The Magnitude of Implied Volatility Smiles: Theory and Empirical Evidence for Exchange Rates

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Abstract

Theoretical methods are used to show that implied volatilities are approximately a quadratic function of the forward price divided by the exercise price when volatility is stochastic, asset price and volatility differentials are uncorrelated and volatility risk is not priced. The curvature of the quadratic function depends on the time to maturity of the option and several volatility parameters including the present level, the long-run median level and the variance of future average volatility. The magnitude of the 'smile effect' is a decreasing function of time to maturity. Empirical evidence for exchange rate options supports the theoretical predictions, although the empirical smiles are approximately twice as large as those predicted by theory.

The magnitude of implied volatility smiles : theory and empirical evidence for exchange rates

1. Introduction

At any moment of time implied volatilities vary for different times to option expiry T and different exercise prices X. A matrix of implied volatilities is frequently available, with columns ordered by T and rows ordered by X. Rational expectations of the average volatility during the next T years will vary with T whenever volatility is believed to be stochastic. Thus the rows of the implied volatility matrix may provide information about the term structure of expected future volatility. Xu and Taylor (1993) have explained how this term structure can be estimated from a row of the implied volatility matrix. In this paper we present theoretical and empirical results for the columns of the matrix.

Hull and White (1987) and Stein and Stein (1991) have shown that it is rational for the implied volatility to vary with X when the asset volatility is believed to be stochastic. Their equations and calculations show that a plot of theoretical implieds against X displays a 'smile' : the function has a U-shape with the minimum implied occurring when X is the forward price of the underlying asset. Shastri and Wethyavivorn (1987) show that a 'smile effect' is predicted by theory when prices follow a mixed jump-diffusion process.

There have been several attempts to decide if option prices have a 'strike bias' but we are only aware of one empirical study which explores the idea that a 'smile effect' is the appropriate form of strike bias. This is the exchange rate paper by Shastri and Wethyavivorn (1987). They concluded that empirical implieds in 1983 and 1984 were a U-shaped function of the exchange rate divided by X. Fung and Hsieh (1991) discuss informally some empirical 'smile' pictures. Traders have told us that they know implieds 'smile' but there evidence is largely anecdotal.

Section 2 reviews notation and definitions, followed by theoretical results in Sections 3 to 5. Stochastic volatility is shown to be a sufficient reason for smiles to exist. The theoretical effects are of economic importance especially when T is a relatively short time. Section 6 describes empirical estimates of the magnitude of the smile effect for spot currency options traded at the Philadelphia Stock Exchange. A general effect is found although it is more pronounced than predicted by the theory we develop. Conclusions are stated in Section 7.

2. Notation and definitions

To develop the theoretical results we consider European options, traded upon an asset which pays dividends at a continuous rate. The fair price for a call option when the asset price follows geometric Brownian motion is represented by

$$c(S,T,X,r,d,V), \quad V=\sigma^2,$$

or simply $c(X,V)$, with S the spot price, T the time to expiry, X the exercise price, r the riskfree interest rate, d the dividend rate and σ the volatility (which is constant for the assumed price model). Time t is measured in years and the present time is $t=0$.

When volatility is stochastic and therefore depends upon t let

$$V(t)=\sigma^2(t),$$

$$\overline{V}=\frac{1}{T}\int_0^T V(t)\,dt$$

and suppose the conditional distribution $\overline{V} \mid V(0)$ has probability density function $f(v)$ for which the mean and variance are respectively $\mu_{\overline{V}}$ and $\sigma^2_{\overline{V}}$.

The fair price in a world of stochastic volatility is represented by

$$C(\ldots X,\ldots).$$

When firstly the price and the volatility both follow diffusion processes within the general structure discussed by Hull and White (1987), secondly volatility risk is not priced and thirdly the price and volatility differentials are uncorrelated, $C(X)$ is given by the following integral :

$$C(X) = \int_0^\infty c(X,v) f(v)\, dv. \tag{1}$$

For each X there is a Black-Scholes implied volatility corresponding to the fair price given by a stochastic volatility process. This implied quantity is defined by :

$$C(X) = c(X, \sigma^2_{imp}(X)). \tag{2}$$

3. Theoretical approximations

Series expansions of $c(X,V)$ around $V = \mu_V$ permit theoretical analysis of the smile effect. A quadratic approximation is required in (1) and a linear approximation in (2) to ensure that σ^2_{imp} depends upon X. Then :

$$C(X) \approx \int_0^\infty f(v)\left[c(X,\mu_V) + (v-\mu_V)\frac{\partial c}{\partial v} + \frac{1}{2}(v-\mu_V)^2 \frac{\partial^2 c}{\partial v^2} \right] dv$$

$$= c(X,\mu_V) + \frac{1}{2}\sigma_V^2 \frac{\partial^2 c}{\partial V^2} \tag{3}$$

and

$$C(X) \approx c(X,\mu_V) + \left[\sigma^2_{imp}(X) - \mu_V\right]\frac{\partial c}{\partial V}. \tag{4}$$

From (3) and (4) :

$$\left[\sigma^2_{imp}(X) - \mu_V\right]\frac{\partial c}{\partial V} \approx \frac{1}{2}\sigma_V^2 \frac{\partial^2 c}{\partial V^2}$$

and so

$$\sigma^2_{imp}(X) \approx \mu_V + \frac{1}{2}\sigma_V^2 \frac{\partial^2 c / \partial V^2}{\partial c / \partial V} \tag{5}$$

with the partial derivatives evaluated at $V = \mu_V$. These partials are functions of X and other variables.

The ratio of the second partial derivative to the first partial derivative of c, with respect to V, can be calculated from the following familiar equations :

$$c(X,V) = S\mathrm{e}^{-dT}\mathrm{N}(d_1) - X\mathrm{e}^{-rT}\mathrm{N}(d_2),$$

$$d_1 = \frac{\ln(S/X) + \left(r - d + \frac{1}{2}V\right)T}{(VT)^{\frac{1}{2}}},$$

$$d_2 = d_1 - (VT)^{\frac{1}{2}},$$

$$\frac{\partial c}{\partial V} = \frac{1}{2}S\mathrm{e}^{-dT}\phi(d_1)T^{\frac{1}{2}}V^{-\frac{1}{2}},$$

$$\frac{\partial^2 c}{\partial V^2} = \frac{1}{4}S\mathrm{e}^{-dT}\phi(d_1)T^{\frac{1}{2}}V^{-\frac{3}{2}}(d_1 d_2 - 1),$$

and hence

$$\frac{\partial^2 c/\partial V^2}{\partial c/\partial V} = \frac{d_1 d_2 - 1}{2V}. \tag{6}$$

Using the forward price $F = S\mathrm{e}^{(r-d)T}$,

$$d_1(X) = \frac{\ln(F/X) + \frac{1}{2}VT}{(VT)^{\frac{1}{2}}}$$

and thus

$$d_1(X)d_2(X) = \frac{[\ln(F/X)]^2 - \frac{1}{4}V^2T^2}{VT}. \tag{7}$$

Also,

$$d_1(X)d_2(X) - d_1(F)d_2(F) = \frac{[\ln(F/X)]^2}{VT}. \tag{8}$$

Substituting (7) into (6), then replacing V by $\mu_{\bar{V}}$ and finally substituting (6) into (5) gives the following approximation :

$$\sigma^2_{imp}(X) \approx \mu_{\bar{V}} + \frac{\sigma^2_{\bar{V}}}{4\mu_{\bar{V}}}[d_1(X)d_2(X) - 1] \tag{9}$$

$$= \mu_{\bar{V}} + \frac{\sigma_{\bar{V}}^2}{4\mu_{\bar{V}}}\left[\frac{[\ln(F/X)]^2 - \mu_{\bar{V}}T - \frac{1}{4}\mu_{\bar{V}}^2T^2}{\mu_{\bar{V}}T}\right]. \tag{10}$$

From (7) and (9) the approximate implied volatility is minimised when $X = F$. Furthermore, the quadratic term $[\ln(F/X)]^2$ predicts a theoretical smile.

The height of the smile can be approximated if it is assumed that the second term on the right of (9) is small compared with the first term. This assumption gives

$$\sigma_{imp}(X) \approx \mu_{\bar{V}}^{\frac{1}{2}}\left[1 + (d_1(X)d_2(X) - 1)\frac{\sigma_{\bar{V}}^2}{8\mu_{\bar{V}}^2}\right]. \tag{11}$$

From (8), an approximate *height* is given by

$$\sigma_{imp}(X) - \sigma_{imp}(F) \approx [\ln(F/X)]^2\frac{\sigma_{\bar{V}}^2}{8T\mu_{\bar{V}}^{2.5}}. \tag{12}$$

From (8) and (11) it can also be shown that an approximate *relative height* is given by

$$\frac{\sigma_{imp}(X)}{\sigma_{imp}(F)} \approx 1 + [\ln(F/X)]^2\frac{\sigma_{\bar{V}}^2}{8T\mu_{\bar{V}}^3}. \tag{13}$$

Several manuscripts, some unpublished, contain subsets of equations (1)-(8), whilst equations (9)-(13) are believed to be new.

A plot of theoretical implied volatilities against the exercise price will display a quadratic function (approximately) of $\ln(F/X)$ with the magnitude of the smile effect being dependent upon T, $V(0)$ and the parameters of the process defining $\{V(t), 0 < t \leq T\}$.

4. Accuracy of the approximations

The approximations have been evaluated for one of the most frequently specified continuous-time stochastic processes for volatility. This process is an Ornstein-Uhlenbeck (O-U) process for the logarithm of volatility, studied by Scott (1987), Wiggins (1987) and Chesney and Scott (1989). These authors have considered discrete-time approximations to this process, as also have Taylor (1986, 1993), Harvey, Ruiz and Shephard (1992) and Jacquier, Polson and Rossi (1992).

Recall $\sigma(t) = \sqrt{V}(t)$ and denote the unconditional mean and variance of $\ln(\sigma(t))$ by respectively α and β^2. The diffusion model for volatility is then :

$$d(\ln\sigma) \quad = \quad \Phi(\alpha - \ln\sigma)dt \quad + \quad (2\Phi)^{\frac{1}{2}}\beta\, dW. \tag{14}$$

with Φ a positive parameter which controls the rate of reversion towards the mean level α and with $W(t)$ a standardised Wiener process. For the 'half-life' h, equal to $(\ln 2)/\Phi$, :

$$\mathrm{E}[\ln\sigma(h) \mid \sigma(0)] \quad = \quad \frac{1}{2}[\alpha + \ln(\sigma(0))].$$

Monte Carlo methods have been used to calculate exact implied volatilities when the volatility logarithm follows the above O-U process. These methods require a discrete-time approximation to (14). The discrete-time process is also required for the calculation of $\mu_{\bar{V}}$ and $\sigma^2_{\bar{V}}$ and hence the approximate implieds given by (11) and the approximate relative smile heights given by (13).

A discrete-time approximation to (14) which matches the mean, variance and half-life is the following AR(1) process :

$$\ln(\sigma_{t+\Delta t}) - \ln(\sigma_t) \quad = \quad (1-\phi)(\alpha - \ln(\sigma_t)) + \beta(1-\phi^2)^{\frac{1}{2}}\varepsilon_{t+\Delta t} \tag{15}$$

with

$$\phi \quad = \quad \exp(-(\ln 2)(\Delta t)/h)$$

and $\{\varepsilon_{\Delta t}, \varepsilon_{2\Delta t}, \ldots\}$ a set of i.i.d. standardised Normal variables. For model (15) and $T = N(\Delta t)$, straightforward mathematics and calculations provide the conditional mean and variance of

$$\bar{V}_{discrete} \quad = \quad \frac{1}{N}\sum_{j=1}^{N} V_{j(\Delta t)} \quad = \quad \frac{1}{N}\sum_{j=1}^{N} \sigma^2_{j(\Delta t)}$$

for any specified initial variance $V(0)$. All the calculations in this paper based upon the discrete process assume Δt = 1 day = 1/365 years.

The accuracy of the approximations derived in Section 3 has been evaluated by considering parameter values similar to the empirical estimates reported for currencies by Taylor (1993) and Harvey, Ruiz and Shephard (1992). The parameter α was chosen so that $\sigma(t)$ has median value equal to 10%. The parameter β was chosen to be 0.4 and results are given for the initial volatility $\sigma(0)$ equal to one of the lower quartile $Q_1 = \exp(\alpha - 0.674\beta)$ =

7.6%, the median $Q_2 = \exp(\alpha) = 10\%$ and the upper quartile $Q_3 = \exp(\alpha + 0.674\beta) = 13.1\%$. The half-life h was set equal to 30 days.

Table 1 presents results when :

$T = 0.5$, 1, 2 or 4 half-lives,

$X/F = 0.92$, 0.96, 1, 1.04 or 1.08,

$\sigma(0) = Q_1$, Q_2 or Q_3,

$r = d = 0.06$, and

$S = F = 100$.

Columns 3 to 6 list the exact implieds $\sigma_{imp}(X)$, the approximate implieds given by equation (11), the exact relative heights $\sigma_{imp}(X)/\sigma_{imp}(F)$ and the approximate relative heights given by equation (13). Sufficient Monte Carlo replications were performed to ensure that the standard errors of the exact implieds were less than 0.00001.

It can be seen from Table 1 that the approximations are close to the exact results when X/F equals 0.96, 1 or 1.04 but the approximations are sometimes inaccurate when X/F equals 0.92 or 1.08. Both the exact and approximate results show that smile magnitudes decrease as either T or $\sigma(0)$ increases.

5. Further results for Ornstein-Uhlenbeck processes

Equation (13) predicts that empirical estimates of $\sigma_{imp}(X)/\sigma_{imp}(F)$ may depend upon X/F and the function

$$R(T) = \frac{\sigma_{\bar{V}}^2}{8T\mu_{\bar{V}}^3}. \tag{16}$$

It is helpful for empirical work to understand how R depends upon T and $\sigma(0)$.

Table 2 presents some relevant results when $\ln(\sigma(t))$ follows an O-U process. The parameters α and h are as before but now β is one of 0.2, 0.4 or 0.6. Once more $\sigma(0) = Q_1$, Q_2 or Q_3. Panel A of Table 2 lists values for both $\sigma_{\bar{V}}^2/(8\mu_{\bar{V}}^3)$ and $\sigma_{\bar{V}}^2$, with $\bar{V}$ approximated by $\bar{V}_{discrete}$. These functions of T are unimodal. Panel B of Table 2 lists the values of T which maximise these functions for the nine combinations of β and $\sigma(0)$. Most of the maxima are at between two and four 'half-lives'.

The function $R(T)$ decreases as T increases for each of the nine combinations. It is difficult to say at what rate the function $R(T)$ decreases. When β is 0.4, $R(T)$ is, very approximately, proportional to $T^{-0.5}$ when T is around one half-life and proportional to T^{-1} when T is around 2.5 half-lives. The function is proportional to T^{-2} for large T.

Table 2 confirms that $R(T)$ decreases as $\sigma(0)$ increases and, for fixed T, $\sigma(0)$ is approximately proportional to $1/\sigma(0)$. Also $R(T)$ increases with β.

Analytic results can be obtained when $V(t)$, rather than its logarithm, follows an O-U process. Of course it is then impossible to guarantee that $V(t)$ is positive. Suppose the unconditional mean and variance of $V(t)$ are μ_V and σ_V^2, and :

$$dV \;=\; \Phi(\mu_V - V)dt \;+\; (2\Phi)^{\frac{1}{2}}\sigma_V dW \tag{15}$$

with Φ positive and $W(t)$ a standardised Wiener process. The 'half-life' h again equals $(\ln 2)/\Phi$, with :

$$E[V(h)\,|\,V(0)] \;=\; \frac{1}{2}(\mu_V + V(0)).$$

Adapting the analysis presented in Cox and Miller (1972, Sec. 5.8) gives the following results for the first two conditional moments :

$$\mu_{\bar{V}} \;=\; E[\bar{V}\,|\,V(0)] \;=\; \mu_V + [V(0) - \mu_V]\left(\frac{1 - e^{-\Phi T}}{\Phi T}\right), \tag{16}$$

and

$$\sigma_{\bar{V}}^2 \;=\; \mathrm{var}(\bar{V}\,|\,V(0)) \;=\; \frac{2\sigma_V^2}{\Phi^2 T^2}\left[\Phi T - (1 - e^{-\Phi T}) - \frac{1}{2}(1 - e^{-\Phi T})^2\right]. \tag{17}$$

The conditional variance as a function of T is unimodal and converges to zero as $T \to 0$ or $T \to \infty$. The maximum of this function is at $T = 1.89/\Phi$ (approx.) which is 2.73 'half-lives' (approx.). The conditional variance depends on neither $\mu_{\bar{V}}$ nor $V(0)$. From (17),

$$\sigma_{\bar{V}}^2 \;\approx\; \frac{2}{3}\sigma_V^2(\Phi T) \qquad \text{as} \quad T \to 0,$$

$$= \; 0.28\sigma_V^2 \qquad \text{when} \quad T = (\ln 2)/\Phi,$$

$$= \quad 0.38\sigma_V^2 \qquad \text{when} \quad T = 1.89/\Phi,$$

$$\approx \quad 2\sigma_V^2/(\Phi T) \qquad \text{as} \quad T \to \infty.$$

The theoretical magnitude of the smile effect as a function of T depends particularly upon σ_V^2/T. This quantity is essentially constant for small T, is proportional to T^{-1} near to 2.73 'half-lives' and is proportional to T^{-2} for large T.

It is concluded that when either $V(t)$ or its logarithm follows an O-U process then the theoretical relative smile height is

(i) a decreasing function of T, and

(ii) a decreasing function of $\sigma(0)$.

It is also concluded that, in theory, 'large' smiles should be found at those markets whose prices have 'high' values of var($V(t)$).

6. Empirical estimates of smile magnitudes

6.1 Currency data

The primary source database for the options prices is the transaction report compiled daily by the Philadelphia Stock Exchange (PHLX). Only the closing call and put options prices and the simultaneous spot exchange rate quotes for the (British) Pound, (German) Mark, (Japanese) Yen and (Swiss) Franc against the US Dollar have been used. All eight datasets start on November 5, 1984; Pound calls and puts, Mark calls and Yen calls end on January 8, 1992 while the other four datasets end on November 22, 1989. This is determined by the availability of the PHLX data. However, the transaction report is not available for some trading days and then prices have been collected manually from the Wall Street Journal (WSJ). Approximately 10% of our implied volatilities are calculated from WSJ prices.

The domestic and foreign interest rates used are London euro-currency rates, collected from Datastream. This source provides overnight, seven days, one month, three months, six months and one year interest rates. For intermediate times, we simply use linear interpolation.

Often applied exclusion criteria have been used to remove uninformative options records from the database. Option prices less than 0.05 cents are eliminated as transaction costs including the bid-ask spread and liquidity premia are then large relative to the options prices. We also remove options with less than 10 calendar days to expiry because of well-known expiration effects.

6.2 Calculation of implied volatility

Implied volatilities have been calculated from American model prices. The model prices are approximated by the very accurate functions derived in Barone-Adesi and Whaley (1987). The calculations of implied volatility used an interval subdivision method, which always converged to an unique solution.

6.3 Estimation of $\sigma_{imp}(F)$

In order to empirically examine the relative smile effect, we require estimates of the implied volatility when the exercise price is exactly equal to the forward price. These estimates are required for all values of T found in the datasets. Estimates have been obtained by fitting the term structure model developed in Xu and Taylor (1993) to nearest-the-money options whose exercise prices minimise $|X - F|$. This methodology provides approximate estimates of the at-the-money implied volatilities.

The volatility term structure model involves two factors representing short-term and long-term volatility expectations. These volatility expectations are assumed to be mean reverting. The average squared volatility over a general time interval is a linear function of squared short- and long-term expected volatilities.

A Kalman filtering method has been applied to obtain both parameter estimates for the term structure model and time series of volatility expectations estimates. Estimates of at-the-money implied volatility are then calculated using equation (7) in Xu and Taylor (1993). These estimates are denoted by

$$\hat{\sigma}_{imp}(F,T).$$

6.4 The regression model

Let M denote the 'moneyness' of an option defined by

$$M \;=\; \ln(F/X).$$

Guided by the general conclusions about theoretical smiles given in Section 5, the following regression model has been estimated using ordinary least squares :

$$\frac{\sigma_{imp}(X,T)}{\hat{\sigma}_{imp}(F,T)} = a_0 + a_1\frac{M}{\sqrt{T}} + a_2\frac{M^2}{\sqrt{T}} + a_3\frac{M}{T} + a_4\frac{M^2}{T}$$

$$+ \quad a_5\frac{M^2}{\sqrt{T}\,\hat{\sigma}_{imp}(F,T)} + a_6\frac{M^2}{T\hat{\sigma}_{imp}(F,T)} + \text{residual.} \qquad (18)$$

The smile theory developed in Section 3 assumes that asset price and volatility differentials are uncorrelated and hence smile effects are symmetric functions of M. This prediction can be assessed by fitting the above regression specification which permits the relationship between implied volatility ratios and M to be asymmetric. Additional variables can be included in the regression model but it has been found that explanatory variables such as M and M^2 do not change the functions fitted to implied volatility ratios.

6.5 Results

Five figures are used to summarise the regression results for the Mark options. These clearly demonstrate the existence of smile effects in the prices of these options. Further figures, available from the authors, make clear that similar implied volatility smiles are found for the other three currencies. These figures all show the fitted regression relationships when the at-the-money volatility is 12%.

Figure 1 summarises the implied volatility smiles given by the regression estimates for Mark calls. Six curves representing six different maturities ranging from 10 days to 360 days are plotted. Implied volatility smiles are most pronounced for short maturity options and become smaller when the time to maturity becomes longer as predicted by theory. There is little evidence of skewness in these implied volatilities. Similar smiles have been obtained for Mark puts as in figure 2, although the magnitude of the effects for the puts is slightly larger than those for the calls. The results presented here are broadly consistent with those of Shastri and Wethyavivorn (1987) for two earlier years although they simply examined average implied volatilities across different maturities and different levels of moneyness.

Figures 3 and 4 plot implied volatility smiles when the regression model is estimated for subsets of the Mark calls data, respectively when the time to maturity is first restricted to be 10 to 30 days and second with the restriction from 31 to 60 days. The magnitude of the smiles on Figures 3 and 4 are extremely close to those on Figure 1. Furthermore, the curves for $T = 30$ days on Figures 3 and 4 are almost identical. These observations confirm that the results are robust against the selection of maturities T.

Figure 5 presents the results for 30-day implied volatility smiles for Mark calls on a year-to-year basis. The variation between years is small during the first six years, although it seems the magnitude of the volatility ratio has increased gradually for in-the-money options. Skewness in the fitted curves is more evident than for the full sample estimates. The skewness changed from negative to positive sometime around 1988.

Comparing Table 1 with Figure 1 we find that the empirical implied ratios are larger than the theoretical ratios. The magnitudes of the empirical smiles are very approximately twice the magnitude of the theoretical smiles. This might be explained by our assumptions about the price process (for example, no jumps) or by market imperfections (for example, transaction costs).

7. Conclusions

The 'smile effect' has been shown, by theoretical methods, to be a logical consequence of stochastic variation in asset volatility. Implied volatilities are approximately a quadratic function of $\ln(F/X)$ when asset price and volatility differentials are uncorrelated and volatility risk is not priced. This conclusion is probably also true when volatility risk is priced although we have not yet proved this conjecture. The curvature of the quadratic function depends on the time to maturity T of the option and several volatility parameters including the present level, the long-run median level and the variance of future average volatility. The magnitude of the smile is a decreasing function of T. It is concluded that implied volatilities ought to be functions of both T and X, even when there are no term structure effects so that at-the-money options have the same implieds for all T.

The empirical evidence for exchange rate options supports the theoretical predictions, although the empirical smiles are larger than predicted by the theory.

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Table 1
Accuracy of smile approximations
when the logarithm of volatility follows an O-U process

Parameters
S=100, r= 0.06, d= 0.06
median $\sigma(t) = Q_2 = 10\%$
standard deviation of $\ln(\sigma(t)) = \beta = 0.4$
lower quartile for $\sigma(t) = Q_1 = 7.6\%$
upper quartile for $\sigma(t) = Q_3 = 13.1\%$
half-life $h = 30$ days

Panel A : Initial volatility $\sigma(0) = Q_1 = 7.6\%$

T (days)	X	exact implied %	approx. implied (11) %	exact ratio	approx. ratio (13)	$\sqrt{\mu_{\bar{V}}}$ %
15	92	n/a				
15	96	8.79	8.93	1.074	1.091	
15	100	8.18	8.17	1.000	1.000	8.30
15	104	8.75	8.87	1.069	1.084	
15	108	n/a				
30	92	n/a				
30	96	9.16	9.26	1.058	1.072	
30	100	8.66	8.62	1.000	1.000	8.87
30	104	9.12	9.21	1.054	1.067	
30	108	n/a				
60	92	10.52	11.03	1.125	1.180	
60	96	9.69	9.70	1.036	1.043	
60	100	9.35	9.28	1.000	1.000	9.67
60	104	9.66	9.67	1.033	1.040	
60	108	10.37	10.77	1.109	1.154	
120	92	10.77	10.88	1.059	1.076	
120	96	10.32	10.28	1.016	1.018	
120	100	10.16	10.09	1.000	1.000	10.50
120	104	10.31	10.26	1.014	1.017	
120	108	10.68	10.77	1.051	1.064	

Panel B : Initial volatility $\sigma(0) = Q_2 = 10\%$

T (days)	X	exact implied %	approx. implied (11) %	exact ratio	approx. ratio (13)	$\sqrt{\mu_{\bar{V}}}$ %
15	92	n/a				
15	96	10.78	10.86	1.047	1.055	
15	100	10.29	10.28	1.000	1.000	10.43
15	104	10.74	10.81	1.044	1.051	
15	108	n/a				
30	92	11.94	12.54	1.136	1.193	
30	96	10.91	10.97	1.038	1.046	
30	100	10.51	10.47	1.000	1.000	10.75
30	104	10.89	10.93	1.036	1.043	
30	108	11.76	12.24	1.119	1.164	
60	92	11.78	12.11	1.092	1.126	
60	96	11.06	11.05	1.025	1.030	
60	100	10.78	10.71	1.000	1.000	11.12
60	104	11.03	11.02	1.023	1.028	
60	108	11.65	11.91	1.081	1.107	
120	92	11.59	11.67	1.048	1.059	
120	96	11.20	11.16	1.012	1.014	
120	100	11.07	10.99	1.000	1.000	11.41
120	104	11.19	11.14	1.011	1.013	
120	108	11.52	11.57	1.041	1.051	

Panel C : Initial volatility $\sigma(0) = Q_3 = 13.1\%$

T (days)	X	exact implied %	approx. implied (11) %	exact ratio	approx. ratio (13)	$\sqrt{\mu_V}$ %
15	92	n/a				
15	96	13.35	13.39	1.029	1.033	
15	100	12.96	12.95	1.000	1.000	13.13
15	104	13.32	13.35	1.027	1.031	
15	108	14.19	14.51	1.094	1.119	
30	92	13.99	14.35	1.093	1.122	
30	96	13.12	13.14	1.025	1.029	
30	100	12.80	12.75	1.000	1.000	13.08
30	104	13.10	13.11	1.023	1.027	
30	108	13.83	14.12	1.080	1.104	
60	92	13.34	13.55	1.066	1.086	
60	96	12.73	12.70	1.017	1.021	
60	100	12.51	12.44	1.000	1.000	12.87
60	104	12.71	12.68	1.016	1.019	
60	108	13.23	13.39	1.057	1.074	
120	92	12.61	12.66	1.037	1.045	
120	96	12.27	12.23	1.009	1.011	
120	100	12.16	12.09	1.000	1.000	12.51
120	104	12.27	12.22	1.009	1.010	
120	108	12.55	12.57	1.032	1.039	

Table 2

Selected conditional moments for $\overline{V}$ when the logarithm of volatility follows an O-U process

The conditional moments depend on the initial volatility $\sigma(0)$

Parameters
median $\sigma(t) = Q_2 = 10\%$
standard deviation of $\ln(\sigma(t)) = \beta$
lower quartile for $\sigma(t) = Q_1 = 10\%/\exp(0.674\beta)$
upper quartile for $\sigma(t) = Q_3 = 10\%*\exp(0.674\beta)$
half-life $h = 30$ days

Panel A : Conditional moment functions

β	T (days)	$\sigma_{\overline{V}}^2/(8\mu_{\overline{V}}^3)$			$10^6\sigma_{\overline{V}}^2$		
		$\sigma(0) = Q_1$	Q_2	Q_3	$\sigma(0) = Q_1$	Q_2	Q_3
0.2	15	0.503	0.395	0.310	2.20	3.39	5.24
0.2	30	0.741	0.596	0.478	3.75	5.34	7.64
0.2	60	0.902	0.755	0.629	5.54	7.09	9.12
0.2	120	0.820	0.724	0.636	6.17	7.07	8.15
0.4	15	2.612	1.614	0.995	7.26	17.20	40.87
0.4	30	3.773	2.449	1.577	15.42	31.00	62.91
0.4	60	4.329	3.060	2.125	29.24	46.99	76.98
0.4	120	3.583	2.831	2.182	39.13	50.39	66.60
0.6	15	7.774	3.788	1.834	16.14	58.20	211.39
0.6	30	11.090	5.846	3.026	48.10	133.79	379.24
0.6	60	11.893	7.192	4.180	130.36	255.84	521.13
0.6	120	8.707	6.262	4.268	218.98	309.18	458.64

Panel B : Values of T which maximise the conditional moment functions

β	$\sigma_{\overline{V}}^2/(8\mu_{\overline{V}}^3)$			$\sigma_{\overline{V}}^2$		
	$\sigma(0) = Q_1$	Q_2	Q_3	$\sigma(0) = Q_1$	Q_2	Q_3
0.2	69	78	87	105	84	66
0.4	58	72	90	132	96	64
0.6	49	65	89	154	109	70

Figure 1: Implied Volatility Smiles (Mark Calls)

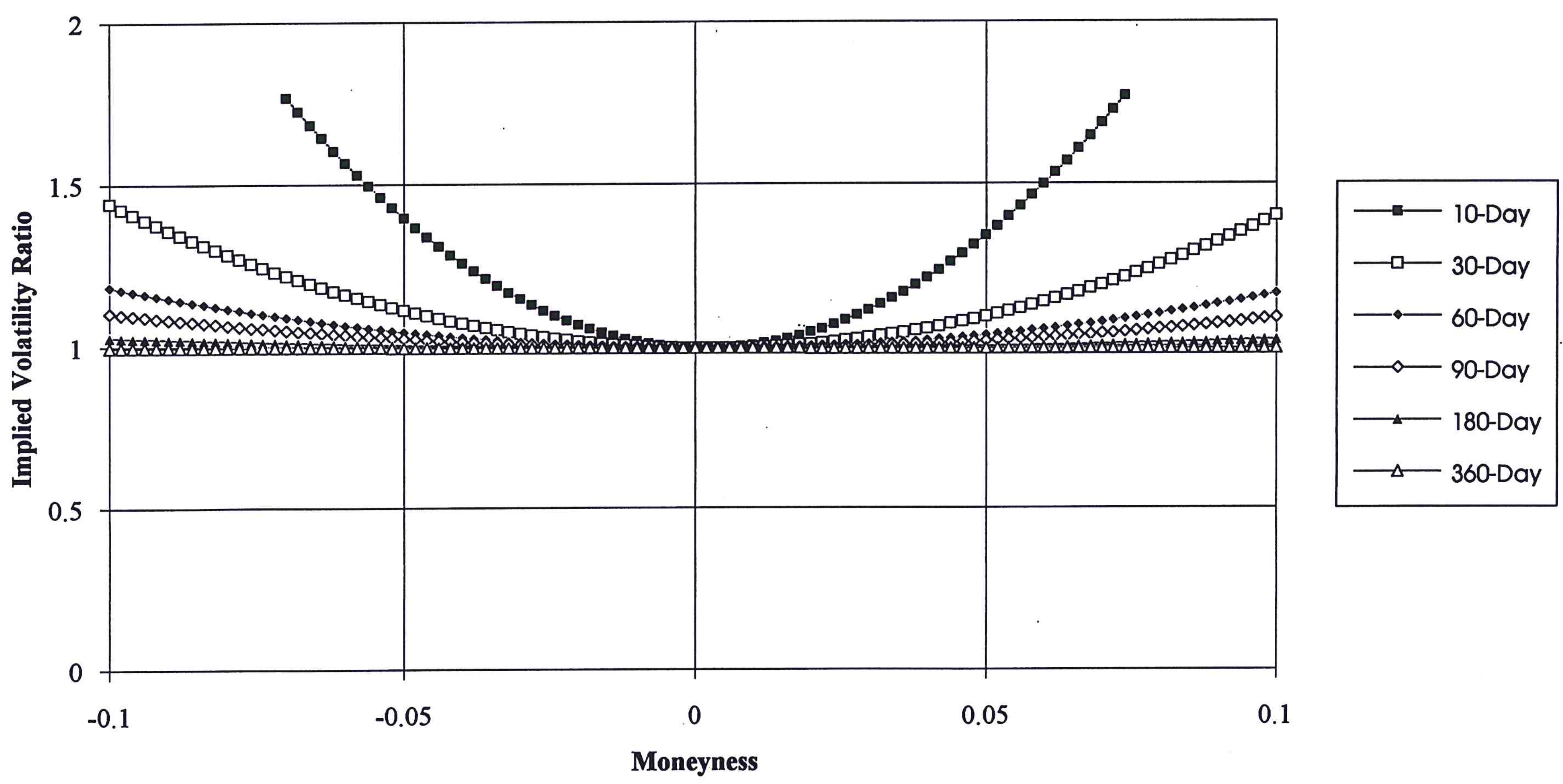

Figure 2: Implied Volatility Smiles (Mark Puts)

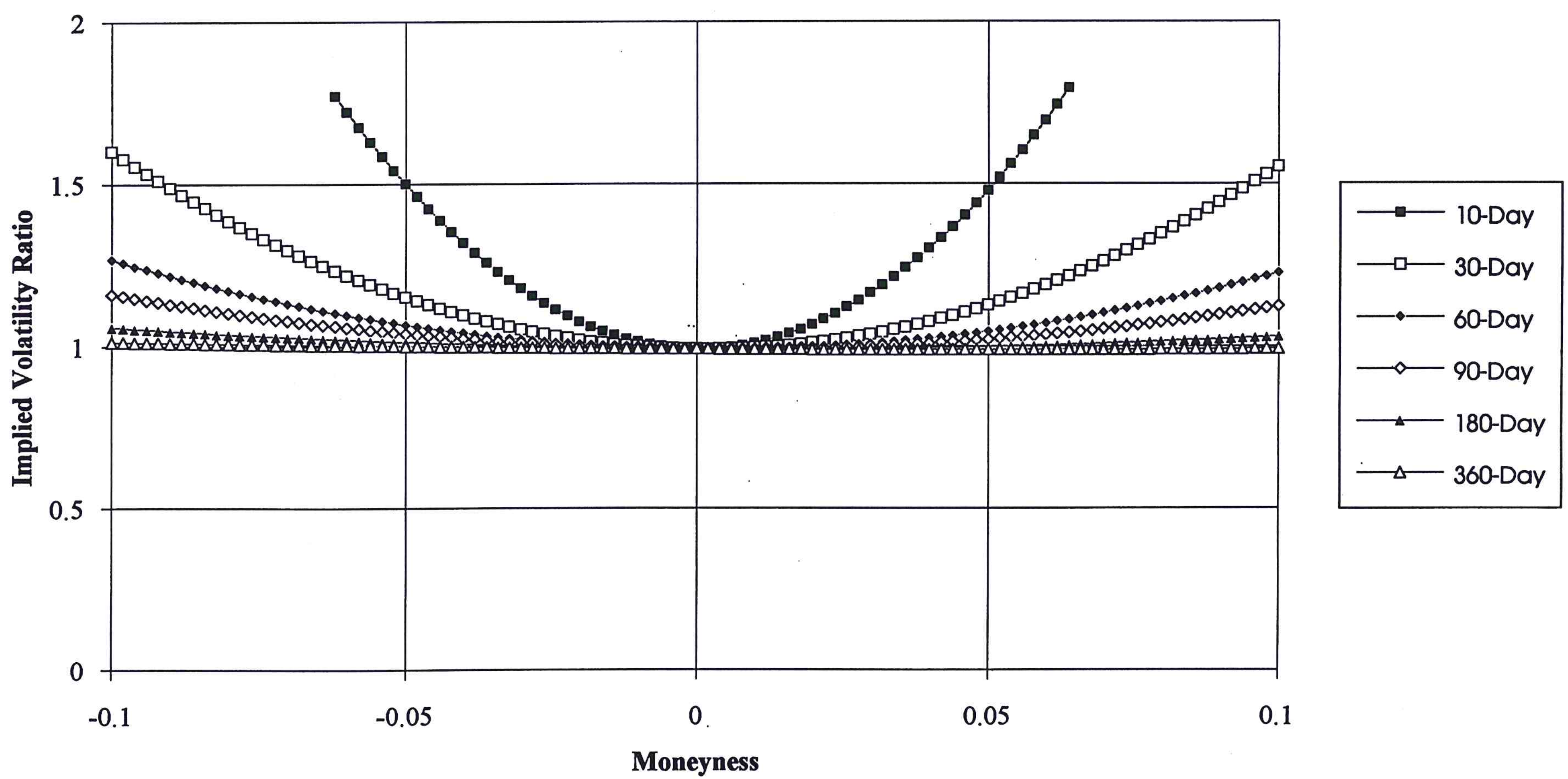

Figure 3: Implied Volatility Smiles (Mark Calls: 10-30 Days)

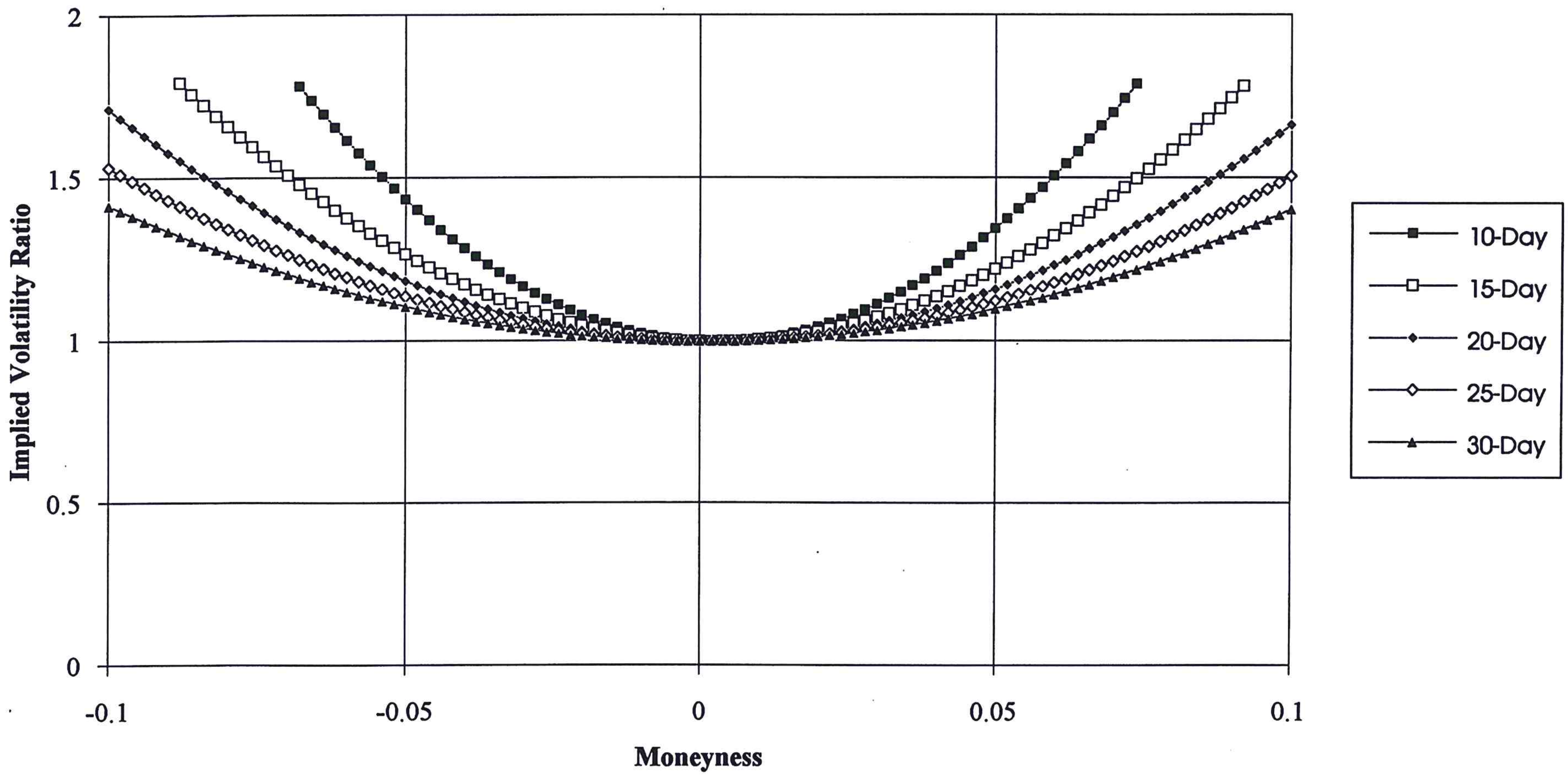

Figure 4: Implied Volatility Smiles (Mark Calls: 31-60 Days)

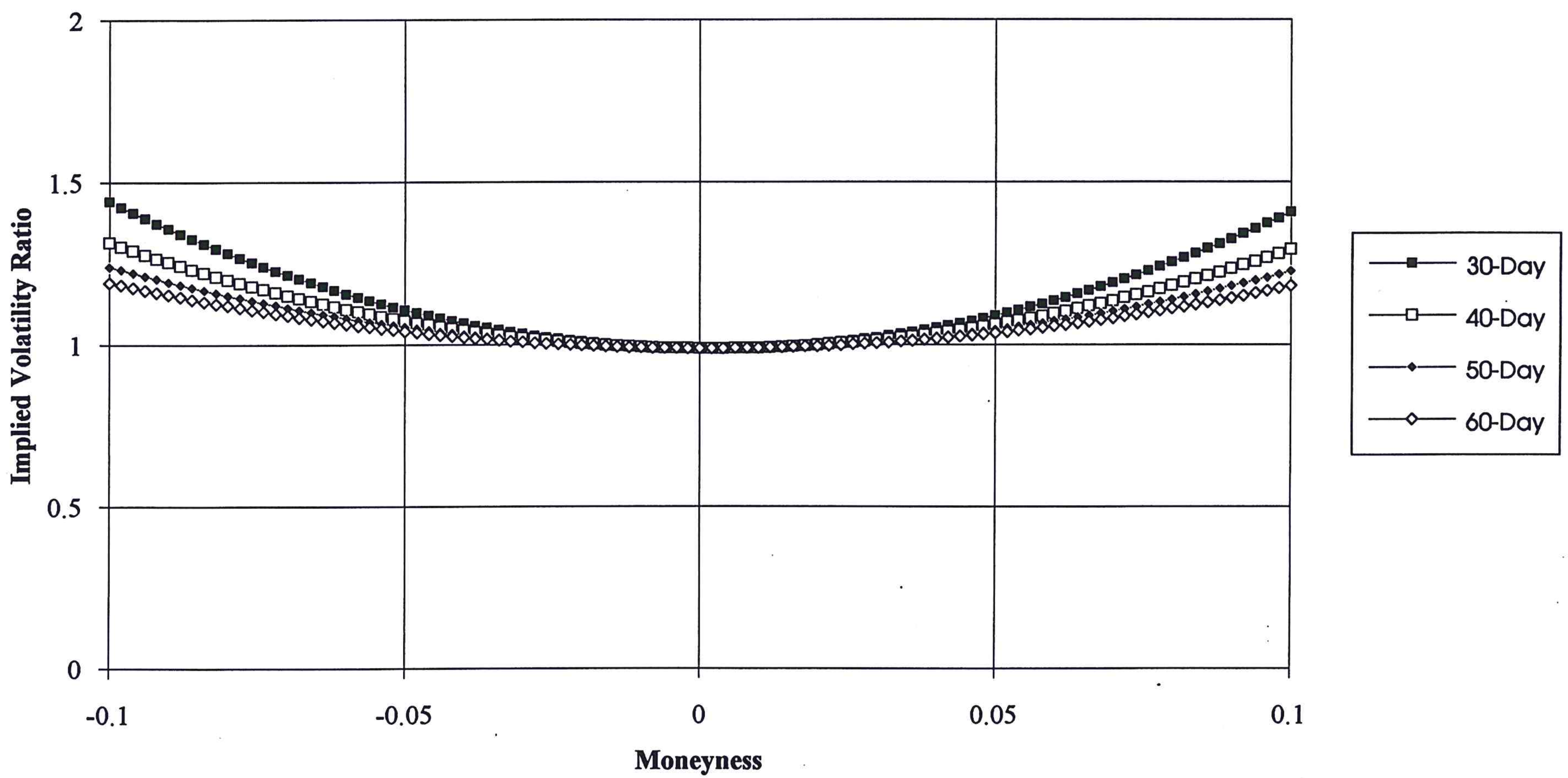

Figure 5: 30-Day Implied Volatility Smiles (Mark Calls: 1985-1991)