A Review of Option Pricing with Stochastic Volatility

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ABSTRACT

In this paper we consider the problem of option pricing when the volatility is changing randomly. We review some of the major advances in option pricing with stochastic volatility which have appeared in the literature. We discuss their advantages and disadvantages from both a theoretical and practical view point. In particular we focus on the key issues of how the models deal with the risk premium on volatility, hedging the volatility risk and the correlation between the asset and its volatility. We consider realistic and tractable choices for diffusion processes driving the volatility. Finally, we attempt to reach some conclusions regarding the most promising and useful approaches.

1 Introduction

The no-arbitrage, risk neutral option valuation approach introduced by Black and Scholes (1973) and Merton (1973) provides the theoretical basis for todays derivative markets. However, none of the assumptions made by Black and Scholes, namely,

- costless trading takes place in continuous time
- the short term interest rate is constant
- the underlying security pays no dividends
- the underlying security follows geometric Brownian motion with constant volatility

hold in real markets. Some of these assumptions have been relaxed by later authors. Merton (1973) showed that an option can be priced in terms of a bond price so relaxing the assumption of a constant short term interest rate. The no dividends assumption was dealt with by Merton (1973), Roll (1977) and Geske (1978). Merton (1973) also showed that volatility which was a deterministic function of time could be handled.

The assumption of geometric Brownian motion for the underlying security implies that it is lognormally distributed or that returns are normally distributed.

Since daily price changes are the sum of many intra day price changes it may be supposed that the central limit theorem would imply they are normally distributed. However, the conditions for the central limit theorem almost certainly do not hold intra day. For example the number of intra day price changes is probably itself random leading to a subordinated process.

The consensus from empirical investigations (Mandelbrot (1963), Fama (1965)) is that the distributions of stock prices are fat-tailed and skewed. However mixtures of normals with different variances appear to fit the data better than stationary distributions with infinite variances for example.

This suggests the need for an option pricing theory in which the volatility itself follows a stochastic process. Recently there has been a great deal of work in this area. The papers Hull and White (1987), Johnson and Shanno (1987), Scott (1987) and Wiggins (1987) established that option prices could be computed if the risk premium for the volatility could be identified. This tells us what options prices should be in equilibrium in an economy in which investors have certain preferences. However it does not indicate how arbitrage profits can be locked in if market prices deviate from the model prices. In this paper we review the development since 1987 of the theory of option pricing when the volatility is changing randomly.

The paper is organised as follows: In section 2 we review the continuous time framework within which the option pricing equation under stochastic volatility can be derived. Section 3 discusses some of the theoretical and empirical properties of stochastic volatility. The distributional approach of McDonald and Bookstaber (1988) is discussed in section 4. The alternative source of randomness for the underlying asset proposed by Madan and Senata (1990) is considered in section 5. In section 6 we consider possible diffusion processes for the volatility. Models based on volatility as a diffusion process are considered in sections 7, 8 and 9. Finally section 10 concludes the paper.

2 The Continuous Time Framework

The basic framework for developing a continuous time model with any number of state variables has been established by Garman (1976) and Cox, Ingersoll and Ross (1985). Consider a world where the n state variables S_1, ..., S_n follow the random processes

$$dS_i = \mu_i dt + \sigma_i dz_i \tag{1}$$

where μ_i is the drift term, σ_i is the volatility of S_i and dz_i is a Wiener process. In general μ_i and σ_i can be functions of all the state variables and time. The correlation between dz_i and dz_j is $\rho_{i,j}$ and $\sigma_{i,j}$ is the instantaneous covariance, $\rho_{i,j}\sigma_i\sigma_j$ between S_i and S_j.

Now suppose we have a contingent claim C which depends on the state variables S_1, ..., S_n and time t. Then, by Ito's Lemma $C(S,t)$ follows the process

$$dC = (C_t + \sum_{i=1}^{n} \mu_i C_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{i,j} C_{i,j})dt + \sum_{i=1}^{n} \sigma_i C_i dz_i \tag{2}$$

where subscripts on C denote partial differentials with respect to time (t) and the state variables (i,j).

Consider that the position in the claim is financed through borrowing its value C at the instantaneous riskless rate r. The dynamic of the value associated with this leveraged position, C^L, is given by

$$dC^L = (C_t + \sum_{i=1}^{n} \mu_i C_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{i,j} C_{i,j} - rC)dt + \sum_{i=1}^{n} \sigma_i C_i dz_i \tag{3}$$

In the special case where the S_i are traded assets which pay a continuous dividend at rate α_i, the dynamic of the value associated with a similar, zero-net wealth, leveraged position is given by

$$dS_i^L = (\mu_i - rS_i + \alpha_i S_i)dt + \sigma_i dz_i \tag{4}$$

Now if we form a portfolio P such that

$$dP = dC^L - \sum_{i=1}^{n} C_i dS_i^L \tag{5}$$

we obtain the dynamic of a portfolio with zero investment and zero risk

$$dP = (C_t - rC + \sum_{i=1}^{n}(r - \alpha_i)S_i C_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{i,j}C_{i,j})dt \tag{6}$$

By economic considerations this portfolio must have zero rate of return. Thus we obtain the general pricing equation for contingent claims

$$C_t + \sum_{i=1}^{n}(r - \alpha_i)S_i C_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{i,j}C_{i,j} = rC \tag{7}$$

Note that the Black-Scholes equation is just a special case of equation (7) with $n = 1$. If some of the state variables S_i are not traded securities then equation (7) can still be applied but with two reservations. Firstly, it is necessary to establish a value for the dividend yield α_i at which the state variable would be accepted as a traded security. This is called its "convenience yield". As we shall see below, some researchers have equivalently worked with the "price of risk" associated with a given untraded state variable. Second, we note that where some state variables are not traded, we can only derive equilibrium prices for contingent claims and riskless arbitrage is not possible if market prices are different from the equilibrium prices.

The general pricing equation (7) may be interpreted as the valuation equation in a risk-neutral setting. If investors were risk neutral then the drift on S_i would have to be $r - \alpha_i$ for S_i to trade in equilibrium. We can interpret the difference between S_i's objective rate of drift μ_i and the "risk-neutral" drift as a risk premium as follows

$$\lambda_i \sigma_i = \mu_i - (r - \alpha_i) \tag{8}$$

We are interested in a world in which the first state variable is the underlying asset and the second is the volatility or variance of that asset. Thus equations (1) become

$$dS = \mu S dt + \sigma S dz_1 \tag{9}$$

$$d\sigma = \mu_2 dt + \sigma_2 dz_2$$

where μ_2 and σ_2 will in general be functions of σ.

Scott (1987) shows how it is possible to replicate an option in a stochastic volatility world using the underlying asset and another option of a different time to maturity. However, although one obtains a replication strategy, the pricing equation obtained does not have a unique solution because an option is not a known function of volatility. Furthermore, the proportions of the underlying security and the second option are functions of the partial derivatives of the option with respect to the security and its volatility so a solution to the GPE is still necessary to make the replication strategy practical.

Recently Dupire (1992) has described a new approach to pricing and hedging volatility. The aim being to develop continuous-time no- arbitrage pricing with stochastic volatility without the need to specify a volatility risk premium. He does this by specifying conditions for the evolution of prices which preclude arbitrage. This parallels recent developments in the pricing of interest rate term structure contingent claims. The prices of European options are taken as traded assets which are priced consistent with no-arbitrage within the continuum of exercise prices and time to maturity. Dupire then shows that the forward variance can be synthesised and priced in terms of the natural logarithm of the forward prices of the underlying security. Since the variance can be priced and synthesised as a traded asset Dupire shows how the risk-neutral process for the instantaneous variance can be derived. It is this which ensures compatibility with the initial volatility term structure. Prices of more complex claims contingent on the underlying asset and volatility can then be computed as the discounted risk neutral expectation of their payoff. This is a promising new approach, however it does not solve the problem of pricing standard options since their prices are taken as given.

The general pricing equation (GPE) which we obtain cannot in general be solved analytically. However, we can write down the general solution as the risk-neutral expectation of the payoff (A)

$$C = e^{-r(T-t)}E^*[A] \tag{10}$$

where t is the current time, T is the time at which the option expires and E^* is the risk-neutral expectation operator. Both analytical and approximate solutions rely on solving the GPE or the integral implicit in the expectation operator.

3 Some properties of stochastic volatility

Stein and Stein (1991) find that the option pricing bias (the difference between the option prices of the stochastic volatility model and the Black-Scholes model) is always positive becoming larger the further away from the money the option becomes. Hull and White (1987, 1988) find that the bias is negative for near the money but becomes positive and increases in agreement with Stein and Stein as the option becomes further away from the money. However the bias tends to zero as the option becomes very far away from the money for the Hull and White models. Hull and White (1988) also examine the case of non-zero correlation and find that when the correlation is negative the bias decreases as exercise price increases. The reverse is true when the correlation is positive. In most of these papers this exercise price effect is clearly non-linear (the so called smile effect) yet we are only aware of one paper (Shastri and Wethyavivorn (1987)) which explicitly mentions the non-linearity. However, it does not seem possible to account for the size of the empirically observed smile effects with any of the diffusion models we will describe using reasonable values for the parameters of the volatility process. Empirically observed smile effects are typically characterised by implied volatilities around 10% higher for away- from-the-moneyness of around 5%. The Hull and White (1988) model gives implied volatilities of only 1% higher for this level of away-from-the-moneyness. This seems to imply that empirically observed smiles are not due to stochastic volatility but other effects such as transaction costs.

As we have already noted, when the drift term is non-zero then the Black-Scholes implied volatilities will vary with time to maturity. The exact shape of this term structure will depend on the stochastic process or equivalently the relationship between the current volatility and its mean. This is confirmed by empirical evidence from currency options (Xu and Taylor (1992)). Generally the smile effect decreases with increasing time to maturity.

There are very few studies which actually examine the pricing ability of stochastic volatility models using actual options data. Wiggins (1987) developed an estimation method for the parameters in his stochastic volatility model and applied it to daily-returns data for eight stocks and two broadly based indices. He finds that the volatilities of two stock indices behave differently from the volatilities of the individual stocks in volatility persistence and the volatility of the volatility and that the correlation between stock returns and volatility is much more pronounced for the indices and the estimates are generally negative. He then uses a finite difference method to price options while the risk premium is assumed to be zero. The option prices from the stochastic volatility model are not very different with those from the Black-Scholes model in most cases, except Black-Scholes may significantly overvalue out-the- money call options relative to in-the-money call options on stock indices for long maturities. This is consistent with other studies (Hull and White (1987, 1988)) as the correlation between stock index returns and volatility is significantly negative.

Chesney and Scott (1989) first apply a stochastic volatility model to study its pricing ability for currency options on the Swiss Franc against the US dollar traded in Geneva. They also apply the method of moments to estimate the parameters. The estimated correlation coefficient between currency returns and volatility is not significantly different from zero. Although Black-Scholes performs very poorly when an historical estimate of volatility or constant volatility is used, it performs much better than the stochastic volatility model if the volatility estimate is revised every day.

Melino and Turnbull (1990) give results for spot exchange rate options written on the Canadian dollar against the US dollar. The correlation coefficient estimate is marginally significant. They tried different values of the volatility risk premium and conclude that it has significant effect on the option prices; when the risk premium is equal to -0.1 the stochastic volatility model prices options much more accurately than the Black-Scholes model, although systematic pricing errors remain.

Scott (1992) gives the results of a general equilibrium model for the S&P 500 index options. He estimates significant negative correlation between stock returns and volatility. His analysis of implied volatility suggest that the magnitude of the risk premium on volatility has a substantial effect on the behaviour of option prices and implied volatilities and that this risk premium should be negative. He concludes that the negative correlation between stock returns and volatility and the potential of a negative volatility risk premium can explain recent empirical observations of implied volatilities for stock index options.

4 A Distributional Approach

Bookstaber and McDonald (1987) and McDonald and Bookstaber (1988) advocate the generalised beta distribution of the second (GB2) kind for security prices. It includes the lognormal, log-t, log- Cauchy, Chi-squared and a wide variety of other distributions as special cases. The GB2 can be interpreted as a generalised gamma mixed by an inverse generalised gamma. The GB2 has the stationarity property of closure under multiplication, that is returns over all time periods have a GB2 distribution. McDonald and Bookstaber (1988) are able to solve the risk-neutral expectation integral to obtain an explicit option pricing formula. However, no known continuous time process leads to the GB2, so McDonald and Bookstaber are not able to identify an equivalent change of measure and have to assume that risk-neutrality prevails. The absence of a continuous time process also means that we have no model for replication of the option and thus locking in arbitrage profits. McDonald and Bookstaber motivation for advocating the GB2 is that its generality allows fitting to a wide variety of security price time series. However, we believe that the estimation of the four parameters will be very unstable. This is because very different values of the parameters can lead to similarly shaped distributions. This together with the lack of a continuous time process and an associated replication strategy limits the usefulness of this model.

5 An Alternative Source of Randomness for Asset Prices

Madan and Senata (1990) suggest that any model of security price returns should satisfy the following criterion,

(1) long tailedness relative to the normal distribution for daily returns

(2) finite moments of at least the lower powers of returns

(3) independent stationary increments, the distribution of increments belonging to the same family irrespective of the length of the period

(4) extendible to multivariate processes

They note that the symmetric stable (Mandelbrot (1963)) fails (2) and (3), the t-distribution (Praetz (1972)) fails (3) and as described above the GB2 has no known process. The model of Press (1967) of normally distributed jumps at Poisson times does satisfy all four conditions but Madan and Senata claim their model has advantages being a pure jump process. The model they choose, which they call the variance gamma model, has a unit period distribution which is normal conditional on a variance which is distributed as a gamma variate. The process is a pure jump process which can be approximated as a compound Poisson process which in the limit has infinite jump frequency and zero jump magnitude. The process can be viewed as Brownian motion evaluated at a random time change. They are able to obtain an incomplete markets equilibrium approximate option pricing formula which is similar to Black- Scholes. However a self-financing continuous trading strategy which replicates the payoff of an option at maturity cannot be constructed for a pure jump process. Thus any solution to the option pricing equation must depend on investor preferences. Also the identification of the change of measure is extremely difficult, Madan and Milne (1991) are only able to obtain an approximate solution.

6 A Diffusion Process for the Volatility

Reasonable and tractable specifications for the volatility or variance process leave us with the following choices: The drift can be constant, constant proportional or mean reverting and the volatility of the volatility can be constant or constant proportional. We believe that any reasonable specification should satisfy the following conditions: the volatility is mean reverting, this is supported by empirical evidence (e.g. Stein (1989), Merville and Pieptea (1989)) and to some extent by the extensive ARCH model literature; it respects the non-negativity of volatility; the parameters should be practical to estimate and the pricing model practical to implement. Therefore for the drift we should choose a mean-reverting term. Choice of a constant volatility of volatility leads to the possibility of negative values. The variance process is the square root process of the Cox, Ingersoll and Ross (1985) (CIR) short term interest rate model. However, if the variance follows the CIR process with mean reverting drift then although the volatility has an arithmetic stochastic term the drift term is such as to exclude the possibility of negative values (the drift term tends to positive infinity as the volatility tends to zero). Note that choice of a constant volatility term for the variance leads to a process for the volatility whose drift and volatility tend to infinity as the volatility tends to zero - a very undesirable property. A constant proportional volatility for either the volatility or the variance leads the other to also be constant proportional volatility with neither allowing negative values. Our preferred process is therefore of the form

$$dV = \alpha(\bar{V} - V)dt + \xi V^{\beta} dz_2 \qquad (11)$$

Equation (11) with $\beta = 1$ is the diffusion limit of the GARCH(1,1) model (Nelson (1990)). Duan (1991) has demonstrated that risk-neutral valuation can be applied to GARCH models of stock returns. However it requires that asset returns are conditionally lognormal. Furthermore since the distribution functions of GARCH models do not have simple analytical forms a practical option pricing formula does not seem possible. Finally this model does not yield an option replication strategy.

In the context of time series models of volatility there appears to be a paradox in that no choice of the conditional distribution seems able to fully account for the kurtosis observed in actual price series. This is not encouraging with regard to the likelihood that tractable choices of the volatility process will be able to fully account for the biases in option prices due to stochastic volatility.

7 Integrating Black-Scholes over the mean variance distribution

Hull and White (1987) make the assumption that the volatility risk premium is zero which is the case if the volatility risk is diversifiable or that volatility is uncorrelated with marginal utility of wealth. They also assume that the correlation between the Wiener processes is zero. In this situation we can assume risk-neutrality prevails and the solution to the pricing equation is the discounted risk-neutral expectation of the option payoff. Assuming the following risk- neutral processes for the underlying asset and its variance

$$dS = rSdt + \sigma S d\tilde{z} \tag{12}$$

$$d\sigma^2 = \alpha\sigma^2 dt + \xi\sigma^2 d\tilde{w} \tag{13}$$

where α and ξ are independent of S and $d\tilde{z}$ and $d\tilde{w}$ are independent Wiener processes. Then, conditional on the mean variance over the interval $[0, T]$

$$\bar{V} = \frac{1}{T}\int_0^T \sigma^2(t)dt \tag{14}$$

the distribution of $\log(S(T)/S(0))$ is normal (mean $rT - \bar{V}T/2$, variance $\bar{V}T$).

The option price is therefore the integral of the Black-Scholes price $c(\bar{V})$ over the distribution of $\bar{V}$. It does not seem possible to obtain an analytical form for the distribution of $\bar{V}$ for any realistic process for V but it is possible to write down the moments when μ (the objective drift of the variance) and ξ are constant. Therefore, if we expand $c(\bar{V})$ as a Taylor series about $\bar{\bar{V}}$ the expected value of $\bar{V}$ and integrate the individual terms in the series we obtain

$$C(S_t, \sigma_t^2) = c(\bar{\bar{V}}) + \frac{1}{2}\left.\frac{\partial^2 c}{\partial \bar{V}^2}\right|_{\bar{\bar{V}}} Var(\bar{V}) + \frac{1}{6}\left.\frac{\partial^3 c}{\partial \bar{V}^3}\right|_{\bar{\bar{V}}} Skew(\bar{V}) + \ldots \tag{15}$$

Only for sufficiently small values of $\xi^2(T-t)$ does the series converge quickly. This is because the Taylor series expansion is only valid if the variation of $\bar{V}$ about $\bar{\bar{V}}$ is small. Hull and White state that a choice of zero drift for the variance is justified since a non-zero drift would lead to markedly different implied volatilities at different maturities. However, term structures of implied volatilities are well documented (Stein (1989), Heynen, Kemna and Vorst (1992), Xu and Taylor (1992)). The problems with this model then are a non-mean-reverting volatility and no correlation between the security and its volatility.

8 A Quasi-analytical Solution

Stein and Stein (1991) assume a mean reverting arithmetic process for the volatility

$$dS = \mu S dt + \sigma S dz \tag{16}$$

$$d\sigma = \alpha(\bar{\sigma} - \sigma)dt + \xi dw \tag{17}$$

They are able to derive a quasi-closed form formula involving a numerical integration for the probability distribution of the security by using Fourier inversion of the characteristic function. Then by assuming the risk premium for the volatility is zero they can solve the risk neutral expectation solution to the GPE numerically (this involves another numerical integration). They also note that a constant risk premium can be handled by adjusting the mean reversion level. A risk premium proportional to the volatility can also be handled by adjusting the mean reversion rate (Hull and White (1988) demonstrate both these adjustments). Furthermore, from the analysis they obtain the relationship between the parameters of the stochastic volatility model and the level of kurtosis observed in stock prices.

Their model specification suffers three drawbacks. Firstly, it does not comply with nonnegativity of the volatility. Although Stein and Stein argue that for empirically reasonable parameter values the probability that the process will ever reach zero is very small and also that Stein (1989) finds no evidence of skewness in implied volatilities of S&P 100 index options. However, this process leads to a reflecting barrier at zero for the variance. This will give the variance distribution, which is the important one, very undesirable properties. Secondly, it assumes that the correlation between stock returns and volatility is zero. Finally, the option pricing formula requires a double numerical integration which is quite computationally expensive.

9 A Realistic and Practical Model

Hull and White (1988) assume the following processes

$$dS = \phi(t)Sdt + \sigma Sdz \tag{18}$$

$$dV = \eta(V)dt + \xi\sqrt{V}dw \tag{19}$$

where dz and dw are Wiener processes with correlation ρ. They can handle a risk premium which is constant or proportional to the square root of the variance by adjusting the parameters of the mean reverting term. The option price under stochastic volatility (C) is assumed to be the Black-Scholes price (c) plus a bias

$$C = c + B \tag{20}$$

B is then assumed to be a power series in ξ

$$B = f_0 + f_1\xi + f_2\xi^2 + ... \tag{21}$$

where the f_i are functions of S, V and t. They then substitute the expression for C (equation (20)) with B replaced by its power series into the GPE. By collecting terms by powers of ξ they obtain a set of differential equations for the functions f_i whose solution is lengthy but straightforward. This model has a realistic process for the variance, allows correlation between the driving Wiener processes and is based on the GPE. Hull and White show that only the first three terms in the power series for ξ are needed to obtain accurate results and since it is a purely closed form approximation it is efficient to compute. The only restriction is on the functional forms allowed for the volatility risk premium.

In a recent paper Scott (1992) essentially assumes the same processes together with stochastic interest rates. The model specification is consistent with the general equilibrium model of Cox, Ingersoll, and Ross(1985). Again there are no restrictions on the correlation between the stock return and volatility. The process for the short term interest rate is a linear combination of the variance and a second CIR stochastic process. This is not a very realistic model of the interest rate and its correlation with volatility. However, it is straightforward to assume that the short term interest rate is constant and the second stochastic factor does not exist. A quasi closed form solution for option prices can be derived by using the Fourier inversion formula for probability distribution functions. To price options only two univariate numerical integrations are needed which is quite efficient to compute.

We thus have two different solutions for a model which has no serious drawbacks in terms of the asumptions it makes. An interesting question is how the two solutions are related to each other.

10 Conclusions

We have reviewed the main approaches to option pricing with stochastic volatility which have appeared in the literature. The approach, characterised by McDonald and Bookstaber (1988), which is concerned with fitting a general distribution to asset prices is not very helpful. It leaves unresolved the key problems of a replication strategy and the risk premium on volatility. The alternative source of randomness proposed by Madan and Senata (1990) is also unsatisfactory. In order that the risk associated with a state variable can be hedged we essentially require sample path continuity. This implies that locally we have Brownian motion. Any alternative must involve a jump process with the associated disadvantage of not being able to form a replicating portfolio for the option and option prices depending on investor preferences.

The most promising way forward seems to be to model the volatility as a diffusion process. We describe the properties which we believe the process must have to be both realistic and tractable. The analyses presented by Hull and White (1987) and Stein and Stein (1991) are helpful in demonstrating techniques which can be used to obtain practical option pricing formulae. However, they both have the disadvantages of requiring the correlation between the underlying asset and its volatility to be zero and having unrealistic processes for the volatility. We then describe the models of Hull and White (1988) and Scott (1992). We note that the interest process in the Scott model is not very realistic but that this facet of the model can be removed to reduce it to the Hull and White model. This model has all the desirable properties which we sought. The option pricing formulae which are obtained are practical to implement and the only restriction is on the functional forms allowed for the risk premium.

Finally, the new approach which Dupire (1992) has introduced and which we briefly discussed in section 2 appears to hold great promise. There is still some work needed to clarify the implications of this model but the prospect of no-arbitrage evolution and pricing of the term structure of volatility is very exciting.

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