A Note on Parameter Estimation in the Two Factor Longstaff and Schwartz Interest Rate Model

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A NOTE ON PARAMETER ESTIMATION IN THE TWO-FACTOR LONGSTAFF AND SCHWARTZ INTEREST RATE MODEL

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In a recent series of articles, Longstaff and Schwartz develop a two-factor model of the term structure of interest rates where the two factors are the level of the short rate and the volatility of changes in the short rate. The authors develop the model using the same general equilibrium framework as Cox, Ingersoll, and Ross [1985] (Longstaff and Schwartz [1992a]); show how the model can be used to price a number of different interest rate instruments (Longstaff and Schwartz [1992b]); and finally discuss a number of issues related to the practical implementation of the model (Longstaff and Schwartz [1993]).

It is with this last issue that this note is concerned. Longstaff and Schwartz outline a parameter estimation method that uses the historical time series of interest rates and estimated time series of interest rate volatilities. Our aim is to test the practical robustness of this estimation technique, given realistic limitations on data available to us. We show that although, in theory, their estimation methodology works, the length of financial time series available to financial researchers and market participants makes its practical implementation problematic.

I. THE MODEL

The two factors of the Longstaff-Schwartz model are the short rate of interest, r, and the variance of changes in the short rate, v. The model starts from a specification not of r and v but of the dynamics of two, unspecified, economic factors representing the component of expected returns on physical investment and the preferences of a representative investor, within a general equilibrium framework:

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$$dx = (\gamma - \delta x)\,dt + \sqrt{x}\,dz_1$$

$$dy = (\eta - \xi y)\,dt + \sqrt{y}\,dz_2 \qquad (1)$$

The equilibrium instantaneous interest rate and the variance of changes in this rate are given, within this framework, as a weighted sum of the original state variables, x and y, where the weights relate to parameters of the return process for physical investment.

$$r = \alpha x + \beta y$$

$$v = \alpha^2 x + \beta^2 y \qquad (2)$$

Rearranging these linear functions, we obtain

$$x = \frac{\beta r - v}{\alpha(\beta - \alpha)}$$

$$y = \frac{v - \alpha r}{\beta(\beta - \alpha)} \qquad (3)$$

implying that, for the original state variables x and y to be positive, we must have

$$\alpha r < v < \beta r \qquad (4)$$

The relationship between r and v allows the user to work either in terms of the original state variables (1) or the more economically intuitive variables given by (2). In either case, in order to use the model for pricing pure discount bonds or other interest rate derivatives, we have to estimate six parameters: α, β, δ, γ, η, and ξ. Longstaff and Schwartz call these parameters "stationary" parameters. One of the parameters, ξ, always enters into the pricing formulas as a sum with the market price of interest rate risk.

II. PARAMETER ESTIMATION

Longstaff and Schwartz show that both r and v have long-run stationary unconditional distributions with means and variances given by:

$$E[r] = \frac{\alpha\gamma}{\delta} + \frac{\beta\eta}{\xi}$$

$$Var[r] = \frac{\alpha^2\gamma}{2\delta^2} + \frac{\beta^2\eta}{2\xi^2}$$

$$E[v] = \frac{\alpha^2\gamma}{\delta} + \frac{\beta^2\eta}{\xi}$$

$$Var[v] = \frac{\alpha^4\gamma}{2\delta^2} + \frac{\beta^4\eta}{2\xi^2} \qquad (5)$$

Longstaff and Schwartz [1993] show that by rearranging the equations in (5), four of the parameters, δ, γ, η, and ξ, can be written in terms of the remaining two, α and β, and the first two moments of the distributions for r and v. It is suggested that the long-run means and variances be estimated from time series data, leaving only the estimation of α and β. Longstaff and Schwartz's suggestion is that the form of Equation (4) implies that historical data can be used to choose parameter values for α and β by looking at the time series of the ratio of v and r.

$$\alpha < \frac{v}{r} < \beta$$

and choosing α and β to be:

$$\alpha = \min\left(\frac{v}{r}\right)$$

$$\beta = \max\left(\frac{v}{r}\right) \qquad (6)$$

Rearranging (5) gives us the remaining four parameters in terms of "observables."

$$\delta = \frac{\alpha(\alpha + \beta)(\beta E[r] - E[v])}{2(\beta^2 Var[r] - Var[v])}$$

$$\gamma = \frac{\delta(\beta E[r] - E[v])}{\alpha(\beta - \alpha)}$$

$$\xi = \frac{\beta(\alpha + \beta)(E[v] - \alpha E[r])}{2(Var[v] - \alpha^2 Var[r])}$$

EXHIBIT 1 ■ Model Parameter Values Estimated by Longstaff and Schwartz and Used in the Simulations

Mean Value of r	0.06717
Mean Value of v	0.0007157
α	0.001149
β	0.3125
γ	3.0493
δ	0.05658
η	0.1582
ξ	3.998

$$\eta = \frac{\xi(E[v] - \alpha E[r])}{\beta(\beta - \alpha)} \qquad (7)$$

Longstaff and Schwartz have therefore used the fact that there is a direct mapping between the six "stationary" parameters of the model, and the parameters of the distributions for r and v.

III. TESTING METHODOLOGY

In order to test the Longstaff and Schwartz methodology for parameter estimation, first we start from the parameters that they have themselves estimated from twenty-five years of monthly Treasury bill rates starting in 1964. These are reproduced in Exhibit 1.

Second, we assume we are living in a Longstaff and Schwartz world, i.e., the short rate of interest and the variance of changes in the short rate do actually evolve precisely according to Equations (1) and (2). We then simulate through time with these parameters and processes, recording the (realized) time series for r and v. Finally, we try to recover the parameters of the process using the expressions given in Equations (6) and (7) and the two simulated time series obtained.

Originally, we performed two types of simulation, each consisting of 10,000 separate simulations. First, we simulated daily data for five years (yielding 1,800 observations), in order to put ourselves, for example, in the place of an investment house that may have access to five years of daily data. Second, we simulated daily data for twenty-five years, "capturing" one observation per month (giving 300 observations). This allows us to replicate the Longstaff and Schwartz estimation, giving us the same number of observations, in each simulation, that they use in their empirical estimation.

IV. RESULTS

Exhibits 2 and 3 present summary statistics for the estimation of the parameters from the two simulation studies described in the previous section. α is estimated well even for the monthly data. β, however, is extremely poorly estimated. This is because of the nature of the distribution of (v/r), which is highly positively skewed (see Exhibit 4), which in turn is due to the nature of the joint distribution for r and v for the particular set of parameter values that Longstaff and Schwartz obtain (see Exhibit 5).

Perhaps a better estimation method for β would be via an estimation of the correlation between r and v, which is given by

$$\rho = -\alpha\beta r + (\alpha + \beta)v$$

Note that if the model is misspecified, and v is not constrained, then it is not clear how either α or β should be estimated.

The estimation of the other parameters, δ, γ, ξ, and η, is also very poor. It could be reasoned that this stems from the poor estimation of β.

In order to test this hypothesis we reestimated these parameters with α and β set to their "true" values. These results, presented in Exhibits 6 and 7, reveal

EXHIBIT 2 ■ Estimation of Parameters from Five Years of Daily Data

	Alpha (α)	Beta (β)	Delta (δ)	Gamma (γ)	Xi (χ)	Eta (ε)	E[r] (*100)	E[v] (*100)	Var[r] (*10,000)	Var[v] (*10,000)
Min.	0.114900	0.020413	−1,156.9492	−37,286.6664	0.388570	0.089900	3.345300	0.026100	0.166100	0.000500
Max.	0.118984	0.107090	104.7691	7,084.6592	5.451580	0.912140	11.165400	0.338600	15.871800	0.239100
Avg.	0.115123	0.058346	−0.2878	−11.8516	1.251657	0.307089	6.829451	0.092057	1.431017	0.015445
Std. Dev.	0.000266	0.013804	11.7923	392.9326	0.366889	0.089655	1.017914	0.038348	1.130077	0.016614

EXHIBIT 3 ■ Estimation of Parameters from Twenty-Five Years of Monthly Data

	Alpha (α)	Beta (β)	Delta (δ)	Gamma (γ)	Xi (χ)	Eta (ε)	E[r] (*100)	E[v] (*100)	Var[r] (*10,000)	Var[v] (*10,000)
Min.	0.114900	0.030998	–511.1335	–36,252.9064	0.357550	0.077030	2.760600	0.044200	0.699900	0.002700
Max.	0.125301	0.121930	771.5937	43,044.2830	5.782960	0.496620	14.731100	0.184900	26.362300	0.090800
Avg.	0.116119	0.070413	–0.0245	0.3986	1.403230	0.239413	6.811431	0.092517	2.963395	0.016965
Std. Dev.	0.001284	0.013728	15.5341	897.6369	0.488112	0.046872	1.526868	0.017863	1.663349	0.008758

that the estimation of β is not the only problem. In fact the problem is twofold.

First, as revealed by Exhibits 8 to 13, the distributions of the observables α, β, E[r], Var[r], E[v], and Var [v] are highly non-normal and in general highly positively skewed. This, in combination with the structure of Equation (5), results in the distributions of the parameters being highly non-normal and in particular highly fat-tailed. Consequently the estimation of the parameters is problematic.

Note, however, that all these problems stem from the fact that the joint distribution of r and v is almost degenerate for this particular set of parameters. Indeed Longstaff and Schwartz do not plot the probability density for the parameters that they estimate, and in Longstaff and Schwartz [1992b] they plot a much better behaved joint probability density for r and v for a very different set of parameters. Yet the fact that their estimated parameters lead to an almost degenerate probability density may indicate that estimation of this model (and indeed any similar two-factor model) from a time series of short-term interest rates alone is not feasible.

V. CONCLUSIONS

Our note focuses on the practicality and robustness of the approach suggested by Longstaff and Schwartz in development of their two-factor model of the term structure of interest rates. This approach is based on use of an historical time series of interest rates and an estimated time series of interest rate volatilities.

We have shown that although, in theory, their estimation methodology works, the nature of financial time series available to financial researchers and market participants makes its practical implementation extremely difficult.

The specific problem with the set of parameters that Longstaff and Schwartz estimate is that they lead to an almost degenerate joint probability density for r and v. This is only one problem. The nature of the processes for r and v also makes the estimation procedure unstable.

This is a generic problem with models of this type. It is simply very difficult to differentiate between different parameter regimes and indeed different models using the market data available.

EXHIBIT 4 ■ Histogram of v/r for a Typical Simulation

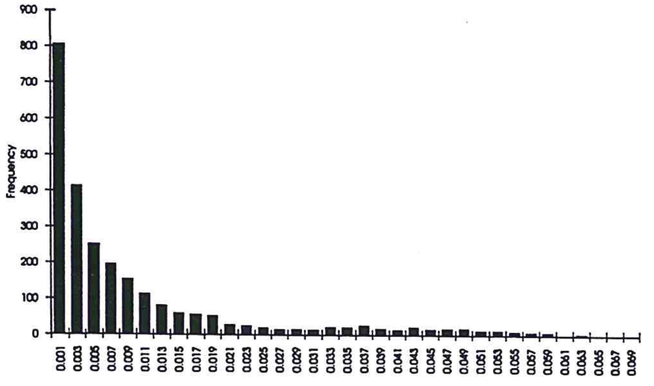


EXHIBIT 5 ■ The Joint Probability Density of r and v for the Longstaff and Schwartz Parameters

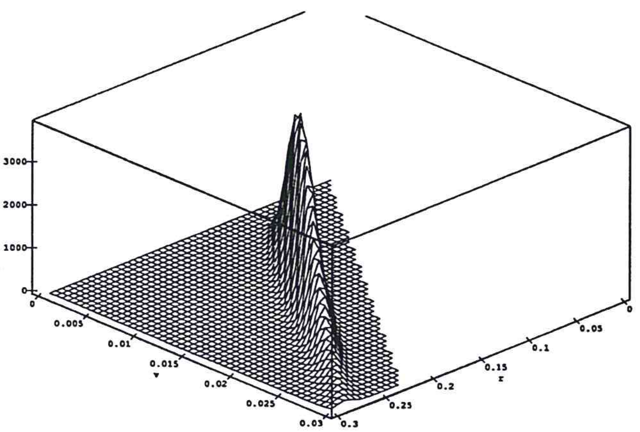


EXHIBIT 6 ■ Estimation of Parameters from Five Years of Daily Data ($\alpha = 0.001149$ and $\beta = 0.1325$)

	Delta (δ)	Gamma (γ)	Xi (χ)	Eta (ε)	E[r] (*100)	E[v] (*100)	Var[r] (*10,000)	Var[v] (*10,000)
Minimum	−3,469.9009	−134,165.4670	0.909690	0.089040	3.345300	0.026100	0.166100	0.000500
Maximum	2,666.7394	156,406.7640	31.846090	0.896090	11.165400	0.338600	15.871800	0.239100
Average	0.7866	51.5718	7.291788	0.299516	6.829630	0.092056	1.431235	0.015444
Standard Deviation	46.4637	2,191.3690	3.521622	0.086702	1.017879	0.038352	1.130259	0.016617

EXHIBIT 7 ■ Estimation of Parameters from Twenty-Five Years of Monthly Data ($a = 0.001149$ and $\beta = 0.1325$)

	Delta (δ)	Gamma (γ)	Xi (χ)	Eta (ε)	E[r] (*100)	E[v] (*100)	Var[r] (*10,000)	Var[v] (*10,000)
Minimum	−20.8753	−1,263.8746	1.293640	0.076650	2.760600	0.044200	0.699900	0.002700
Maximum	12.3797	543.7800	13.416820	0.477070	14.731100	0.184900	26.362300	0.090800
Average	0.2637	13.9318	5.067369	0.235713	6.811756	0.092516	2.963673	0.016964
Standard Deviation	0.3557	19.3437	1.486706	0.045537	1.527028	0.017861	1.663446	0.008758

EXHIBIT 8 ■ Histogram of Estimates of α

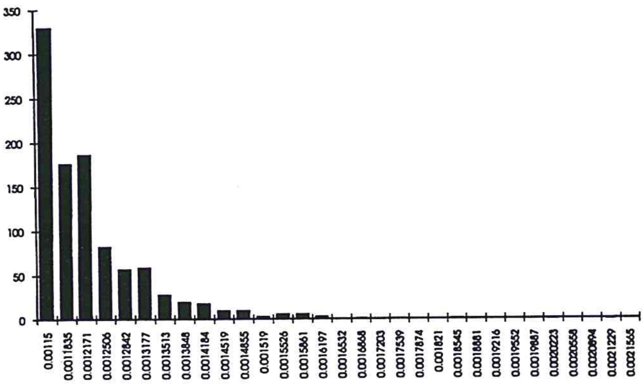


EXHIBIT 10 ■ Histogram of Estimates of E[r] (100)

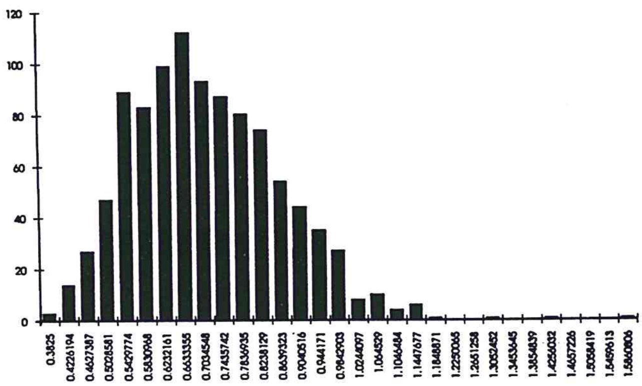


EXHIBIT 9 ■ Histogram of Estimates of β

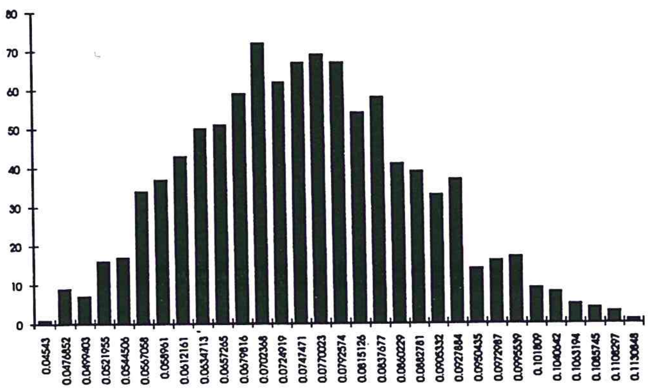


EXHIBIT 11 ■ Histogram of Estimates of Var[r] (10,000)

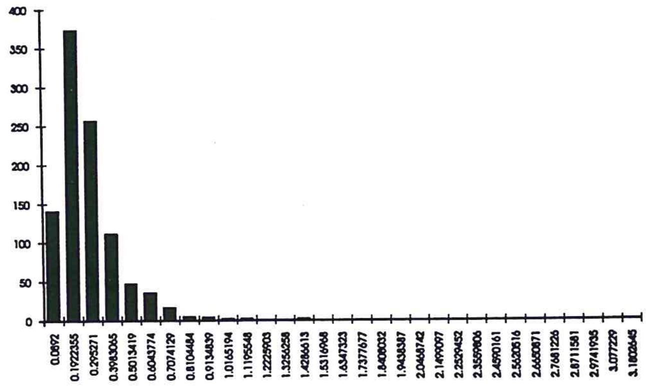


EXHIBIT 12 ■ Histogram of Estimates of E[v] (100)

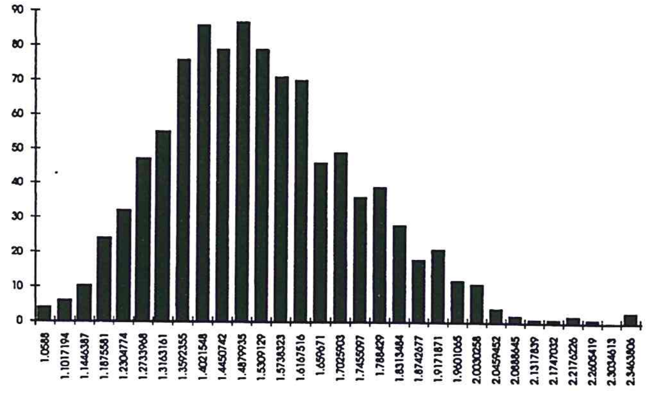
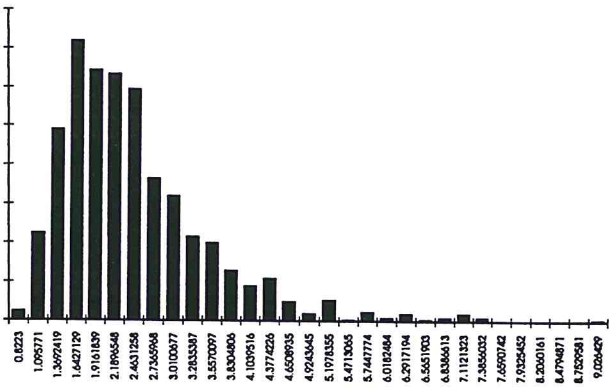


EXHIBIT 13 ■ Histogram of Estimates of Var[v] (10,000)



ENDNOTE

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