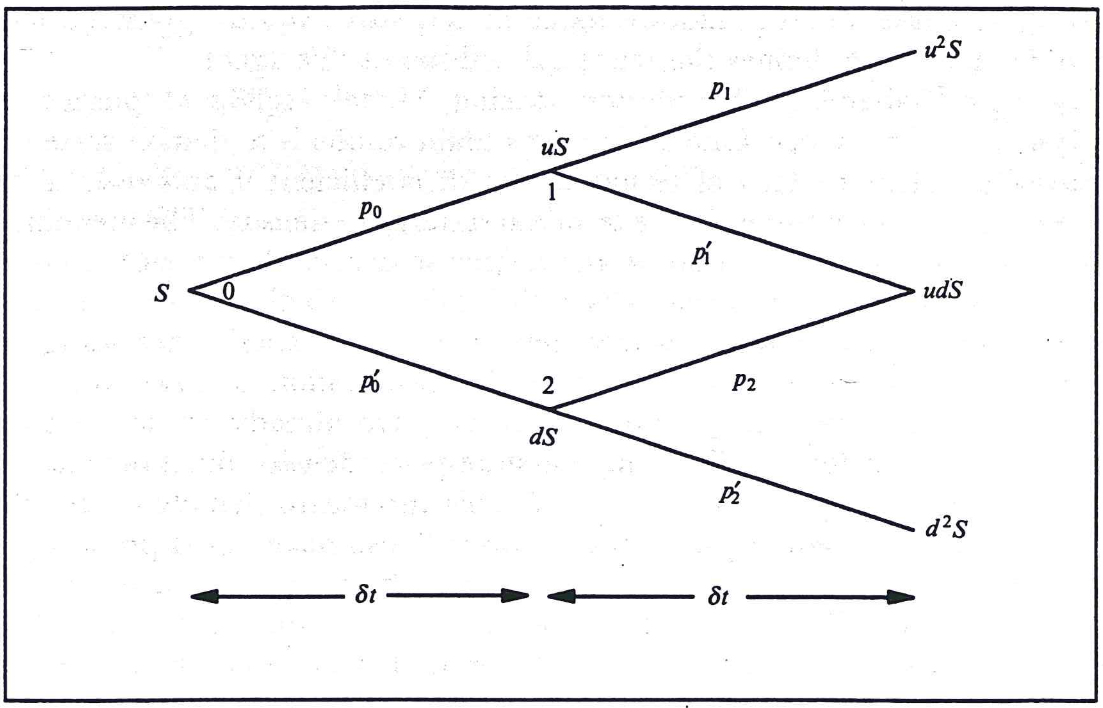

Fig. 1. Binomial tree.

but if the index has fallen to £90 the risk premium becomes 6%. We might perhaps have thought that the state 0 risk premium could be anything, or maybe a simple average of 4%. Both conjectures are false. The path independence condition implies that

$$p_0 \times 0{\cdot}4 = (1-p_0) \times 0{\cdot}8,$$

so $p_0 = \frac{2}{3}$, giving a state 0 risk premium of $3\frac{1}{3}$%. In fact p_0 is a weighted average of p_1 and p_2, with p_0 itself providing the weighting:

$$p_0 = p_0 p_1 + (1-p_0) p_2.$$

If at some horizon date H we can specify how the risk premium depends on the level of the market index, then it is uniquely determined for all earlier dates under this averaging equation.

The General Derivation

It is convenient to choose the parameters u and d so that the risk neutral probabilities q_s from each state are all equal to $\frac{1}{2}$. We may do this by choosing

$$u = \exp\left[(r-\tfrac{1}{2}\sigma^2)\,\delta t + \sigma\sqrt{(\delta t)}\right], \tag{4}$$

$$d = \exp\left[(r-\tfrac{1}{2}\sigma^2)\,\delta t - \sigma\sqrt{(\delta t)}\right]. \tag{5}$$

(see for example, Jarrow and Rudd (1983)).

Next we need to find the relationship between p_s and the price of risk, α_s. The expected return from state s is given by

$$1+\mathrm{E}(R_s)\,\delta t = p_s \exp\left[(r-\tfrac{1}{2}\sigma^2)\,\delta t + \sigma\sqrt{(\delta t)}\right] + (1-p_s)\exp\left[(r-\tfrac{1}{2}\sigma^2)\,\delta t - \sigma\sqrt{(\delta t)}\right]. \tag{6}$$

Expanding this as a Taylor series gives

$$\begin{aligned} 1+\mathrm{E}(R_s)\,\delta t &= p_s[1+(r-\tfrac{1}{2}\sigma^2)\,\delta t+\sigma\surd(\delta t)+\tfrac{1}{2}\sigma^2\,\delta t+O(\delta t^{\frac{3}{2}})] \\ &\quad +(1-p_s)\,[1+(r-\tfrac{1}{2}\sigma^2)\,\delta t-\sigma\surd(\delta t)+\tfrac{1}{2}\sigma^2\,\delta t+O(\delta t^{\frac{3}{2}})] \\ &= 1+r\,\delta t+(2p_s-1)\,\sigma\surd(\delta t). \end{aligned} \tag{7}$$

Hence the price of risk, α_s, is given as

$$\left.\begin{aligned} \alpha_s &= \frac{\mathrm{E}(R_s)-r}{\sigma} \\ &= \frac{(2p_s-1)}{\surd(\delta t)} \\ \text{so}\qquad p_s &= \tfrac{1}{2}[1+\alpha_s\surd(\delta t)]. \end{aligned}\right\} \tag{8}$$

We are now in a position to derive the difference equation which restricts the evolution of α_s through time.

For our tree, the crucial path independence condition is simply that

$$p_0 p_1' = p_0' p_2. \tag{9}$$

Substituting our previous expression for p_s gives:

$$[1+\alpha_0\surd(\delta t)]\,[1-\alpha_1\surd(\delta t)] = [1-\alpha_0\surd(\delta t)]\,[1+\alpha_2\surd(\delta t)] \tag{10}$$

and solving for α_0, we find that

$$\alpha_0\surd(\delta t)\,[1-\alpha_1\surd(\delta t)+1+\alpha_2\surd(\delta t)] = 1+\alpha_2\surd(\delta t)-1+\alpha_1\surd(\delta t), \tag{11}$$

$$\alpha_0 = \frac{\alpha_1+\alpha_2}{2-\surd(\delta t)\,(\alpha_1-\alpha_2)}. \tag{12}$$

This is our key difference equation, which determines the evolution of α_s throughout the entire binomial tree. We notice that it is similar to the usual diffusion equation which would give

$$\alpha_0 = \frac{\alpha_1+\alpha_2}{2}, \tag{13}$$

but the extra $\surd(\delta t)\,(\alpha_1-\alpha_2)$ term makes a significant difference.

Note that path independence in every up-down down-up 'diamond' of the binomial lattice ensures path independence between every pair of points of the lattice. Thus the difference equation we have obtained is necessary and sufficient for path independence.

The difference equation we have obtained can be used as a numerical scheme if we are able to specify the values of α_s at some later date H. We shall report some numerical results later in the paper. It remains in this section to take the limit of our difference equation as $\delta t\to 0$ to obtain the corresponding partial differential equation for α.

Burgers' Equation

It is convenient, first of all to introduce a change of variables. Instead of working with S_t we shall work with the transformed variable

$$x_t = \ln S_t-(r-\tfrac{1}{2}\sigma^2)\,t. \tag{14}$$

Whereas S_t evolves as

$$S_{t+\delta t} = S_t \exp\left[(r-\tfrac{1}{2}\sigma^2)\,\delta t \pm \sigma\sqrt{(\delta t)}\right] \tag{15}$$

x_t evolves as

$$x_{t+\delta t} = x_t \pm \sigma\sqrt{(\delta t)}. \tag{16}$$

We now examine the evolution of $\alpha(x, t)$ from the difference equation (12).

$$\alpha(x, t-\delta t) = \tfrac{1}{2}[\alpha(x+\sigma\epsilon, t)+\alpha(x-\sigma\epsilon, t)]\,[1-\tfrac{1}{2}\epsilon\alpha(x-\sigma\epsilon, t) + \tfrac{1}{2}\epsilon\alpha(x+\sigma\epsilon, t)+O(\epsilon^2)], \tag{17}$$

where $\epsilon = \sqrt{(\delta t)}$.

Expanding further as a Taylor series, ignoring terms in ϵ^3 and higher, and using the notation α_t, α_x, α_{xx} to denote the appropriate partial derivatives of α evaluated at (x, t), we obtain:

$$\alpha-\epsilon^2\alpha_t = (\alpha+\tfrac{1}{2}\sigma^2\epsilon^2\alpha_{xx})\,(1+\sigma\epsilon^2\alpha_x). \tag{18}$$

The zero order terms cancel, and collecting the terms in ϵ^2 we conclude that

$$\alpha_t = -\tfrac{1}{2}\sigma^2\alpha_{xx} - \sigma\alpha\alpha_x. \tag{19}$$

This is Burgers' equation.

A more elegant and rigorous derivation can be obtained in a continuous time framework, using the formula for the change of measure between the objective probability measure and the risk-neutral one Q. This is shown in the Appendix at the end of the paper.

III. BURGERS' EQUATION

The equation we have derived was proposed by Burgers in 1948 as a model for the one dimensional flow of a viscous fluid. It also occurs in a number of other applications including modelling traffic flows. Properties of this equation and solution methods may be found in Bland (1988), Klevorkian (1990) and Whitham (1974). As with the usual diffusion equation, for stability we must be solving the equation backwards in time. In other words, for the signs of the equation to look more conventional, we should measure time from our horizon date as

$$\tau = H-t. \tag{20}$$

With time measured in this direction the equation now becomes:

$$\alpha_\tau = \tfrac{1}{2}\sigma^2\alpha_{xx} + \sigma\alpha\alpha_x. \tag{21}$$

We will look at some simple properties of this equation and see how solutions may be obtained from initial conditions for α specified as $\alpha(x, 0)$. Note first that if $\alpha(x, 0)$ is a constant then all the space derivatives are zero and so it keeps this same constant value for all time. It is well known that this solution corresponds to a representative investor with constant proportional risk aversion (see Bick (1987)). The coefficient of σ attached to the final expression of the equation is necessary in order for the equation to take the same form independently of the scale to which τ is measured. For example, if we make a further change of the time variable (for example to measure time in months instead of years) so we introduce $\tau' = k\tau$, the natural dimensions in which to measure μ become $\mu' = k\mu$, while the natural dimensions in which to measure σ are as $\sigma' = \sqrt{k}\,\sigma$. Thus the rescaled α is scaled by $\sqrt{k}$ to become $\alpha' = \sqrt{k}\,\alpha$. The equation becomes transformed to:

$$\alpha'_\tau = \tfrac{1}{2}\sigma'^2\alpha'_{xx} + \sigma'\alpha'\alpha'_x \tag{22}$$

and has the same form independently of the time metric. For small time steps the effect of the diffusion term is also small, and the last expression can propagate discontinuities backwards in time. Over larger time spans the diffusion element becomes more important.

Another property of Burgers' equation is that $\alpha(x, 0) > 0$ is sufficient to ensure that $\alpha(x, \tau) > 0$ for all $\tau > 0$. This means that the monotonicity condition of the state-price density function is automatically satisfied: provided the boundary condition makes the state price density monotonic in x, the same monotonicity is assured at all earlier dates.

Solution Method

In 1950 and 1951, Hopf and Cole showed independently how an analytic solution to Burgers' equation could be derived using a clever transformation which reduces the problem to a conventional diffusion equation. We shall show how that may be applied to the equation in our notation.

For the Cole–Hopf transformation we set

$$\alpha = \sigma\frac{v_x}{v}. \tag{23}$$

Cole and Hopf showed that if $v(x, \tau)$ satisfies the diffusion equation

$$v_\tau - \tfrac{1}{2}\sigma^2 v_{xx} = 0 \tag{24}$$

then the resulting $\alpha(x, \tau)$ from equation (23) solves Burgers' equation (21).

To solve for $\alpha(x, \tau)$ from a particular boundary condition giving values for $\alpha(x, 0)$ we note that

$$\alpha(x, 0) = \sigma\frac{v_x(x, 0)}{v(x, 0)} = \sigma\frac{d}{dx}[\ln v(x, 0)]. \tag{25}$$

Therefore

$$\int_0^x \alpha(x, 0)\, dx = \sigma \ln v(x, 0) \tag{26}$$

so $v(x, 0)$ is defined up to an arbitrary constant c as $cg(x)$ in the equation

$$v(x, 0) = c\exp\left[\frac{1}{\sigma}\int_0^x \alpha(s, 0)\, ds\right] = cg(x). \tag{27}$$

Finally then, $\alpha(x, \tau)$ is obtained from equation (23), after evaluating $v(x, \tau)$ as

$$v(x, \tau) = \int_{-\infty}^{\infty} g(s)\exp\left[-\frac{(x-s)^2}{2\sigma^2\tau}\right] ds. \tag{28}$$

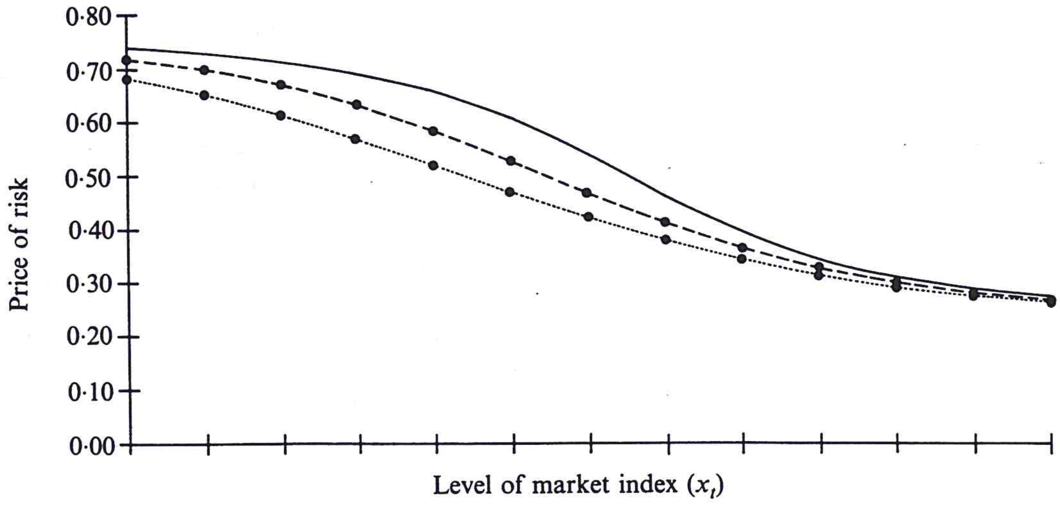

Fig. 2. Solution of Burgers' equation.

Some Examples

We can take as our first example the case where $\alpha(x, 0)$ is given as

$$\alpha(x, 0) = a - bx, \tag{29}$$

where b is a non-negative constant. The $g(x)$ function is therefore determined by integrating equation (29) as

$$g(x) = \exp\{(ax - \tfrac{1}{2}bx^2)/\sigma\}. \tag{30}$$

Next, $v(x, \tau)$ is found by integrating $g(x)$ over the normal kernel as in equation (28). This has a convenient (though rather messy) closed form solution and it can also be differentiated to give $v_x(x, \tau)$. Their ratio multiplied by σ gives $\alpha(x, \tau)$ which simplifies to

$$\alpha(x, \tau) = \frac{a - bx}{1 + b\tau\sigma}. \tag{31}$$

It is easily verified that this satisfies Burgers' equation with our initial conditions.

Fig. 2 shows the results of numerical calculations starting from more complicated S-shaped initial conditions. The initial condition for alpha is shown with a solid line, the dashed line shows alpha three years earlier, and the dotted one three years before that. Note that the direction in which alpha moves depends on the slope of alpha as a function of x.

IV. CONCLUSIONS AND POSSIBLE EXTENSIONS

We have shown that a necessary condition for economic equilibria which support the Black–Scholes option valuation formula (in its usual lognormal form) is that the risk premium on the market portfolio must follow a non-linear partial differential equation known as Burgers' equation. The equation can be solved analytically in some cases and numerically in others. As far as we know, this is the first finance application of the equation, though Merton's (1971) paper on optimum rules for consumption and portfolio investment contains some equations with related non-linear terms. The analysis provides new insights into how the price of risk can behave within an efficiently functioning market in equilibrium. This has potential applications in empirical work, for example on observed mean reversion in capital markets. Representative agent equilibrium implies that although the risk premium can vary it must evolve according to Burgers' equation. It may be possible to devise econometric tests of this restriction.

The assumptions we have made are quite strong, and it is worth discussing briefly how far they might be relaxed. Our derivations could obviously be generalised to deterministic but non-constant r and σ. The path independence results derived by Cox and Leland (1982) include dividend payments, and thus suitable extensions can probably be found to include at least some kinds of dividends and intermediate consumption. It also appears that in some cases we can handle a stochastic interest rate. These extensions are not very different from the generalisations to Black and Scholes given by Merton (1973). However, it seems much more doubtful that a worthwhile extension to the case of stochastic volatility could be found.

Understanding the behaviour of the market risk premium is a problem of fundamental importance for fields such as portfolio management. Even if empirical work did not support a single representative agent equilibrium it might be difficult to refute an overlapping generations model which could introduce more complex time dependent effects. An example will illustrate this. We could imagine an economy which has two classes of investors: class 1 are risk neutral and their investment horizon is at time 1; class 2 have constant proportional risk aversion and a longer investment horizon, time 2. In such an economy alpha will be zero until time 1, when it will jump to the level required to attract the second risk averse investors. It is clear that there is no conservation equation which can hold across the time 1 event, and the market portfolio is no longer an undominated investment for long term investors.

Financial Options Research Centre, University of Warwick

Hong Kong University of Science and Technology

References

Bick, A. (1987). 'On the consistency of the Black–Scholes model with a general equilibrium framework.' *Journal of Financial and Quantitative Analysis*, vol. 22, pp. 259–75.

Bick, A. (1990). 'On viable diffusion price processes of the market portfolio.' *Journal of Finance*, vol. 45, pp. 673–89.

Black, F. and Scholes, M. (1973). 'The pricing of options and corporate liabilities.' *Journal of Political Economy*, vol. 81, pp. 637–59.

Bland, D. R. (1988). *Wave Theory and Applications*. Oxford: Oxford University Press.

Cechetti, S. G., Lam, P. and Mark, N. C. (1990). 'Mean reversion in equilibrium asset prices.' *American Economic Review*, vol. 80, pp. 398–418.

Cox, J. C. and Leland, H. E. (1982). 'On dynamic investment strategies', Proceedings of the Seminar on the Analysis of Security Prices, Center for Research in Security Prices, University of Chicago.

Dybvig, P. H. (1988*a*). 'Inefficient dynamic portfolio strategies, or how to throw away a million dollars.' *Review of Financial Studies*, vol. 1, pp. 67–88.

Dybvig, P. H. (1988*b*). 'Distributional analysis of portfolio choice.' *Journal of Business*, vol. 61, pp. 369–93.

Fama, E. F. and French, K. R. (1988). 'Dividend yields and expected stock returns.' *Journal of Financial Economics*, vol. 22, pp. 3–35.

Harrison, J. M. and Kreps, D. M. (1979). 'Martingales and arbitrage in multiperiod securities markets.' *Journal of Economic Theory*, vol. 20, pp. 381–408.

He, H. and Leland, H. (1991). 'Equilibrium asset price processes.' Finance Working Paper no. 221, Haas School of Business, University of California at Berkeley, December.

Huang, C-F. and Litzenberger, R. H. (1988). *Foundations for Financial Economics*, Amsterdam: North-Holland.

Jarrow, R. A. and Rudd, A. (1983). *Option Pricing*, Homewood, Illinois: Dow Jones-Irwin.

Klevorkian, J. (1990). *Partial Differential Equations: Analytical Solution Techniques*, Belmont, California: Brooks/Cole.

Kreps, D. M. (1982). 'Multiperiod securities and the efficient allocation of risk: a comment on the Black–Scholes option pricing model.' In *The Economics of Information and Uncertainty*, (ed. J. J. McCall), Chicago: University of Chicago Press.

Lo, A. W. and MacKinlay, A. C. (1988). 'Stock prices do not follow random walks: evidence from a simple specification test.' *The Review of Financial Studies*, vol. 1, pp. 41–66.

Merton, R. C. (1971). 'Optimum consumption and portfolio rules in a continuous model.' *Journal of Economic Theory*, vol. 3, pp. 373–413.

Merton, R. C. (1973). 'The theory of rational option pricing.' *Bell Journal of Economics and Management Science*, vol. 4, pp. 141–81.

Oksendal, B. (1989). *Stochastic Differential Equations*, 2nd Edition, Berlin, Heidelberg: Springer-Verlag.

Poterba, J. M. and Summers, L. H. (1988). 'Mean reversion in stock prices: evidence and implications.' *Journal of Financial Economics*, vol. 22, pp. 27–59.

Whitham, G. B. (1974). *Linear and Nonlinear Waves*. New York: Wiley.

Appendix: continuous time derivation

Again we shall work in terms of the transformed variable x_t. While x_t is a Martingale under Q, it follows

$$dx_t = \alpha(x, t)\,\sigma\,dt + \sigma\,dB \tag{32}$$

under the objective measure P.

The Girsanov formula (see, for example, Oksendal (1989), p. 102) tells us that the change of measure is characterised by the Radon–Nikodym derivative (or state-price density function)

$$\left.\begin{aligned} \frac{dQ}{dP} &= \exp(-Z_t), \quad \text{where} \\ Z_t &= \int_0^t [\tfrac{1}{2}\alpha^2(x,\tau)\,d\tau + \alpha(x,\tau)\,dB_\tau]. \end{aligned}\right\} \tag{33}$$

The path independence property which holds within our economy requires Z_t to be a single valued function of x and t and it is not allowed to depend on the path followed to a particular (x, t) combination. Substituting dx into the integrand of this equation in place of dB, and equating to the expression for dZ from Itô's Lemma we find:

$$\alpha\frac{dx}{\sigma} - \tfrac{1}{2}\alpha^2\,dt = Z_t\,dt + Z_x\,dx + \tfrac{1}{2}\sigma^2 Z_{xx}\,dt. \tag{34}$$

We can therefore identify

$$Z_x = \alpha/\sigma, \quad \text{and} \quad Z_t + \tfrac{1}{2}\sigma^2 Z_{xx} = -\tfrac{1}{2}\alpha^2. \tag{35}$$

We need to manipulate these two equations to eliminate Z and provide instead a partial differential equation for α.

Differentiating Z_x again by x we get:

$$Z_{xx} = \alpha_x/\sigma, \quad \text{and hence} \quad Z_t = -\tfrac{1}{2}\sigma\alpha_x - \tfrac{1}{2}\alpha^2. \tag{36}$$

Finally, we can differentiate Z_t by x and Z_x by t and then eliminate Z_{tx} from each equation

$$Z_{tx} = -\tfrac{1}{2}\sigma\alpha_{xx} - \alpha\alpha_x = \alpha_t/\sigma = Z_{xt}. \tag{37}$$

This again gives us a complete derivation for Burgers' equation:

$$\alpha_t = -\tfrac{1}{2}\sigma^2\alpha_{xx} - \sigma\alpha\alpha_x. \tag{38}$$