Computing the Fong and Vasicek Pure Discount Bond Price Formula

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Abstract

Fong and Vasicek [1992] developed a model of the term structure which results in closed-form expressions for pure discount bond prices, and which reflects both the current level of interest rates as well as the level of interest rate volatility.

Although a number of articles have appeared detailing the Fong-Vasicek model (Fong and Vasicek [1991], Fong and Vasicek [1992a], Fong and Vasicek [1992b]), there appears to be very little reported practitioner interest. One of the reasons for this relatively minor practitioner approval, we feel, is the computational difficulty in obtaining prices from their formula. Two of the three functions of time involved in the model's solution require computation of the confluent hypergeometric function. As described by Fong and Vasicek, this involves complex (as opposed to real) algebra.

In this article we propose an alternative solution to the ordinary differential equations which lead to the need to solve the confluent hypergeometric function. Our method involves a series solution which is both computationally efficient and can be easily implemented in a programming language or spreadsheet.

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1/ Introduction

There are essentially two main approaches to modeling the term structure. The traditional approach involves starting with a plausible process, or processes, for the sources of uncertainty that drive the evolution of the yield curve. Prices of pure discount bonds and discount bond yields are then determined as functions of the state variables and the risk-adjusted parameters of the assumed processes. This approach is represented in the single factor case by the papers of Vasicek [1977] and Cox, Ingersoll, and Ross [1985]. Models that assume that the term structure is driven by more than one state variable are represented by the papers of Brennan and Schwartz [1979], Schaefer and Schwartz [1984], Longstaff and Schwartz [1992] and Fong and Vasicek [1991]. In the alternative approach, models seek to describe the dynamics of the entire term structure in a way that is automatically consistent with the initial (observed) yield curve, and is represented by the papers Ho and Lee [1986], Hull and White [1990], and Heath, Jarrow, and Morton [1992].

Two of the traditional two factor models of the first approach are of particular interest, especially to practitioners. Both Longstaff-Schwartz and Fong-Vasicek develop models of the term structure which result in closed-form expressions for pure discount bond prices, and which reflect both the current level of interest rates as well as the level of interest rate volatility. Much has been written about the Longstaff-Schwartz model (see for example Longstaff and Schwartz [1992a], Longstaff and Schwartz [1992b]) with anecdotal evidence suggesting that it is becoming popular amongst practitioners. In contrast, although a number of articles have appeared detailing the Fong-Vasicek model (see for example Fong and Vasicek [1991], Fong and Vasicek [1992a], Fong and Vasicek [1992b]), there appears to be very little reported practitioner interest. One of the reasons for this relatively minor practitioner approval, we feel, is the computational difficulty in obtaining prices from their formula. Two of the three functions of time involved in the model's solution require computation of the confluent hypergeometric function. As described by Fong and Vasicek, this involves complex (as opposed to real) algebra. In this article we propose an alternative solution to the ordinary differential equations which avoids the need to solve the confluent hypergeometric function. Our method involves a series solution which is both computationally efficient and can be easily implemented in a programming language or spreadsheet.

The plan of this paper is as follows. Section 2 looks at the volatility of interest rates and the ability of single factor traditional models, described in the first approach of the introduction, to represent that uncertainty. In section 3 we present a necessary and brief description of the Fong and Vasicek term structure model and the solution as proposed by the authors. Section 4 contains the major part of the paper. In it we develop a series solution to the ordinary differential equations to which the partial differential equation, governing the evolution of bond prices in the Fong-Vasicek formulation, reduces. Finally, section 5 contains our summary and conclusions.

2/ Interest Rate Volatility

Interest rate volatility is a key element when valuing fixed income securities which have options or any kind of contingent payoff attached (for example puttable, callable, or convertible bonds). These option features can significantly affect the price of the bond to which they are attached. Many investors are also aware that the uncertainty affecting interest rate moves, as well as the level of rates, affects the relative attractiveness of bonds which differ in their time to maturity and the level of their coupon payments. Fixed income academics and practitioners have long noted that the volatility of interest rates changes over time. A casual analysis of some of the empirical evidence illustrates their intuition. Figure 1 shows the implied volatility calculated from LIFFE short sterling futures options using the Black [1976] formula over the period November 1987 to February 1990. It is obvious that volatility is changing through time, with periods of high volatility, periods of low volatility and sudden changes between the two.

The traditional approach to modelling the term structure has a number of advantages. Early, single factor models of the term structure can be characterised by their tractability and ease of use[1] - they are computationally cheap and there exist relatively few parameters to estimate. Closed form solutions are also desirable since they provide greater insight than numerical techniques and are usually much faster to evaluate. Although single factor models that allow for the volatility of the short rate to depend, in some way, on the level of the rate itself are consistent with the empirical evidence that rates are typically more volatile when their level is high, and less volatile when their level is low, the assumed stochastic

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1 Vasicek [1977] and Cox, Ingersoll, and Ross [1985] are regarded as the benchmark models of this approach.

processes of these models are unrealistic[2]. A further disadvantage of one factor models is the limited family of possible term structures that are permissible under the assumption of a single source of uncertainty. Typically these are monotonic increasing, monotonic decreasing, or slightly humped. Models in which bond prices depend on two sources of uncertainty lead to families of yield curves that can take a greater variety of shapes and can better represent reality. Dybvig [1989] provides evidence that the two most important variables in determining interest rate derivative prices are the level of interest rates and the level of interest rate volatility.

3/ The Fong and Vasicek Pure Discount Bond Price Formula

Fong and Vasicek's two factor model of the term structure explicitly recognises interest rate volatility as a stochastic factor, and derives the pure discount bond price function under the equilibrium condition of no arbitrage. The diffusion processes for the two sources of uncertainty, the short term interest rate *r*, and the instantaneous variance of the short rate *v*, are given as:

$$dr = \alpha(\bar{r} - r)dt + \sqrt{v}dz_1 \tag{3.1}$$

$$dv = \gamma(\bar{v} - v)dt + \xi\sqrt{v}dz_2 \tag{3.2}$$

$\bar{r}$ and $\bar{v}$ are the long-term mean of the short rate and the long-term mean of the variance of the short rate respectively. The short rate has instantaneous volatility *v* with the random component of the volatility process having a variance

2 See Strickland [1993]

proportional to the current level of the volatility. Both processes tend to revert to a long term mean value, with the strength of the reversion being proportional to the variables current deviation from the mean. The two Weiner processes, their increments represented by dz_1 and $dz_2 dy$ are assumed to be correlated with a coefficient of ρ. The model also assumes that the prices of risk are proportional to the risk level

$$q = \lambda\sqrt{v}$$
$$p = \eta\sqrt{v} \tag{3.3}$$

where λ and η are constants.

Let the price at time t of a pure discount bond which matures with value 1 at time s be given under this analysis by $P = P(t,s,r,v)$. Under the equilibrium condition of no arbitrage the partial differential equation governing the price of a pure discount bond is given as

$$P_t + (\alpha\bar{r} - \alpha r + \lambda v)P_r + (\gamma\bar{v} - (\gamma + \xi\eta)v)P_v + \frac{1}{2}vP_{rr} + p\xi vP_{rv} + \frac{1}{2}\xi^2 vP_{vv} - rP = 0 \tag{3.4}$$

subject to the maturity boundary condition $P(s,s,r,v) = 1$.

The solution to the partial differential equation has the form

$$P(t,s,r,v) = \exp(-rD(\tau) + vF(\tau) + G(\tau)) \tag{3.5}$$

where $\tau = s - t$. The functions of time D, F, and G are solutions to ordinary differential equations to which the partial differential equation (3.4) reduces. The duration measure is the same as in Vasicek [1977]

$$D(\tau) = \frac{1}{\alpha}\left(1 - e^{-\alpha\tau}\right) \tag{3.6}$$

The function determining the sensitivity of the bonds price to the level of the short rate volatility is shown to be:

$$F(\tau) = -\frac{2i\alpha\kappa\delta}{\xi^2} e^{-\alpha\tau}$$
$$+\frac{2\alpha}{\xi^2} \frac{\sum_{j=1}^{2} K_j e^{-\beta_j \alpha\tau}\left(\beta_j M\left(d_j, e_j, i\kappa e^{-\alpha\tau}\right) + i\kappa e^{-\alpha\tau} \frac{d_j}{e_j} M\left(d_j + 1, e_j + 1, i\kappa e^{-\alpha\tau}\right)\right)}{\sum_{j=1}^{2} K_j e^{-\beta_j \alpha\tau} M\left(d_j, e_j, i\kappa e^{-\alpha\tau}\right)} \tag{3.7}$$

where

$$\kappa = \frac{\xi}{\alpha^2}\sqrt{1-\rho^2}$$
$$\delta = \frac{1}{2} - \frac{i}{2}\frac{\rho}{\sqrt{1-\rho^2}}$$
$$d_j = \frac{1}{2}e_j - \frac{i}{2}\frac{\rho}{\sqrt{1-\rho^2}}\left(\frac{\xi}{\alpha^2}(1-\alpha\lambda) - \rho(\theta - 1)\right) \qquad \textit{for } j = 1,2$$
$$e_j = 2\beta_j - \theta + 1$$
$$\beta_1 = \beta$$
$$\beta_2 = \theta - \beta$$
$$i = \sqrt{-1}$$

and where

$$\beta = \frac{1}{2}\theta - \frac{1}{2}\sqrt{\theta^2 - \frac{\xi^2}{\alpha^4} + 2\lambda\frac{\xi^2}{\alpha^3}}$$

$$\theta = \frac{\gamma\xi\eta}{\alpha} + \frac{\rho\xi}{\alpha^2}$$

The constants K_1 and K_2 of equation (3.7) are chosen to satisfy the boundary condition

$$F(0) = 0$$

The function M is the confluent hypergeometric function with

$$M(d,e,z) = 1 + \sum_{n=1}^{\infty} \frac{d(d+1)\ldots(d+n)}{e(e+1)\ldots(e+n)} \frac{z^n}{n!} \tag{3.8}$$

The expression for G is found as the solution to

$$G(\tau) = -\alpha\bar{r}\int_t^s D(u)du + \gamma\bar{v}\int_t^s F(u)du \tag{3.9}$$

and is closed-form, involving the solution to the confluent hypergeometric function, once D and F are determined.

In order to use the formula, the difficult part is the calculation of the functions F and G. One possibility is the use of a computer algebra package such as *Mathematica* or *Maple*, but it is not easy to incorporate such packages into the usual investment house trading systems. The next section proposes a series solution, to these functions, that can be easily implemented in a programming language and which the authors have also implemented in an *Excel* spreadsheet.

4/ A Series Solution to Fong and Vasicek's Ordinary Differential Equations.

The numerical complications to computing the pure discount bond price formula (3.5) under Fong and Vasicek involve the computation of the confluent hypergeometric function M, which occurs in the time-dependent functions F and G. In this section we propose an alternative, series solution, which by-passes the need for the complex algebra that Fong and Vasicek's M involves.

The ordinary differential equation that is satisfied by F is a Riccati equation involving the square of the function. Making the substitution

$$H(s) = \exp\left(-\frac{1}{2}\xi^2 \int_t^s F(u)du\right)$$

reduces the equation to a linear second order equation for H. Under this substitution we have

$$F(\tau) = -\frac{2}{\xi^2}\frac{H'(\tau)}{H(\tau)} \tag{4.1}$$

$$G(\tau) = \bar{r}(D(\tau) - \tau) - \frac{2\gamma\bar{v}}{\xi^2}\ln H(\tau) \tag{4.2}$$

Therefore the solution to the bond pricing formula reduces to evaluating $H(\tau)$ and $H'(\tau)$. A further substitution

$$\begin{aligned} \tau &= -\frac{1}{\alpha}\log x \quad 0 \le x \le 1 \\ H(\tau) &= x^{\beta}Q(x) \end{aligned} \tag{4.3}$$

reduces the equation, for H, to a homogeneous linear differential equation of second order for Q.

$$xQ''(x)+\left((2\beta-\theta+1)+\frac{\rho\xi}{\alpha^2}x\right)Q'(x)+\left(\frac{\xi\rho\beta}{\alpha^2}-\frac{\xi^2}{2\alpha^4}(1-\alpha\lambda)+\frac{\xi^2}{4\alpha^4}x\right)Q(x)=0 \quad (4.4)$$

where β and θ are defined in section 3.

The point $x = 0$ is a regular singular point. We shall develop a Frobenius solution for $Q(x)$ and substitute back into expression (4.3) for H, allowing us to calculate F and G via equations (4.1) and (4.2) respectively. Put

$$Q(x)=x^c\sum_{n=0}^{\infty}a_n x^n \qquad (4.5)$$

therefore,

$$Q'(x)=\sum_{n=0}^{\infty}(n+c)a_n x^{n+c-1} \qquad (4.6)$$

and $$Q''(x)=\sum_{n=0}^{\infty}(n+c)(n+c-1)a_n x^{n+c-2} \qquad (4.7)$$

Let

$$\bar{\alpha}=2\beta-\theta+1$$
$$\bar{\beta}=\frac{\rho\xi}{\alpha^2}$$
$$\bar{\gamma}=\frac{\xi\rho\beta}{\alpha^2}-\frac{\xi^2}{2\alpha^4}(1-\alpha\lambda)$$
$$\bar{\delta}=\frac{\xi^2}{4\alpha^4}$$

Substituting the expressions for Q and its derivatives into (4.4) we obtain

$$\sum_{n=0}^{\infty}\left((n+c)(n+c-1+\overline{\alpha})a_n x^{n+c-1}\right)+\sum_{n=0}^{\infty}\left(\overline{\beta}(n+c)+\overline{\gamma}\right)a_n x^{n+c}+\sum_{n=0}^{\infty}\overline{\delta}a_n x^{n+c+1}=0 \quad (4.8)$$

The coefficients of x^{c+n}, $n = -1, 0, 1, \ldots$, must be identically equal to zero. Thus

x^{c-1}: $\quad c(\overline{\alpha}+c-1)=0 \quad$ if $a_0 \neq 0$

This is the incidial equation. Thus

$$c=0$$

or

$$c=1-\overline{\alpha}$$

When $\overline{\alpha}$ is not an integer and also not equal to 1, we have two independent solutions directly. We shall assume that this is the case, therefore

x^{c}: $\quad (1+c)(\overline{\alpha}+c)a_1+(\overline{\beta}c+\overline{\gamma})a_0=0$

implying

$$a_1=\frac{-\left(\overline{\beta}c+\overline{\gamma}\right)a_0}{(1+c)(\overline{\alpha}+c)} \quad (4.9)$$

where a_0 is the seed value. Equating coefficients for $n > 0$,

$$x^{c+1}: \qquad (2+c)(\overline{\alpha}+c+1)a_2+(\overline{\beta}+\overline{\beta}c+\overline{\gamma})a_1+\overline{\delta}a_0=0$$
$$x^{c+2}: \qquad (3+c)(\overline{\alpha}+c+2)a_3+(2\overline{\beta}+\overline{\beta}c+\overline{\gamma})a_2+\overline{\delta}a_1=0$$
$$x^{c+3}: \qquad (4+c)(\overline{\alpha}+c+3)a_4+(3\overline{\beta}+\overline{\beta}c+\overline{\gamma})a_3+\overline{\delta}a_2=0$$

$$x^{c+n}: \qquad ((n+1)+c)(\overline{\alpha}+c+3n)a_{n+1}+(\overline{\beta}n+\overline{\beta}c+\overline{\gamma})a_n+\overline{\delta}a_{n-1}=0$$

Rearranging the coefficients of the powers of x, and defining $\overline{\varepsilon}=\overline{\beta}c+\overline{\gamma}$ we obtain:

$$a_2=\frac{(\overline{\beta}+\overline{\varepsilon})\overline{\varepsilon}-\overline{\delta}(1+c)(\overline{\alpha}+c)}{(1+c)(2+c)(\overline{\alpha}+c)(\overline{\alpha}+c+1)}a_0 \qquad (4.10)$$

$$a_3=\frac{-\overline{\varepsilon}(\overline{\beta}+\overline{\varepsilon})(2\overline{\beta}+\overline{\varepsilon})-[(2\overline{\beta}+\overline{\varepsilon})(1+c)(\overline{\alpha}+c)+\overline{\varepsilon}(2+c)(\overline{\alpha}+c+1)]\overline{\delta}}{(1+c)(2+c)(3+c)(\overline{\alpha}+c)(\overline{\alpha}+c+1)(\overline{\alpha}+c+2)}a_0 \qquad (4.11)$$

For values of $n>3$, we obtain the recursive relationship

$$a_n=\frac{-((n-1)\overline{\beta}+\overline{\varepsilon})a_{n-1}-\overline{\delta}a_{n-2}}{(n+c)(\overline{\alpha}+c+(n-1))} \qquad (4.12)$$

The full equation for (4.5) corresponding to the two independent solutions $c=0$ and $c=1-\overline{\alpha}$ is given by

$$Q(x)=aQ(x,c=0)+bQ(x,c=1-\overline{\alpha})$$

where a and b are constants. [NB. $Q(x,c=0)$ means $Q(x)$ evaluated at $c=0$]. Substituting back into expression (4.3) for H we obtain

$$H(\tau)=e^{-\alpha\beta\tau}\left[aQ(e^{-\alpha\tau},0)+bQ(e^{-\alpha\tau},1-\overline{\alpha})\right]$$

Differentiating with respect to τ gives

$$H'(\tau) = -\alpha\beta H(\tau) - \alpha e^{-\alpha(\beta+1)\tau}\left[aQ'(e^{-\alpha\tau},0) + bQ'(e^{-\alpha\tau},1-\overline{\alpha})\right]$$

We now have the elements needed to calculate $F(\tau)$ and $G(\tau)$ once we have determined the values for the constants a and b. In order to do this we need to solve for two boundary conditions. We choose to solve for the conditions:

$$H(0) = 1$$
$$H'(0) = 0$$

from which we obtain

$$b = \frac{Q'(1,0) + Q(1,0)\beta}{Q(1,1-\overline{\alpha})Q'(1,0) - Q(1,0)Q'(1,1-\overline{\alpha})}$$

$$a = \frac{1 - Q(1,1-\overline{\alpha})b}{Q(1,0)}$$

The authors have implemented this series solution in both a programming language (Turbo Pascal) and a spreadsheet (Excel). To create a term structure going out to 30 years maturity at six-monthly intervals takes about 9 seconds for the former and 15 seconds for the latter on a 386 notebook with a maths co-processor.

Fong and Vasicek do not discuss the issue of the valuation of contingent claims on their term structures. In their related series of papers Longstaff and Schwartz develop a pricing formula for discount bond options. Their formula has the familiar 'Black-Scholes' format i.e. the weighted difference between the underlying instrument and the present value of the exercise price. The weights are dictated by a cumulative bivariate distribution, which is a lot harder to evaluate than the cumulative normal of Black-Scholes. Longstaff and Schwartz [1992a] admit that "...computation of the cumulative distribution functions can be cumbersome.....we have found that numerically solving the partial differential equation for the discount bond option price is as easy as evaluating (the formula) directly"[3].

Given the lack of computational efficiency of (directly) solving the option pricing formula, it does not seem unreasonable to calculate term structure derivative values, under Fong and Vasicek, by monte-carlo simulation. For example to calculate European discount bond options, firstly the risk-adjusted processes (3.1) and (3.2) are simulated until the end of the life of the option. Secondly, pure discount bond prices are calculated using the results of section 4, and the option maturity condition evaluated. Finally, the maturity payoff is discounted to the present time using the 'realised' time series for the short rate.

5/ Summary and Conclusions

Interest rate volatility is an important determinant in the pricing of bonds which differ in their time to maturity with or without imbedded options. Motivated by evidence that the two most important variables in determining interest rate derivative prices are the level of interest rates and the level of interest rate volatility, Fong and Vasicek develop a two-factor model of the term strcuture with these as their state variables.

We develop an easily implementable, and computationally efficient, series solution to their discount bond pricing formula avoiding the need for complex algebra, and

3 Footnote 15, page 1271.

hence computer algebra packages of the Fong and Vasicek solution. We show that our algorithm can be computed in real-time.

References

Black F, 1976, "The Pricing of Commodity Contracts", *Journal of Financial Economics*, Vol 3, pp 167-179.

Brennan M J, and E S Schwartz, 1979, "A Continuous Time Approach to the Pricing of Bonds", *Journal of Banking and Finance*, Vol 3, pp 133-155.

Cox, Ingersoll, and Ross. 1985, "A Theory of the Term Structure of Interest Rates", *Econometrica*, Vol 53, March.

Dybvig P H, 1989, "Bond and Option Pricing Based on the Current Term Structure", Working Paper, Washington University, St. Louis, Missouri.

Fong, H. G., and O. A. Vasicek, 1991, "Fixed Income Volatility Management", *Journal of Portfolio Management*, Summer, pp 41-46.

Fong, H. G., and O. A. Vasicek, 1992a, "Omission Impossible", Risk, Vol 5 (2), pp 62-65.

Fong, H. G., and O. A. Vasicek, 1992b, "Interest Rate Volatility as a Stochastic Factor", Gifford Fong Associates Working Paper.

Ho T S Y, and S-B Lee, 1986, "Term Structure Movements and Pricing Interest Rate Contingent Claims", *Journal of Finance*, Vol 41 (5), December pp 1011-1029.

Heath, D., R Jarrow, and A Morton. 1992, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claim Valuation", *Econometrica*, Vol 60, (1), pp 77-105.

Hull J and White A, 1990, "Pricing Interest Rate Derivative Securities", *The Review of Financial Studies*, Vol 3 (4).

Longstaff F, and E Schwartz, 1992a, "Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model", *The Journal of Finance*, Vol XLVII (4), September, pp 1259-1282.

Longstaff F, and E Schwartz, 1992b, "A Two-Factor Interest Rate Model and Contingent Claims Valuation", *The Journal of Fixed Income*, Vol 3, December, pp 16- 23

Schaefer S, and Schwartz E, 1984, "A Two Factor Model of the Term Structure: An Approximate Analytical Solution", *Journal of Financial and Quantitative Analysis*, Vol 19 (4), pp 413-424.

Strickland C, 1993, "A Comparison of Models of the Term Structure", Unpublished, Financial Options Research Centre, University of Warwick

Vasicek O,, 1977, "An Equilibrium Characterisation of the Term Structure" *Journal of Financial Economics*, No. 5.

Figure 1

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