Recent Developments in Derivative Securities

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Abstract

The paper reviews recent developments in modelling derivative securities. It describes new models for valuing interest rate derivatives based on 'whole term structure models' which are consistent with an arbitrary initial term structure. It also discusses the tension between this approach and other recent studies of the relationship between the shape of the yield curve and internal rate volatility. The issue of stochastic volatility in pricing and hedging other kinds of options is also examined in its own right.

The usual Black-Scholes replication argument requires frictionless markets. New results for valuing options when transactions costs affect trading in the underlying asset are reviewed. The fourth topic examined concerns the role of general equilibrium analysis in the field of contingent claims. Brief mention is made of some other topics in the options literature. The paper concludes with the thought that work on options has focused major new resources at the problems of understanding the nature of the stochastic processes for underlying security market assets.

Recent Developments in Derivative Securities: 20 years on from Black and Scholes.

Introduction

It is now twenty years since Black and Scholes (1972) presented an empirical test of their option pricing formula within the context of determining the efficiency of the pre-CBOE options market. In that paper they state "we hope to establish the empirical validity of our model as a forerunner to more complicated models of other option forms". As every reader will know, the degree of their success was almost unprecedented in the field of finance. There is no need for us to recount the speed and variety of much subsequent work: the explosion of theoretical work including the binomial model, leading to general notions of risk neutral valuation alongside the innovation of new financial products (financial engineering) featuring derivatives on interest rates and FX as well as on equities[1]. In this paper we survey what we regard as the most striking developments in this area over the last twelve months or so[2].

Significantly, most of the topics we see as of current research importance were flagged within the early Black-Scholes paper. The relaxation of the important assumptions of a constant term-structure of interest rates and constant instantaneous volatility of returns has led to important areas not explicitly addressed during the 1970's and not really considered until the last five years. These developments have been motivated both by the implications of empirical studies, and by the growing variety of risk-management instruments, initiated in part by the swaps market. This relaxation of the original Black-Scholes assumptions has been continued with regard to transactions costs, and has proved to be an area which has brought financial economists in closer contact with stochastic optimal control theorists, whose tools of stochastic and semi-martingale calculus had been used by financial economists since the original contributions of Black & Scholes (1973), Merton (1973) etc. Finally, from a period in which valuation formulae were developed from strong assumptions of the (exogenously defined) stochastic processes for the underlying asset prices, we are now seeing much more emphasis on the idea of 'consistency with equilibrium' as an additional constraint to the usual the 'no-arbitrage' one.

The focus of our survey is on the areas we have just described: we believe these to have been the most exciting (and that they will continue to be so) for the foreseeable future.

The paper is organised as follows:
In the next section we review developments in models for interest rate derivatives. This includes the role of volatility in the term structure, and the second section reviews other recent work on stochastic volatility in option pricing. The third section describes work on option valuation when there are transactions costs involved in trading the underlying asset. The fourth section describes equilibrium issues related to option pricing. Finally, we comment briefly on other important topics which lack of space prevents us from covering in a more detailed way.

Interest Rate Derivatives

In this section we survey alternative models for pricing interest rate derivative securities and also look at the relationship between the volatility of interest rates and the shape of the term structure.

Early Models

During the late 1970's and the early 1980's models for interest rate derivatives were based on models developed to explain the term structure of interest rates. The earliest models (eg. Vasicek (1977)) started from an assumed process for the short term interest rate, and an assumed form for the price of interest rate risk and used the now conventional no-arbitrage condition to solve for the form and process of the entire term structure. Cox, Ingersoll and Ross (CIR) (1985) pointed out the inherent dangers in closing such models by assuming a specific functional form for the risk premium, and developed their term structure model from an intertemporal general equilibrium asset pricing model. Both models can be characterised by their assumptions about the short term interest rate, which is assumed to be the single source of uncertainty:

$$dr(t) = \kappa\left(\theta - r(t)\right)dt + \sigma\left(r\right)dz \qquad (1)$$

where r(t) is the short rate, at time t, and dz is a Wiener process. The instantaneous drift represents the process as mean reverting. In the Vasicek paper the short rate is assumed to follow the Ornstein-Uhlenbeck diffusion process with the volatility of the process σ(r) set equal to a constant σ. In the CIR paper the volatility of the short rate increases with the square root of the rate itself.

Although this approach to pricing interest rate derivatives has the important advantage that all derivatives are valued on a common basis (ie. it is a whole term structure approach) it has the severe disadvantage that the term structures provide a limited family which do not correctly price many traded bonds.

By valuing interest rate derivatives with reference to a theoretical yield curve rather than the actually observed curve, equilibrium models produce contingent claims prices that disregard key market information affecting the valuation of any interest rate derivative security. The most obvious market data that could be used to price interest rate derivatives is the term structure of interest rates and the term structure of interest rate volatilities. Many models currently appearing in the literature are what we term "whole term structure models", and an underlying motivation for the models has been to provide a unifying framework; unifying in both the sense of using as much market data as possible from which to build the models, and in trying to price and hedge the whole range of interest rate dependent securities.

Models Which Fit an Initial Term Structure

Ho and Lee (1986) were the first authors to build a model that set out to model the dynamics of the entire term structure in a way that was automatically consistent with the initial (observed) term structure of interest rates. In much the same way that the Black-Scholes model, for pricing options on stocks, can be inverted to obtain the implied volatility of stock prices consistent with the option price, Ho and Lee reasoned that the same principle could be applied to the pricing of bonds. The Ho and Lee model is developed in the form of a binomial tree relating future movements of the yield curve explicitly to it's initial state. The authors estimate the parameters of the discrete time binomial process including the risk neutral probability. Unfortunately for large step sizes the parameters are not independent, making estimation problematic (see Heath, Jarrow and Morton(1990b)).

Although the authors do not discuss the issue of convergence, a number of other authors (eg Dybvig (1988c) and Jamshidian (1988)) show that the continuous time limit can be characterized by the short rate process:

$$dr(t) = \theta(t)dt + \sigma dz \tag{2}$$

where $\theta(t)$, the drift during the short time interval *dt*, is a function of time in order to make the model consistent with the initial term structure of interest rates. This model attracted a great deal of attention: practitioners in particular liked being able to fit any term structure exactly, and liked the binomial form of the model. Another advantage is that the valuations are preference independent. However, there were some severe drawbacks too. As can be seen from equation (2) the model describes the whole volatility structure by a single parameter σ, implying that spot rates and forward rates that differ in their maturity are all equally variable, all future spot rates are normally distributed and all possible yield curves at a future time are parallel to each other. A further difficulty of the model is that it incorporates no mean reversion, and as a result there is a high probability that future interest rates will become negative. Subsequent work in this area has therefore aimed to keep the advantages of the Ho and Lee approach while eliminating these kinds of problems[3].

Black, Derman and Toy (1990) develop a model to match the observed term structure of spot interest rate volatilities as well as the term structure of interest rates, and which is currently popular. As with Ho and Lee, the model developed describes the evolution of the entire term structure in a discrete-time binomial lattice framework. The continuous time limit of the model is given by the stochastic differential equation:

$$d\log r(t) = \left[\theta(t) - \frac{\sigma'(t)}{\sigma(t)}\log r(t)\right]dt + \sigma(t)dz \tag{3}$$

This model incorporates two functions of time; the first, $\theta(t)$, is chosen so that the model fits the term structure of spot interest rates, and the second, $\sigma(t)$, so that it fits the term structure of spot rate volatilities. In this model changes in the short rate are lognormally distributed, with the advantage that interest rates cannot become negative. Once $\theta(t)$ and $\sigma(t)$ are chosen the future short rate volatility is entirely determined, and an unfortunate consequence of the model is that for certain specifications of the volatility function $\sigma(t)$, the short rate can be mean-fleeing rather than mean-reverting. Unfortunately neither analytic solutions for the prices of bonds or the prices of bond options are available.

Heath, Jarrow and Morton (HJM) (1990a), (1990b), and (1992) seek to construct a family of continuous time stochastic processes for the term structure, consistent with the observed initial term structure data. In order to model the dynamics of the term structure we can chose between equivalent formulations in terms of bond prices, short term interest rates or forward rates. HJM chose to model forward instantaneous interest rates, due to volatility considerations concerned with the maturity of zero coupon bonds, and so take as given the initial forward rate curve.

The forward rate curve's dynamics are exogeneously given by the equation:

$$f(t,T) = f(0,T) + \int_0^t \alpha(v,T,\omega)dv + \sum_{i=1}^{n} \int_0^t \sigma_i(v,T,\omega)dW_i(v) \qquad (4)$$

where $f(0,T)$ is a fixed initial forward rate curve,

$\alpha(v,T,\omega)$ is the instantaneous forward rate's drift, with ω signifying an arbitrary path of the evolution of interest rates, σ_i are the volatilities of the forward rates, and W_i denotes the ith Weiner process (for $i = 1, ..., n$).

Equation (6) is expressed in its most general form, with n independent Brownian motions determining the stochastic fluctuation of the forward rate curve. HJM (1992) show that prices of pure discount bonds satisfy a stochastic differential equation that states that the instantaneous return on the T-th maturity bond has a drift rate equal to the spot rate plus a term premium which is a function of the forward rate's drift and volatility, and volatilities which are also a function of the forward rate volatilities. It can be shown that the HJM model is consistent with bond prices converging to par at maturity, and the behaviour of the instantaneous rate. By applying the insights of Harrison and Kreps (1979) the process is shown to be arbitrage free, and contingent claim values are obtained via an application of Harrison and Pliska (1981).

The HJM paper highlights two examples which aid the readers understanding of the model's practical applications. The first is the, single factor, continuous time limit of HL, which leads to a modified Black-Scholes formula for the value of a European bond option, with the second example involving two sources of randomness also leading to a closed form solution.

Motivated by practical considerations, and the desire to value all interest rate contingent claims on a consistent basis, whilst retaining the ability for the model to provide a perfect fit to observed market data, Hull and White (HW) (1990) derive two one state variable models for the short rate. HW seek to reconcile the tractability of the Vasicek and CIR models with the consistency of a model that fits the initial yield curve. Their proposed models can be seen, and are presented as, extensions to the Vasicek and Cox-Ingersoll-Ross models due to the similarity of the nature of the short rate processes to those presented in the original papers:

$$dr = \left[\theta(t) - \phi(t)r\right]dt + \sigma(t)dz \tag{5}$$

$$dr = \left[\theta(t) - \phi(t)r\right]dt + \sigma(t)\sqrt{r}dz \tag{6}$$

Hull and White propose that the three functions of time $\theta(t)$, $\phi(t)$ and $\sigma(t)$ are chosen so that the models, determined by equations (5) and (6), fit the initial term structure of interest rates, the term structure of spot rate volatilities, and the anticipated variability across time of the instantaneous spot rate. The first model leads to normally distributed interest rates and lognormally distributed bond prices with the resulting disadvantage that interest rates can become negative. The model leads to analytical solutions for European option prices. The second model eliminates the possibility of negative interest rates but is not as analytically tractable as the first. Jamshidian (1990) independently analysed the general Gaussian model, highlighting the important role played by the variance structure of the model.

The Variance of Rates and the Term Structure

Attractive though the class of models just described may be, they are not without their dangers. At any date we must estimate functions for the term structure of interest rates, the term structure of volatility and the time path of volatility. However, just as with implied volatility in a conventional options model, when we look at market prices at a later date, there is no guarantee that they will be consistent with the previously estimated functions. In particular, choosing volatility functions to some extent independently of fitting to the term structure of interest rates poses the danger that we may ignore the information that the shape of the term structure contains about the anticipated volatility of interest rates. Convexity considerations or other more formal analysis lead very easily to relationships between the concavity of spot rates with respect to maturity and the prospective volatility of interest rates.

The relationship between the shape of the yield curve and interest rate volatility has been studied empirically by Brown and Dybvig (BD) (1986), and by Litterman, Scheinkman and Weiss (LSW) (1991), and theoretically by Brown and Schaefer (BS) (1991).

BD's principle motivation is to test the single factor CIR model of the term structure. They discovered that an implied volatility of the short term rate could be identified from the fitted term structure at any date, and they demonstrated that this seemed to have some predictive power, even though the single factor model is mis-specified. LSW find that a measure of volatility obtained from the yield curve, as a linear function of pure discount bond yields, explains nearly 70% of the variation in the volatility implied from the prices of options on Treasury bond futures. Prompted by this seemingly robust feature of the relationship between the shape of the term structure and the level of interest rate volatility, BS develop a relatively simple relation between the forward interest rate curve, the level of interest rate volatility and the level of bond price volatility, or duration, for a general class of equilibrium type models which they refer to as the 'affine yield class'. This class includes both the Vasicek and CIR models.

Longstaff and Schwartz (1991) develop a remarkable two-factor model of the term structure of interest rates drawing upon the same general equilibrium framework as CIR (1985). The choice of the model's two factors, the short rate and the volatility of the short rate - both of which are observable and intuitively appealing - allow contingent claim prices to reflect both the current level of interest rates and the current level of interest rate volatility. The authors obtain closed-form solutions for the prices of pure discount bonds and options on pure discount bonds in this two-factor setting, (a strong attraction of the model), but for other more complex applications numerical solutions are required. The two factors allow a richer variety of possible yield curves than the one factor equilibrium models discussed earlier, but generally will be inconsistent with an observed initial curve. Although a model based on the two factors of the short term interest rate and its instantaneous volatility is a most exciting step forward, their joint processes were necessarily chosen for their analytical tractability rather than their empirical realism. Substantial problems still remain and we can hope for further work in this area.

Stochastic Volatility

Motivation

Ever since the early empirical work of Black and Scholes (1972) it has been evident that volatility is a central problem in option pricing. That early study found that historical measures of volatility gave rise to an errors in variables problem. By now there is massive evidence that for most asset processes we have distributions with fat tails, due in part at least to the volatility changing through time. Option prices, too, exhibit related effects of implied volatilities which vary through time, and non-linear exercise price ('smile') effects, of higher implied volatilities for in- or out-of-the-money options. (See, for example, Shastri and Wethyavivorn (1987)).

This has provided a clear motivation for the development of option pricing models for asset processes where the volatility itself is stochastic. Since options effectively enable the volatility to be traded it is also unsurprising that we have simultaneously seen considerable new work on modelling and forecasting asset volatility. ARCH models, especially their generalisations to GARCH and EGARCH have become standard tools of analysis in this field. We refer the reader to the special journal issues edited by Campbell and Melino (1990) and by Engle and Rothschild (1992).

The natural way to develop option pricing models under stochastic volatility, is to write down diffusion equations for the joint process of the underlying asset and its volatility, and use some kind of numerical technique to integrate (or simulate) the required solutions. The early papers by Hull and White (1987), Johnson and Shanno (1987), Scott (1987) and Wiggins (1987) all make use of this, and the idea that option prices could be computed if the risk premium for the volatility could be identified. The difficulty is to find functional forms for the processes which combine a degree of realism with reasonably simple calculations for option values. There is also a tension between this approach and ARCH models for which there is no clear correspondence to diffusion equations (see Nelson (1990)).

Recent Models

This is an awkward literature to survey. There are now quite a few different approaches none of which seems as yet to dominate the others or to provide quite the ideal solution. Table 1 provides a summary of recent models we are aware of. All these models enable volatility to command a risk premium. The models differ in whether changes in volatility can be correlated with asset returns, in the computational approach and on the plausibility of the process for the volatility.

Duan (1991) provides a model for a GARCH process. He assumes HARA utility in order to obtain a risk neutral valuation relationship, and uses simulation to compute numerical values from the model. Hull and White (1988) extend their earlier work to allow the volatility to be instantaneously correlated with the security price. They assume a square root process for the variance. Madan and Seneta (1990) and Madan and Milne (1991) propose a model of a continuous-time pure jump process which provides the variance with a gamma distribution, and the resulting compound distribution enables high levels of kurtosis to be generated.

Stein and Stein (1991) assume a mean reverting arithmetic process for the volatility. They are able to derive a quasi-closed form formula for the probability distribution of the by security by using Fourier inversion of the characteristic function. Option values are computed by integrating the stock price distribution numerically. This model specification suffers two drawbacks. First, it is possible for negative volatility to occur, and second it assumes that the correlation between stock returns and volatility is zero.

Scott (1992) also uses a square root process for the variance in a model which also includes stochastic interest rates There are no restrictions on the correlation between the stock return and volatility and between the interest rates and volatility. A quasi closed form solution for prices of index options has been derived by also using the Fourier inversion formula for probability distribution functions. To value options only univariate numerical integration is needed.

Empirical Work

Recent empirical studies have tended to work with equity index options (eg. Heynen, Kemna and Vorst (1991), Hull and White (1987, 1988) and Scott (1992)) or currency options (Chesney and Scott (1989), Melino and Turnbull (1990)). The general conclusions from these studies are that equity returns and volatility are negatively correlated, that stochastic volatility models price options more accurately than the alternative simpler models. The risk premium on volatility risk is generally hard to estimate, though Melino and Turnbull found that it has significant effect on the option prices. It does not seem to be known whether these models enable more accurate hedging to be undertaken.

A New Approach

An entirely new approach to volatility hedging and pricing has recently been described by Dupire (1992). The aim is to develop continuous-time no-arbitrage pricing with stochastic volatility, without the need to specify any volatility risk premium. It does this by working with conditions for the evolution of option prices to preclude arbitrage, in a way which is entirely parallel to the Ho & Lee approach to term structure modelling. A key assumption is that a continuum of options in exercise price and time to maturity are traded and are priced consistent with no arbitrage within the continuum. All that we require in reality is that those options which are traded are priced consistent with the continuum. Since the variance can be traded it will not command a risk premium and we can assume the drift is the risk-neutral drift (which Dupire shows how to compute). Option prices can then be computed as the discounted risk-neutral expectation of the payoff.

Transactions Costs

Early Work

The construction of hedging strategies which best replicate the outcomes from options (and other contingent claims) in the presence of transactions costs is an important problem which has seen significant recent advances. Delta hedging is central to the theory of option pricing. Arbitrage valuation models, such as that of Black and Scholes (1973), depend on the idea that an option can be perfectly hedged using the underlying asset, so making it possible to create a portfolio which replicates the option exactly. Hedging is also widely used to reduce risk, and the kind of delta hedging strategies implicit in Black and Scholes are commonly applied, at least approximately, by participants in options markets. Optimal hedging strategies are therefore of direct practical interest. Much of the theory of options assumes that markets are frictionless. The analysis of delta-hedging strategies under transactions costs also provides more general insights into the valuation issues which arise where the nature of the market dictates that trading is discontinuous, or that the asset processes are such that the market is incomplete and contingent claims are not spanned by existing securities.

The first paper to consider the problem of replicating options' payoffs using delta hedging under transactions costs was Leland (1985). The issue is particularly interesting because the usual Black-Scholes strategy, implemented as rebalancings at discrete intervals, tends to an infinite quantity of expected transactions as the frequency of rebalancings is increased[4]. Leland's analysis is set in a continuous-time framework, and assumes proportional transactions costs. It describes how by making an adjustment to the variance (depending on the exogeneously specified revision frequency) the Black-Scholes formula can be used to hedge with a zero expected replication error, and with a standard deviation which tends to zero with the length of the rebalancing interval. Neuhaus (1989) contributes some further insights to this approach. However, this method is in no sense an optimal one. It is worth explaining briefly the intuition behind the variance adjustment. The seller of an option (either a call or a put), hedging risk by delta-hedging, is forced to buy stock whenever the stock price has increased, and conversely sell it when it has fallen. The spread on trading the stock makes it as if the stocks movement was greater than its actual movement, leading to a need to increase the volatility used for valuation and hedging. On the other hand, someone delta-hedging a long option position has the opposite experience and must make a downward adjustment to the volatility.

Discrete-time Formulations

Recent contributions to the transactions costs literature may conveniently be classified according to whether the analysis is set in a discrete or continuous-time framework, and whether or not it involves a utility function. Merton (1990) provides a single period working of exact replication of an option under proportional transactions costs in a binomial tree. Boyle and Vorst (1992) have extended this in an elegant way to many periods in what amounts to a reworking of Leland's analysis in a discrete binomial tree. Interestingly, they obtain a variance adjustment which differs from Leland's by a factor of

$\sqrt{\frac{2}{\pi}}$: this is essentially because although the binomial tree provides the same variance of return as the continuous process which it approximates, it distorts the expected absolute change over any small time interval.

Edirisinghe, Naik and Uppal (1991) develop replication approaches based on both linear programming and on dynamic programming. In their analysis they minimize the initial cost of obtaining a terminal payoff at least as large as that from the option being hedged. They show that allowing a cash surplus in some states of the world can significantly reduce the replication cost compared to the "exact replication" of Boyle and Vorst, particularly when transactions costs are high. Another advantage of this approach is that the model is also applicable to contingent claims whose payoff is non convex.

Portfolio Selection with Transactions Costs

The continuous-time literature on replication under transactions costs, represents the outcome of extensions to Merton's (1971) work on portfolios which maximize expected utility over an infinite horizon. Constantinides (1986) and Magill and Constantinides (1976) were the first papers to consider this problem under transactions costs, albeit in a heuristic manner. Important control theory formulations and solutions were subsequently derived in the papers by Davis (1988), Davis and Norman (1990), Eastham and Hastings (1988), Taksar, Klass and Assaf (1988), and Dumas and Luciano (1991). The papers by Davis show rigorously, that whereas in Merton's world without frictions the ratio of risky to safe assets is managed to remain constant, under proportional transactions costs it should be managed to remain between a pair of constants (which bracket the constant for the zero cost case). The pair of constants define a *no-transaction region*: transactions are only undertaken to return to the region when the portfolio strays outside it. While these papers are concerned with optimal policies, they are not directly concerned with the problems of replicating (or similarly hedging) contingent claims by means of the underlying asset.

Continuous-Time Option Valuation Under Costs

A continuous-time formulation for option valuation under transactions costs was first developed by Hodges and Neuberger (1989). The mathematical analysis has been treated in a more rigorous fashion by Davis and Panas (1991), and by Davis, Panas and Zariphopoulou (1992), and further work is also presented in Hodges and Clewlow (1992). The approach is a preference dependent one, using optimal control theory techniques to maximize the expected utility of the delta-hedged option payoff.

For numerical tractability an exponential utility function is used as this reduces the number of state variables to two. For a highly risk averse parameter value this gives very tight replication similar to the dominance criteria of Edirisinghe, Naik and Uppal, and with correspondingly widely set reservation prices for buying or selling the option. For low risk aversion the replication is much looser with reservation prices which are closer together, and tend to the Black-Scholes value for the limiting risk neutral case. The solutions exhibit properties of both Leland's variance adjustment and also the *no-transaction region* of the continuous-time portfolio selection papers. Only with a fixed cost component as well as (or instead of) a variable one would it be appropriate to jump into the interior of the control region for delta (which is now a function of both the asset price and time). Both the width and shape of the control region for delta depend on whether a short or long options position is being hedged. As in Leland's analysis, short options imply a flattening of delta as a function of the underlying and long options positions the converse. The technique can handle portfolios of options without any requirement for monotonicity in the payoff profile.

General Equilibrium and Contingent Claims

A major strength of the option theory initiated by Black and Scholes, and by Merton, is that because it is a relative pricing theory, when dynamic spanning can be implemented, then the value of a derivative security can be obtained on the basis of 'no-arbitrage', and, therefore, a full equilibrium analysis is unnecessary.

The full power and elegance of this approach was developed by Cox and Ross (1976), Harrison and Kreps (1979), and Harrison and Pliska (1981). A key feature of these papers is that the securities markets are what we now term 'dynamically complete' (see, e.g. Duffie and Huang (1985). Further, they all assume that prices are consistent with equilibrium. Indeed, Harrison and Kreps introduce the term "viable" to refer to this consistency and undertake a considerable analysis focusing on the properties of viability. It should, therefore, not be surprising that there is a significant and important linkage between derivative securities pricing and general equilibrium analysis. However, until the last few years, the explicit consistency and role of an equilibrium has been relegated in importance because of the power of the 'no-arbitrage' approach to pricing.

The 'no-arbitrage' approach to securities pricing relies heavily upon the "consistency with equilibrium" assumption. The pitfalls in this area were explicitly discussed by Cox, Ingersoll and Ross (1985), as we have already mentioned. However, until quite recently very little explicit work had been undertaken which addressed the question "can a general equilibrium model be constructed in which a given stochastic behaviour of prices will result from more primitive assumptions on agents' preferences, endowments, information structures, and beliefs?" (Bick (1987), p. 259).

The importance of understanding the exact nature of equilibrium depends critically on the context. The pricing of an option via the Harrison and Pliska paradigm, will give the correct value and hedging information for that derivative security. It is not usually necessary to understand the nature of equilibrium risk premia in order to price and hedge options, or to minimize risks. One the other hand, information about equilibrium expected returns is essential for efficient investment decisions. The use of derivative securities for implementing dynamic asset allocation schemes of a variety of types is becoming a day-to-day activity for many fund managers in the United States and elsewhere.

Investment decisions clearly depend simultaneously both on preferences and on risks and expected rates of return. Recent work has highlighted the simplifications which occur when markets are dynamically complete. Dybvig (1988a) has shown how the opportunity to synthesize Arrow-Debreu pure securities (using options or dynamic replication) enables multiperiod problems to be reduced to single period ones, and means that with the assumption of state-independent utility maximization (or something slightly weaker) a pricing model for probability distributions is obtained. Under such assumptions there are also strong implications for portfolio management. Dybvig (1988b) shows that a necessary condition for efficient portfolio management is that wealth at future dates should be monotonic decreasing in the state-price density function, and he provides a quantification of the losses from path dependent strategies[5] which violate this condition.

The characterization of a market equilibrium is a topic of continuing fundamental importance in financial economics. Papers by Bick (1987), Bick (1990), He and Leland (1991), Hodges and Carverhill (1992), and Stapleton and Subrahmanyam (1990) advance our knowledge of the nature of general equilibria which are consistent with Black-Scholes option valuation.

The issue is motivated by three aspects. First, in option pricing the dynamics for the underlying asset is generally taken as a "primitive assumption", rather than being derived endogenously within the model. Second, because so much of option pricing has essentially concerned itself with relative valuation (with information concerning tastes and preferences embedded within the price of the underlying assets) it was not possible to identify assumptions that would give rise to a general diffusion process. Third, the dynamics of the market portfolio and the behaviour of its risk premium are of particular interest.

Bick (1987) and Stapleton and Subrahmanyam (1990) derive necessary conditions for the process of a market index asset to describe a Geometric Brownian Motion and show (perhaps unsurprisingly) that this involves a representative agent with constant proportional risk aversion (CPRA) utility. Bick (1990) considers the relationship between more general diffusion processes and the possible utility functions of the representative agent. He provides both necessary and sufficient conditions for viability and demonstrates that not only is the set of viable diffusions very restricted, but quite common stochastic processes such as the Ornstein-Uhlenbeck (that is used in the Vasicek term-structure of interest rates model) are not consistent with equilibrium if applied to the market portfolio. This line of inquiry has been extended in the work of Hodges and Carverhill (1992) and He and Leland (1991) which derives necessary conditions for the market risk premium in general equilibrium economies which support Black-Scholes option pricing.

This work begins to shed new light on the growing literature which suggests that equity market returns are to some extent predictable (see, for example, Poterba and Summers (1988) and Fama and French (1988)). Apparent observed mean reversion could be explained by consistently time varying equilibrium risk premia, and in order to test whether this applies to the markets that we observe, we must first be able to characterize what kinds of variation in risk premia are consistent with a general equilibrium model.

Other Areas of Research

Before concluding, we shall briefly comment on some of the many topics which we have been unable to discuss in detail. There are various reasons for the omissions.

First, some topics such as the valuation of American style and exotic options (such as average-rate or look-backs) are, in our opinion, a continuation of the original Black-Scholes/Merton work. Also, for these in particular, there are already excellent up-to-date surveys by Myneni (1992) and Rubinstein (1991) respectively. Further, much of the work represents the presentation of standard approaches in various areas in applied mathematics, such as variational inequalities, in order to give alternative characterisations of the finance problems. Although this often gives a numerical analytical approach new to the finance literature, there do not appear to be any significant financial economic insights. Frequently the 'discussions' are very mathematical. Indeed, may of the articles are being written by mathematicians/probabilists and are being published in mathematics/probability journals. In fact we express some concern over this direction as frequently this work increases the divide between academia and the practitioners.

Second, the focus of this article is on traded instruments. We therefore feel that this is not the appropriate place to discuss the growing interest in the value of real options. This is an area which has received an impetus over the last two or three years due to the interest of mainstream economists such as Pindyck (1991) and Dixit (1992), and includes a recent interesting (and important) paper by Ingersoll and Ross (1992) on project valuation with interest rate uncertainty within a Cox, Ingersoll and Ross framework. One key aspect of much of the work in this area is to recognise the value of the embedded options, particularly with respect to the option to invest now or delay.

There still seems to be a dearth of good and innovative empirical work in the area. Econometric analysis in the field of derivative securities is particularly difficult because of the joint hypotheses relating to the validity of the derivative pricing model and the derivative and underlying asset price processes. The other vital point, which has also led to some strange results, is the need for simultaneous observation of the underlying and derivative prices. Harvey and Whaley (1991, 1992) point out that this can easily lead to inconsistent results. However, it is also encouraging to see papers such as Longstaff (1990) and Bates (1991) which develop theoretical models designed to explain securities prices and then have tested them in a traditional way. We cannot leave empirical work without commenting on the many papers based on the use of ARCH and its generalisations for examining prices and volatility. Although there are tensions between tractability for option modelling and for econometric estimation it is important that mainstream econometricians are now bringing their knowledge and experience to bear, and this augers well for the future.

Although Black and Scholes motivated their formula by corporate liability valuation, until recently there were relatively few signs of interest in this area. However, possibly motivated by developments in the swaps markets, there is renewed interest in the valuation of 'risky' corporate securities and financial products where the option to default is now being explicitly recognised. Cooper and Mello (1991) is a good example. The original Black and Scholes (1973) paper held out high hopes in this sort of area: "Since almost all corporate liabilities can be viewed as combinations of options, the formula and the analysis that led to it are also applicable to corporate liabilities such as common stock, corporate bonds, and warrants". The hopes of pricing credit risk in an options framework have not really been fulfilled. The option theoretic approach generally predicts significantly smaller credit premia than actually observed. Authors such as Ramaswamy and Sundaresan (1986), Babbs (1991), Jarrow and Turnbull (1990) etc have to assume some premium expected return on risky debt securities emanating from an unstated imperfection in markets. (ie. seems to admit arbitrage). While a number of plausible reasons for such imperfections can be stated, they should be encompassed within a more complete theory.

Finally, we have not commented on contract design nor market micro-structure. Although they are key areas for practitioners, and have received some academic attention we feel that the problems are so difficult that the insights from existing analyses are very limited. For example, we know very little about what makes some contracts more liquid than others, or the affect of different market structures (eg. different forms of screen-based trading or open-outcry). We think over the next five years both of these areas will develop and believe that the embedded options for market makers will have to be analysed.

Conclusions

The paper has reviewed recent developments in modelling derivative securities. In particular we have focussed on developments in:

- 'whole term structure models' for valuing interest rate derivatives,
- models of stochastic volatility in pricing and hedging other kinds of options,
- valuing options when transactions costs affect trading in the underlying asset, and
- the role of general equilibrium analysis in the field of contingent claims.

Brief mention is made of some other topics in the options literature.

It is noteworthy that work on options has focused major new resources at the problems of understanding the nature of the stochastic processes for underlying security market assets, both through the key focus on volatility and asset distributions, and through the new insights contingent claims theory provides into market equilibrium. In this sense we believe that the derivative securities area has, within the last year or two, come full circle in feeding back into those parts of the finance literature from which it came.

Table 1

Summary of Stochastic Volatility Models

AUTHORS	CORRELATION	SOLUTION TECHNIQUE
Duan (1990)	n/a	Monte Carlo simulation
Hull and White (1988)	Non-zero	Series solution
Madan and Senata (1990)	Zero	Approximate formula or Single integration.
Scott (1992)	Non-zero	Single integration
Stein and Stein (1991)	Zero	Double integration

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Footnotes

1 The definition of a derivative security does not seem to be generally agreed. Merton (1990) page 10 and page 429 defines derivative securities to be 'securities with contractual payoff structures that are contingent on the prices of one or more traded securities'. In this article we shall be less specific and interchangeably use contingent claim contracts with derivative securities. One reason for this is that most, if not all, of the technology that we use for pricing a security, is actually independent of whether the underlying source of value is a traded asset. Of course, when the underlying is a traded asset, we are able to obtain the (desired) required prices without making explicit assumptions about tastes and preferences. However, a very large part of today's options markets, for example interest related securities, do not meet the requirement that the underlying be tradable. Practitioners, however, refer to interest-rate based options as interest-rate derivatives. It seems preferable to use a language which will be understood explicitly by both academics and practitioners.

2 This is now an extremely large field which continue to grow rapidly. Our selection is necessarily a subjective one which in part reflects our own research priorities.

3 Regretfully, for the purposes of this survey, we have no space to comment on the analyses of Black and Karasinski (1991), Jamshidian (1989, 1991a, 1991b), and Turnbull and Milne (1991).

4 It is well known that with discrete rebalancings at equal time intervals, the expected transactions turnover is proportional to the square root of the number of rebalancings.

5 Cox and Leland (1982) seem to have the first to comment on the relationship between path independence and portfolio efficiency.