Minimax Hedging Strategy[1]

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Abstract

We present several variants of a robust risk management strategy based on minimax for the writer of a European call option on a stock and show that it performs at least as well as the standard hedging strategy, delta hedging. When using the minimax strategy, the hedger specifies a worst case scenario in terms of the price of the underlying stock. The minimax strategy recommends the number of shares in the underlying stock the hedger should hold in order to minimize the hedging error against the worst case occurring. The minimax hedging error may correspond to an extreme point of the price range being considered or to a mid-range solution. Simulation and empirical results suggest that the minimax strategy is particularly powerful for hedging the risk of writing an option when the price of the underlying stock is both highly volatile and crosses over the exercise price frequently.

1 Introduction

We present minimax strategies to solve a hedging problem in risk management and discuss the performance of these strategies. We define the generic minimax formulation of a hedging strategy, hereafter **the minimax hedging strategy** or **minimax**, and develop it into a number of specific strategies which we call **variants**. The minimax hedging strategy is a risk management policy: just as with any other hedging strategy, it is a policy designed to match the desired risk profile of the hedger. The term minimax differentiates it from other hedging strategies by giving some information on what type of policy it is: it is a policy based on the worst-case potential hedging error. Just as with any other hedging strategy, the minimax hedging strategy is implemented over a time horizon, say, a nine-month period corresponding to the life of an option, and it may involve more than one rebalancing date. At such a date, the hedge is adjusted to reflect the hedger's desired risk profile. At a rebalancing date, the minimax hedging strategy uses the minimax algorithm as a computation tool to find the worst-case potential hedging error and the corresponding solution.

The minimax strategy determines simultaneously the optimal policy and the (bounded) uncertain variables that define the worst state of the world. While inherently pessimistic in outlook, we believe a decision maker cannot afford to overlook the "worst case" at least for contingency planning. In addition, any excessive pessimism of the strategy may be avoided by refining the variability bounds of the uncertainties. As a further justification of minimax, we should point out that expected value optimization calculations have to some extent been reconciled with, or justified in view of, the worst case scenario.

In Sections 2 and 3, we give a general introduction to options and to an option pricing model. In Section 2, we describe call options on a stock from the point of view of the hedger who writes[3] one. In Section 3, we describe an option pricing model and a dynamic hedging

[3]When an option contract is sold, the seller of the contract is called the "writer" of the contract and the act of selling is referred to as "writing".

strategy. In Section 4, we define the minimax hedging strategy. In Sections 5 and 6, we present and discuss results when minimax and delta hedging are used by the writer of a European call option[4] which can be regarded as a special case of the American call option[5]. In Section 5, we present a simulation study designed to identify the properties of the variants of minimax and to ascertain whether minimax performs best for a set of options for which it is designed to perform best. In Section 6, we present an empirical illustration showing the performance of minimax when real data are used.

The following notation will be used in this paper:

$B = B(S,t)$	call price
S	stock price
X	exercise price
r	risk-free interest rate
t	current date
T	expiration date
$T-t$	time to maturity
σ	volatility
$\Theta(d)$	the cumulative normal distribution function
Δt	hedging interval
N	the contracted number of shares of stock
n	number of shares to hold
k	the number of written call options
K	transaction cost as percent of transaction volume

The following *subscripts* will be used:

0	time 0, the initiation date of the contract
t	time t, any time such that $0 < t < T$
T	time T, the expiration date

The following *superscript* will be used:

i	refers to stock i or option i

2. Introduction to options and the hedging problem

2.1 Stock options

A **stock option**, hereafter referred to as an **option**, is a contract that entitles the holder to buy or sell a specific number of shares of a given stock, at or within a certain period of time, for

[4]An option which can be exercised only at maturity.

[5]An option which can be exercised any time during the life of the option.

an agreed price; the price which a buyer of an option pays is called the **premium**. The contract under which the option is bought or sold specifies:

- B_0, the premium, the price paid by the buyer,
- the underlying stock,
- N, the contracted number of shares,
- the date of the contract,
- T, the expiration date,
- X, the exercise price, the price at which the offer to buy or sell is to be made.

The market for traded options is the option exchange[6]. The two most widely traded options are called **calls** and **puts**: a **call** gives the holder the **right to buy** a specific number of shares; a **put** gives the holder the **right to sell** a specific number of shares. If the exercise of the option can take place only at the expiration date, it is called a **European option**. If the exercise can take place at any time on or before the exercise date, it is called an **American option**.

We focus on European call options. The hedging problem that we shall describe below pertains to the writing or selling of these types of options. Our analysis also applies to American call options provided that the American calls' underlying stock does not pay dividends. In this case, it is never optimal to exercise an American call before the expiration date.[7]

Our analysis does not apply to put options. Although there is a certain symmetry between puts and calls, it is not perfect. The potential profit or loss, respectively, from buying or selling a call is unlimited whereas that from buying or selling a put is limited. In particular, when transaction costs are introduced, separate analyses are necessary.[8]

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[6]In London, this is the London International Financial Futures Exchange (LIFFE).
[7]This result is discussed in Merton[10].
[8]This is illustrated in Neuhaus[11] where separate option pricing models were designed based on whether the call option is bought or sold.

2.2 Determinants of call premium

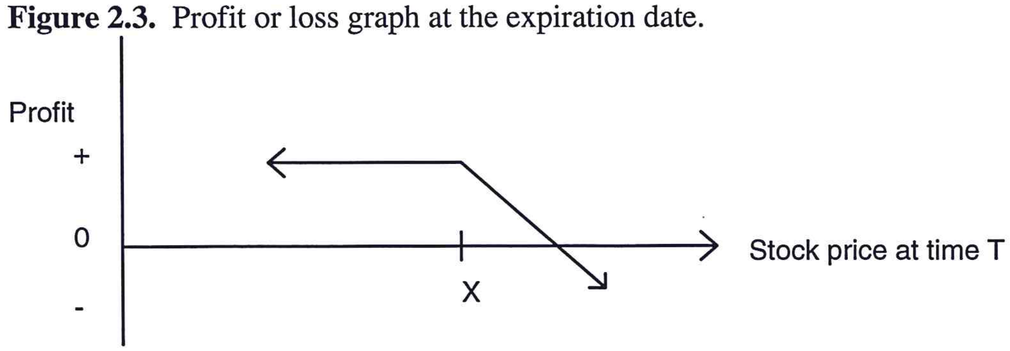
The call premium is the price of the option determined by the market. Some of the factors that determine the call premium are the exercise price, the exercise date, and the volatility of the underlying stock. In Section 3.2 we discuss the Black and Scholes[1] option pricing model and the factors that affect the option price within the framework of their model. In the Black and Scholes world, the option price will be higher when:

- X, the exercise price is lower: this is because the lower the X, the higher the probability that the call will be exercised.
- T, the time to maturity is longer: this is because uncertainty is greater the longer the time between the contract date and the expiration date and, for calls, there is a higher probability of a big reward.
- σ, the volatility of the stock is higher: this is because high stock volatility increases the uncertainty about the final level of the stock price, S_T, and therefore increases the probability that the final stock price will exceed the exercise price.

2.3 The hedging problem: hedging the risk of writing a European call option

A writer of a European call option receives the call premium but incurs a potential liability in case of exercise by the buyer at the expiration date. If the writer of the option does not own the contracted amount of stock, the potential liability is unlimited; this is shown by a graph of profit or loss against final stock price in Figure 2.3.

Figure 2.3. Profit or loss graph at the expiration date.



The writer, hereafter also called the **hedger**, wishes to modify his exposure to risk: he would like to avoid a potentially large loss in case the final stock price is above the exercise price. He would hedge this risk by holding part or all of the contracted number of shares. If he chooses to hold all of the contracted number of shares at the time the contract was made, then he has implemented a **covered write strategy**. This is a **static hedging strategy** where a decision is made at one point in time only. If he chooses to hold part of the contracted number of shares and, in particular, adjust his holding based on the option's "**delta**"[9], then he is implementing a **delta hedging strategy**. This is a **dynamic hedging strategy** where a decision is made at several points in time.

In practice, hedgers can choose from a variety of strategies ranging from ad hoc strategies to sophisticated ones based on option pricing theories. As hedging strategies with theoretical foundations are widely studied and generally acknowledged to be efficient, we assume that hedgers use them. In particular, we assume that delta hedging is also used.

We shall address this hedging problem in Section 4 where we develop the minimax hedging strategy: this strategy is based on the notion of a "**minimax hedging error**".[10]

3 The Black and Scholes option pricing model and delta hedging

In a dynamic strategy the hedger modifies his position in response to movements in the stock price. In Section 3.1, we present the Black and Scholes[1] (BS) option pricing model[11], which is the basis of delta hedging. The BS model will also be used in the minimax hedging strategy to be described in Section 4. In Section 3.2, we present a method of adjusting the BS option pricing model when transaction costs are included in the option valuation. In Section 3.3, we discuss delta hedging.

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[9]The "delta" of an option is the marginal change in the value of the option for a marginal change in the underlying stock's price. This is discussed in more detail in Section 3.

[10]The minimax hedging error is defined and discussed in Section 4.

[11]Other option pricing models, such as the model developed by Cox and Ross[3], may be used but this possibility is not explored here.

3.1 Black and Scholes Option Pricing Model

Readers familiar with the modern theory of option pricing can move directly to Section 3.2. Black and Scholes[1] derived a formula for the value of a European call option. Based on the assumptions underlying the BS model, the writer can create a hedged position consisting of a position in the option and a position in the stock that is instantaneously riskless. At any time t, he can set up a riskless portfolio because S_t and B_t are affected by the same underlying source of uncertainty and so are instantaneously perfectly correlated. This riskless portfolio can also be self-financing, which implies that it is not necessary to introduce, or take out, cash from the system to maintain its riskless nature. He can create a self-financing portfolio by financing any necessary purchases of shares of stock by sales of options. This can be seen as follows: let V_H be the value of a "hedge" portfolio consisting of n shares of stock held long[12] and N written call options,

$$V_H = nS - NB. \tag{3.1}$$

The dynamics of the stock price can be described by the following stochastic differential equation:

$$dS = \mu_S S dt + \sigma_S S dz \tag{3.2}$$

where

μ_S is the instantaneous rate of return on the stock

σ_S is the volatility of the rate of the return on the stock

dt is an increment of time

dz is an increment of Brownian motion.

Assuming $n = n(S,t)$ and $N = N(S,t)$, and using Ito's formula, the change in the value of the portfolio is

$$dV_H = dnS + ndS + dndS - [dNB + NdB + dNdB] \tag{3.3}$$

or

$$dV_H = ndS - NdB + [dn][S + dS] - [dN][B + dB]. \tag{3.4}$$

[12]The purchaser of shares of stock is said to be long in the stock.

If the hedge portfolio is self-financing, then the sum of the third and fourth terms on the right hand side of (3.4) equals zero, i.e. $[dn][S+dS]-[dN][B+dB]=0$ because all purchases or sales of assets are made at "new end of period prices". We now show that such a portfolio can also be riskless.

The dynamics of the option price can be described by a stochastic differential equation similar to (3.2) given by

$$dB = \mu_B B dt + \sigma_B B dz \qquad (3.5)$$

where

μ_B is the instantaneous rate of return on the option

σ_B is the volatility of the rate of the return on the option.

A key assumption of Black and Scholes is that the option price is solely determined by the underlying asset price and time. Therefore, the dependency of the option price on the stock price, (3.5) can be expressed as

$$dB = \left[\mu_S S\frac{\partial B}{\partial S} + \frac{\partial B}{\partial t} + \tfrac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 B}{\partial S^2}\right]dt + \sigma_S S\frac{\partial B}{\partial S}dz. \qquad (3.6)$$

Eqn (3.4) becomes:

$$dV_H = n[\mu_S S dt + \sigma_S S dz] - N[\mu_B B dt + \sigma_B B dz]. \qquad (3.7)$$

In order to construct a riskless portfolio, the dz terms in Eqn (3.7) must cancel out. This can be achieved by choosing n as follows:

$$n = N\frac{\partial B}{\partial S}. \qquad (3.8)$$

Therefore, if we have N call options, we should choose $N\frac{\partial B}{\partial S}$ shares to immunize the risk of writing the calls. In other words, if we sell N call options, we should purchase $N\frac{\partial B}{\partial S}$ shares to make the portfolio riskless. However, the portfolio is only instantaneously riskless; in order to maintain a riskless portfolio, the writer has to rebalance continuously. $\frac{\partial B}{\partial S}$ is called the **delta** and the strategy for maintaining a riskless portfolio using it is called **delta hedging**. We discuss delta hedging further in Section 3.3.

If the portfolio is riskless, then the instantaneous return is r. Substituting (3.6) and (3.8) into (3.7) leads to the Black and Scholes partial differential equation:

$$\tfrac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 B}{\partial S^2} + rS\frac{\partial B}{\partial S} - rB + \frac{\partial B}{\partial t} = 0. \qquad (3.9)$$

The boundary conditions and initial conditions are:

$$B(S,T) = \max(S-X,0)$$

$$B(S,t) \le S \quad \forall t$$

$$B(0,t) = 0 \quad \forall t.$$

The solution to this system is the Black and Scholes Formula.

The Black and Scholes Formula:

$$B = S\Theta(d_1) - Xe^{-r(T-t)}\Theta(d_2) \qquad (3.10)$$

$$d_1 = \frac{\ln(S/X) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \qquad (3.11)$$

$$d_2 = \frac{\ln(S/X) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \qquad (3.12)$$

where

$\Theta(d)$ the cumulative normal distribution function , i.e.

$$\Theta(d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \qquad (3.13)$$

The above analysis only assumes **non-satiation,** i.e. investors prefer more wealth to less, and that the option is neither a dominant[13] nor a dominated security. Explicit assumptions about equilibrium or about investors' preferences are not necessary. The fundamental requirement is the non-existence of arbitrage opportunities. In this sense, the above result is **preference independent** because all assets are perfect substitutes for each other instantaneously and the strategy for maintaining a riskless portfolio is independent of the hedger's attitude to risk.

[13]From Merton[10], Security A is dominant over Security B if, on some known date in the future, the return on A will exceed the return on B for some possible states of the world, and will be at least as large as on B, in all possible states of the world.

3.2 The Leland Model : introduction of transaction costs

The BS option pricing model does not include transaction costs. Leland[9] developed an option pricing model that includes transaction costs[14]. In his model, the hedging errors, including transaction costs, will almost surely approach zero as $\Delta t \to 0$. His model was extended by Neuhaus[11]. Leland[9] and Neuhaus[11] both incorporated transaction costs in their models by modifying the volatility. We present below Leland's[9] modification, which is more closely related to discrete delta hedging than is the Neuhaus[11] modification.

The revised volatility $\hat{\sigma}$ is given by

$$\hat{\sigma} = \sqrt{\sigma^2 \left[1 + \sqrt{\frac{2}{\pi}} \frac{K}{\sigma\sqrt{\Delta t}} \right]} \qquad (3.14)$$

where K[15] is the roundtrip[16] transaction cost expressed as a proportion of trading volume.

We replace σ by $\hat{\sigma}$ in the BS option pricing model when we include transaction costs in the analysis. We define $K = 2\hat{K}$, where $\hat{K}$ is half the roundtrip transaction cost.

3.3 The delta of a call option and delta hedging

Delta and delta hedging are closely related to the BS option pricing model[17]. Delta, D, is the change in option price per unit change in stock price, that is

$$D = \tfrac{\partial B}{\partial S}. \qquad (3.15)$$

Using the BS option pricing model Eqn (3.10), delta is given analytically by

$$D = \Theta(d_1). \qquad (3.16)$$

Following the analysis in Section 3.1, Black and Scholes[1] argue that the writer can realize a riskless portfolio by delta hedging; the writer finds the delta of an option and bases the

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[14]Other option pricing models with transaction costs have been developed by other authors. We used Leland's[9] model because it is applicable to discrete rebalancing and it fits in the framework of the minimax hedging strategy. See Boyle and Emanuel[2], Gilster and Lee[6], Panas[12], Neuhaus[11], and Davis and Norman[5].

[15]In Leland[9], transaction costs varied from $K = 0.0$ to $K = 0.04$.

[16]From Rosenberg[13], a roundtrip trade is defined as any complete transaction made up of a buy followed by a sale of the same stock or vice versa.

[17]The model and its relevance to delta hedging are described in Section 3.1.

number of shares to hold on it. For N, the contracted number of shares per option, and n_t, the number of shares to hold at any time t, under delta hedging,

$$n_t = D_t N \,. \tag{3.17}$$

Because delta changes with time, the hedge portfolio is only instantaneously riskless. He must rebalance continuously to keep the portfolio riskless. Such a portfolio strategy is called a "**delta-neutral**" strategy. However, because he cannot rebalance continuously, he uses discrete delta hedging where he rebalances at discrete intervals of time. With discrete delta hedging, he incurs a **hedging error** which, for an interval of time, is the net position of a hedge portfolio brought about by changes in S_t. The hedging error (HE) for a portfolio of a **written call option and stock held long**, for the interval t to $t+1$ is:

$$HE = N(B_t - B_{t+1}) + n_t(S_{t+1} - S_t). \tag{3.18}$$

Large hedging errors tend to increase the cost of rebalancing the hedge.

4 Minimax hedging strategy

In this section, we develop a strategy to solve the hedging problem which was introduced in Section 2.3. This strategy is based on the concept of a **worst case scenario**, which the hedger specifies in terms of movements in stock price, and it finds a hedge that minimizes the effect of such a scenario[18]. In Section 4.1, we formulate the minimax problem. In Section 4.2, we present the worst case scenario and its two variants. In Section 4.3, we present the hedging error which is the underlying cost to be minimized. In Section 4.4, we present the objective function. In Section 4.5, we define the minimax hedging error. In Section 4.6, we discuss the treatment of transaction costs. In Section 4.7, we present the variants of the minimax hedging strategy. In Section 4.8, we discuss the minimax solution to the problem given in Section 4.1.

[18]In practice, the Black and Scholes assumption of constant instantaneous standard deviation of returns is rarely satisfied. Indeed, a referee has pointed out that the effect of volatility variations is frequently as bad, if not worse, than that of the variations of the underlying variable and neither delta hedging nor the minimax hedging approach in this paper can hedge against these variations. We intend to address this problem in a further paper where we explore extensions of the basic minimax strategy to a multi-period setting.

4.1 Minimax problem formulation

The problem is to minimize an objective function under a worst case scenario. The minimizing variable is n_t and the maximizing variable is S_{t+1}, which is allowed to take any value within predefined bounds.

The minimax problem is

$$\min_{n_t} \max_{S_{t+1}} \quad f(n_t, S_{t+1}) \tag{4.1}$$

$$\text{subject to} \qquad S_l \le S_{t+1} \le S_u \tag{4.2}$$

where $f(n_t, S_{t+1})$ is the objective function, presented in Section 4.4, and $S_l \le S_{t+1} \le S_u$ is either the range defined under Worst Case 1, presented in Section 4.2.1, or under Worst Case 2, presented in Section 4.2.2 .

There are no constraints on n_t, the number of shares to be held at time t: non-negative n_t implies a long position in shares[19]; negative n_t implies a short position in shares[20].

It may be conjectured that the worst case corresponds either to $S_{t+1} = S_u$ or to $S_{t+1} = S_l$, and consequently the minimax problem can be replaced by two simple minimization problems, especially in the case where only one stock is being considered. As minimax is being advocated as a strategy, its value is not diminished even if the conjecture were true. We disprove the conjecture by a counter-example given in Section 6.3 where the worst case value of the objective function corresponds to a mid-range solution.[21] It also includes a further discussion on this topic.

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[19]The hedger bought the shares.

[20]The hedger sold the shares. This situation is possible in the minimax context.

[21]As the emphasis in this paper is a characterization of the properties of the basic minimax strategy applied to single options, we limit the counter-example to a single option. In the more general case of portfolios of options with both long and short positions, a mid-range solution is more common.

4.2 The worst case scenario

4.2.1 Worst Case 1

In Worst Case 1, the hedger defines the worst case in terms of extreme movements in stock price. The range of S_{t+1} has upper and lower bounds that delimit the 95% confidence interval[22] of all possible values of the future stock price, i.e. **two standard deviations** about the expected value of the stock price at time $t+1$. This 95% confidence interval is based on an estimate of the volatility of the stock price and on the assumption that the stock price follows a lognormal distribution function. Worst Case 1 is hereafter referred to as the **95%-Level.**

4.2.2 Worst Case 2

We define the **state** of the option as one of three possibilities:

- in-the-money[23]
- at-the-money[24]
- out-of-the-money[25].

Most of the business in exchange traded options is with respect to options that are at-the-money. The prices of the underlying stocks of these options usually oscillate about the exercise price[26]. That is, they move from sometimes being in-the-money to sometimes being out-of-the-money. In this scenario we focus on movements of the stock price which may result in a switch in the state of the option , i.e. from being in-the-money at time t , to being out-of-the-money at time $t+1$, and vice versa. This switch in the state of the option means that there is the danger of a higher hedging error being incurred from t to $t+1$. In contrast to Worst Case 1, we define the range of S_{t+1} as the range whose upper and lower bounds delimit the possible values of the future stock price within **one and three standard deviations** from the expected value of the stock price at time $t+1$, in the direction of the

[22]We consider the 95% confidence interval as a reasonable range to consider under normal market conditions. Under abnormal market conditions, the 99% confidence interval may be more appropriate.
[23]If the current stock price is greater than the exercise price, then the option is said to be in-the-money.
[24]If the current stock price is equal to the exercise price, then the option is said to be at-the-money.
[25]If the current stock price is lower than the exercise price, then the option is said to be out-of-the-money.
[26]This is the region of greatest elasticity (curvature) and therefore the area about which market makers are mostly concerned. Formally, we are talking about the effects of the option's "gamma".

exercise price X. This means that if $S_t > X$, the relevant range would be on the left side of the distribution of future stock price; if $S_t \le X$, the relevant range would be on the right side. Worst Case 2 is hereafter referred to as the **Abrupt-Change**.

4.3 The hedging error

From Section 3.3, under delta hedging, the hedging error is given by Eqn (3.18). When **actual** values of B_t, S_t, B_{t+1} and S_{t+1} are substituted into Eqn (3.18), we have the actual hedging error under delta hedging. When **actual** values of B_t and S_t and **potential** values of B_{t+1} and S_{t+1} are substituted into Eqn (3.18), we have the potential hedging error under delta hedging. The potential hedging error under delta hedging is the basis of the objective function in the minimax hedging strategy. In minimax, potential S_{t+1} is taken from a predefined range that maximizes the objective function; potential B_{t+1} is the value of the call option based on the pricing model[27] given potential S_{t+1}, i.e. potential $B_{t+1} = B_{t+1}(S_{t+1})$. The minimax strategy minimizes the maximum potential hedging error plus interest payments on borrowed money[28]. In Section 4.5, we define the minimax hedging error and give the definition of actual hedging error and potential hedging error in the context of minimax.

4.4 The objective function

4.4.1 The objective of minimax hedging

In any dynamic hedging strategy, hedging errors are incurred; in order to correct for these errors, the hedge is rebalanced, with the cost of rebalancing being added to the cost of hedging. At time t, the hedger can attempt to minimize the potential hedging error between t and $t+1$. His decision at time t on n_t, the number of shares to hold, affects the actual hedging error between t and $t+1$. The minimax hedging strategy aims to minimize the maximum potential hedging error between t and $t+1$.

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[27] We use the Black and Scholes[1] option pricing model, Eqn (3.10), with a modified volatility, Eqn (3.14).
[28] This is discussed in detail in Section 4.4.2.

4.4.2 The objective function defined

The minimax strategy aims to minimize the potential hedging error by using it as the objective function. In discrete delta hedging, where rebalancing is done at discrete intervals, we expect that the desirable properties of delta hedging given in Section 3.3 will not be observed consistently in time. By minimizing the maximum potential hedging error plus interest payments on borrowed money[29], should the worst case occur, the hedger adopts a cautious strategy. If the worst case occurs, he has effectively minimized its worst effect; if it does not occur, he may incur a hedging error higher than that using delta hedging.

This direct way of minimizing the potential hedging error is based on the no-arbitrage argument of Merton[10] where he considered a portfolio containing an option, the underlying stock and a riskless bond (i.e. riskless in the sense of default) that is suitably chosen such that the aggregate investment in the portfolio is zero. He demonstrated that there is a strategy of finding the mix of option, stock and bond that would ensure that the return on the portfolio would be nonstochastic. Because of the condition of zero aggregate investment, he argued that in order to avoid arbitrage profits, the return on this portfolio must be zero. In the case of a portfolio of written call options, underlying stock and bonds, given Merton's[10] assumptions, the return on this particular portfolio must be zero. We shall work with such a portfolio and we shall call it the "**ideal portfolio**". This "ideal portfolio" is the benchmark we used in defining the objective function. We derive basic properties of the minimax hedging strategy on the basis of a self-financing portfolio; conditional on these results, we add the effect of costs. We will return to this when we discuss Eqn (4.7a) below.

We define $U_1 : \Re^k x \Re^k \rightarrow \Re^1$, $U_2 : \Re^k \rightarrow \Re^k$, $U : \Re^k x \Re^k \rightarrow \Re^{k+1}$, $n_t \in \Re^k$, $S_{t+1} \in \Re^k$ and Q is a $(k+1)*(k+1)$ positive definite diagonal weighting matrix. $U^d \in \Re^{k+1}$ is the vector of desired values for the potential hedging error and the transaction cost terms: we use a desired value of zero, i.e. the desired hedging error is zero[30] and the desired transaction cost is zero.

[29]Refer to Eqn (4.7a) below.

[30]In minimax, we have the opportunity to adopt any desired value. We adopted a desired value of zero because in delta hedging, the expected value of the hedging error is zero.

The objective function is given by

$$f(n_t, S_{t+1}) \;=\; \tfrac{1}{2} < U - U^d, Q(U - U^d) > \qquad (4.4)$$

where

$$n_t = \begin{bmatrix} n_t^1 \\ \vdots \\ n_t^k \end{bmatrix} \quad \text{and} \quad S_{t+1} = \begin{bmatrix} S_{t+1}^1 \\ \vdots \\ S_{t+1}^k \end{bmatrix} \qquad (4.5)$$

$$U(n_t, S_{t+1}) = \begin{bmatrix} U_1(n_t, S_{t+1}) \\ \ldots\ldots\ldots\ldots \\ U_2(n_t) \end{bmatrix} \quad \text{and} \quad U^d = \begin{bmatrix} U^d{}_1 \\ \ldots\ldots \\ U^d{}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \ldots\ldots \\ 0 \end{bmatrix} \qquad (4.6)$$

$$U_1(n_t, S_{t+1}) = \sum_{i=1}^{k} n_t^i (S_{t+1}^i - S_t^i) + \sum_{i=1}^{k} N^i (B_t^i - B_{t+1}^i(S_{t+1}^i)) + \sum_{i=1}^{k} \left(-(n_t^i - n_{t-1}^i) S_t^i + C_{t-1}^i (1 + r\Delta t) \right) r\Delta t \qquad (4.7a)$$

where $C_{t-1}^i = C_{t-2}^i (1 + r\Delta t) - (n_{t-1}^i - n_{t-2}^i) S_{t-1}^i - \hat{K} |(n_{t-1}^i - n_{t-2}^i) S_{t-1}^i|$. (4.7b)

$$U_2(n_t) = \begin{bmatrix} U_2^1(n_t^1) \\ \vdots \\ U_2^k(n_t^k) \end{bmatrix} \qquad (4.8)$$

where[31] $U_2^i(n_t^i) = \hat{K} \left(n_t^i - n_{t-1}^i \right) S_t^i$. (4.9)

We first identify all the variables in Eqns (4.4) to (4.9) and then give the economic interpretation of Eqn (4.7a).

C_{t-1}^i is the cumulative value of **cash inflow minus cash outflow** at time $t-1$. $C_{t-1}^i(1 + r\Delta t)$ is C_{t-1}^i with interest payments. The first term of (4.7b) is the cumulative value of cash inflow minus cash outflow from the previous period with interest payments. The second term is a cash outflow if the $n_{t-1}^i > n_{t-2}^i$; otherwise, it is a cash inflow. The third term is

[31]This is discussed in detail in Section 4.6.

always a cash outflow. We note that C_{t-1}^{i} will normally be a negative number. At time t, C_{t-1}^{i} is a constant: all the variables in (4.7b) have actual values.

Because transaction costs introduce nondifferentiability into the equation, they do not come into the objective function directly as part of U_1. Instead, we introduce a penalty term, U_2^i to represent a penalty for transaction costs for each option i at time t. The treatment of transaction costs is discussed in detail Section 4.6.

In the weighting matrix Q, the weights q_i, $i=1,..,k+1$, which are specified by the hedger, represent his preferences: with a high q_1 he prefers to minimize the potential hedging error that may be incurred from time t to time $t+1$; with a high q_i, $i=2,..,k+1$, he prefers to minimize the penalty term.

For each option i, $i=1,..,k$, $B_{t+1}^{i}(S_{t+1}^{i})$ is determined using Eqns (3.10), (3.11) and (3.12) with the modified volatility estimate Eqn (3.14).

We now give an economic interpretation of (4.7a). U_1 represents the potential hedging error, inclusive of interest payments on borrowed money, between time t and time $t+1$: it comprises the potential shift in the stock position, the potential shift in the option position and the potential interest payment. The first two terms of (4.7a) give the return on a portfolio of written call options and underlying stocks. The third term represents the opportunity cost of money, i.e. the interest payments on borrowed money, because the portfolio is not self-financing. We wish to find the mix of options and stocks that minimizes the deviation of the return on the portfolio, including opportunity cost, from the return on the "ideal portfolio", the value of which is zero, based on Merton's[10] conditions of zero aggregate investment and no-arbitrage.

4.5 The minimax hedging error

In minimax, we distinguish actual from potential hedging error. **Actual hedging error, inclusive of interest payments on borrowed money,** is calculated when **actual** B_t, S_t, B_{t+1} and S_{t+1} are substituted into Eqn (4.7a). **Potential hedging error, inclusive of interest payments on borrowed money,** is calculated when **actual** values of B_t and S_t and **potential** values of B_{t+1} and S_{t+1} are substituted into Eqn (4.7a). Potential S_{t+1} is taken from a predefined range that maximizes the objective function; potential B_{t+1} is the value of the call option based on the pricing model given potential S_{t+1}, i.e. potential $B_{t+1} = B_{t+1}(S_{t+1})$. The minimax strategy minimizes the maximum potential hedging error, inclusive of interest payments on borrowed money.

We define the **minimax hedging error** at time t as

$$\textbf{minimax hedging error} = U_1(n_{t*}, S_{t+1*}). \tag{4.10}$$

The minimax hedging error is the potential hedging error, inclusive of interest payments on borrowed money, given the solution n_t* and $S_{t+1}*$.

4.6 Transaction Costs

In this section, we discuss the treatment of transaction costs in the minimax hedging strategy. The roundtrip transaction cost K is used in valuing the option[32] and $\hat{K}$, with $\hat{K} = \frac{1}{2}K$, is used as part of the cumulative value of cash inflow minus cash outflow and as part of the penalty term in the objective function. In Section 4.6.1, we discuss transaction costs as part of the cumulative value of cash inflow minus cash outflow; in Section 4.6.2, we discuss transaction costs as part of the objective function.

4.6.1 Transaction costs in the cost of hedging

The performance of delta hedging and the variants of minimax is measured by the final cumulative value of cash inflow minus cash outflow at the maturity of the option. After

[32] See Section 3.2.

finding n_t by solving the minimax problem using Eqns (4.4) to (4.9), we can evaluate the **actual** cumulative value of cash inflow minus cash outflow at time t. This is given by

$$C_t^i = C_{t-1}^i(1+r\Delta t) - (n_t^i - n_{t-1}^i)S_t^i - \hat{K}|(n_t^i - n_{t-1}^i)S_t^i| \qquad (4.11)$$

The last term is the transaction cost at time t: **this is always incurred and it is always a cash outflow**. At time $t=0$, the actual cumulative value of cash inflow minus cash outflow includes the option premium which is a cash inflow. This is given by

$$C_0^i = -n_0^i S_0^i + NB_0^i - \hat{K}|n_0^i S_0|. \qquad (4.12)$$

All variants of the minimax hedging strategy will use Eqns (4.11) and (4.12) to compute the actual cumulative value of cash inflow minus cash outflow.

4.6.2 The transaction cost term in the objective function

From Eqns (4.4), (4.8) and (4.9), the transaction cost term (TC) of the objective function can be expressed as

$$TC = \sum_{i=1}^{k} q_i (U_2^i - U^{d_2^i})^2 = \sum_{i=1}^{k} q_i (\hat{K}(n_t^i - n_{t-1}^i)S_t^i)^2. \qquad (4.13)$$

Eqn (4.13) should be interpreted as a penalty term. Because we chose a desired value of zero for U_2^d, the right hand side equality holds. The effect of this term on the solution is dependent on the level of transaction cost $\hat{K}$ and on the weights q_i. We use a uniform weighting system for this study: $q_i = q$, $i = 1,..,k$, where q is held constant.

For low values of $\hat{K}$, we need a high value of q so that the transaction cost term TC is not dominated by the other terms in the objective function. Conversely, for high values of $\hat{K}$, we need a low value of q to ensure that TC does not dominate the objective function. For the simulation and empirical illustration, we set[33] the roundtrip transaction cost to $K = 0.02$ and the weighting coefficient to $q = 100$.

[33]This value is based on simulation results, reported in Howe[8], showing the variation of K with q.

4.7 The variants of the minimax hedging strategy

We consider delta hedging[34] and 5 variants of minimax, given as variants A, B, B*, C, and D, where a variant has a specific objective function and worst case scenario. Transaction costs are included in computing the cumulative value of cash inflow minus cash outflow for all the strategies considered.

Table 4.7. The hedging strategies to be used in the empirical illustration and simulation study.

Code	Strategy	Objective Function	Condition of S_{t+1}	Transaction Costs in Obj. Function
	Delta	delta neutrality	n.a	No
A	Minimax	potential hedging error	95% Level	No
B	Minimax	potential hedging error	Abrupt Change	No
B*	(Described in the next paragraph.)			
C	As **A**	As **A**	As **A**	Yes
D	As **B**	As **B**	As **B**	Yes

The version of **B**, **B***, is a weighted version in which the minimax recommendation on the number of shares to hold is weighted by a factor ranging from 0 to 1 that represents the hedger's assessment of the information contained in changes in the underlying stock.[35]

4.8 The minimax solution

The solution is obtained using an algorithm[36] to solve problem (4.1)-(4.2). The algorithm is based on generating successive directions of descent for $f(n,S)$ in n, while ensuring that the direction chosen maximizes $f(n,S)$ with respect to S. The direction chosen is therefore one that iteratively progresses towards the minimax solution.

Because the hedge recommendation under minimax is different from the hedge recommendation under delta hedging, in the Black and Scholes[1] world, the hedge

[34]Delta with gamma hedging is not considered here.
[35]This is a heuristic method of adjusting the hedge recommendations. The method is given in the Appendix.
[36]Further details of the algorithm and numerical experiments are discussed in Howe[8].

recommendation under minimax is suboptimal. In the minimax hedging strategy, for any fixed n_t, we determine S_{t+1}, from the predefined range $S_l \leq S_{t+1} \leq S_u$, that maximizes the hedging error. We therefore can identify theoretically all the maxima corresponding to all the possible values of n_t. The strategy calculates the n_t that minimizes over these maxima. Although the number of shares n_t introduces some risk into the portfolio because it is not the same as the hedge recommendation under delta hedging, n_t ensures that if the *actual* S_{t+1}, as opposed to the minimax value, falls within the range $S_l \leq S_{t+1} \leq S_u$, the absolute value of the *actual* hedging error, inclusive of interest payments on borrowed money, will not be worse (higher) than the absolute value of the minimax hedging error. This is the **minimax robustness property**. The n_t value thus computed results in a robust strategy that is non-inferior in performance for any stock price within the predefined range.

Given the 95%-Level as the worst case scenario, $S_l \leq S_{t+1} \leq S_u$, the minimax algorithm ensures that either

- n_t is chosen such that, for extreme point maximizers, the objective function value is the same for both upper and lower limits, or
- the objective function value corresponds to a worst case price that is in the middle of the range.

Given the Abrupt-Change as the worst case scenario, $S_l \leq S_{t+1} \leq S_u$, the minimax algorithm ensures that either

- n_t is chosen such that, for extreme point maximizers, the objective function value for the upper limit is as close as possible to the value for the lower limit, or
- the objective function value corresponds to a worst case price that is in the middle of the range.

In all cases, it can be shown that the chosen n_t places an upper bound on the absolute value of the hedging error that can be incurred for any price in the given range.

5 Simulation

In this section we describe the simulation of the performance[37] of the minimax variants against that of delta hedging[38] when they are used to hedge the risk of writing a European call

[37]Defined in Section 5.6.
[38]A hedging strategy that we shall use as a benchmark in the empirical illustration and simulation study.

option[39]. Options with their underlying stock series are generated and then categorized under five general groups of options. Hypotheses on the performance of minimax and delta hedging are tested on an intra-group and inter-group basis.

5.1 Objective of the simulation

The objective of the simulation is to identify which variants, for which groups of options, outperform[40] delta hedging, and so establish the characteristics of options, in terms of crossovers and abrupt changes[41] in the price of the underlying stock, for which the different variants are particularly suited.

5.2 Generation[42] of the full data set for the simulation

We used 5 volatility levels, namely, 0.20, 0.30, 0.40, 0.50 and 0.60. Holding the volatility constant, 1250 sets of option and stock time series were generated which were then screened based on a selection procedure described in Section 5.3. Selection is based on whether a particular set of option and stock time series falls within any of the specified **Groups** identified below.

5.3 Groups

For all sets having the same volatility, each set is placed into one of 5 option groups. These groups are defined based on two events : the **crossover** and the **abrupt change**.

- A **crossover** is an event such that $S_t \leq X$ and $S_{t+1} > X$.
- An **abrupt change** is an event such that $S_{t+1} \in \mathit{Abrupt\ Change}$[43].

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[39]In this paper we focus on European call options. Although call options have unlimited upside potential while puts have a maximum possible profit, because of the European Put-Call parity theorem, we believe that the essence of our results will not change if we investigate European calls. However, the early exercise feature of American puts does lead to a significantly more complex analysis which we intend to address in a further paper.
[40]Defined in Section 5.6.
[41]Crossovers and abrupt changes are defined in Section 5.3.
[42]Details of the data generation can be found in Howe[8].
[43]See Section 4.2.2.

Table 5.3a The idealized groups.

	Crossovers	Abrupt Changes
Group 1:	*None*	*None*
Group 2:	*Several*	*None*
Group 3:	*None*	*Several*
Group 4:	*Several*	*Several*
Group 5:	*Few*	*Few*

The allocation of a set (an option/stock time series) to any of the above groups is determined by the distribution of total number of crossovers per series, the distribution of total number of abrupt changes per series and the distribution of total number of simultaneous crossovers and abrupt changes per series.

<u>The method of allocation of a stock price series to a group</u>

Given the exercise price and the Abrupt-Change range defined in Section 4, the following numbers are known for each stock series:

J_X total number of crossovers

J_{AC} total number of abrupt changes

$J_{X\&AC}$ total number of simultaneous crossovers and abrupt changes

The distributions of J_X, J_{AC} and $J_{X\&AC}$, each based on a sample of 1250 stock series (given a constant volatility σ) are ascertained. These distributions are used to allocate a time series of a specified volatility into one of the 5 option groups. Given the means ,μ_{J_X}, $\mu_{J_{AC}}$, $\mu_{J_{X\&AC}}$, and standard deviations ,σ_{J_X}, $\sigma_{J_{AC}}$, $\sigma_{J_{X\&AC}}$, of these distributions, a stock price series is allocated to a group if it has the properties given below for that group:

Group 1:

$$J_X < \mu_{J_X} - a\sigma_{J_X}, \quad J_{AC} < \mu_{J_{AC}} - b\sigma_{J_{AC}}, \text{ and } \quad J_{X\&AC} < \mu_{J_{X\&AC}} - c\sigma_{J_{X\&AC}}$$

Group 2:

$$J_X > \mu_{J_X} + a\sigma_{J_X}, \quad J_{AC} < \mu_{J_{AC}} - b\sigma_{J_{AC}}, \text{and} \quad J_{X\&AC} < \mu_{J_{X\&AC}} - c\sigma_{J_{X\&AC}}$$

Group 3:

$$J_X < \mu_{J_X} - a\sigma_{J_X}, \quad J_{AC} > \mu_{J_{AC}} + b\sigma_{J_{AC}}, \text{and} \quad J_{X\&AC} < \mu_{J_{X\&AC}} - c\sigma_{J_{X\&AC}}$$

Group 4:

$$J_{X\&AC} > \mu_{J_{X\&AC}} + c\sigma_{J_{X\&AC}}$$

Group 5:

$$\mu_{J_X} - a\sigma_{J_X} \le J_X \le \mu_{J_X} + a\sigma_{J_X}, \quad \text{and} \quad \mu_{J_{AC}} + b\sigma_{J_{AC}} \le J_{AC} \le \mu_{J_{AC}} + b\sigma_{J_{AC}}$$

In the simulation, the coefficients of the standard deviations are: $a=1$, $b=1$ and $c=2$. We chose these values in order to ensure that the groups are sufficiently differentiated and that there are a large number of elements within a group. We set $c=2$ in order to ensure that all groups, except Group 4, have a low incidence of simultaneous crossovers and abrupt changes. Each group has a total of 250 options, representing 50 options for each of the 5 volatility levels. We refer to one simulation run for one option as a **replication**. A total of 1250 replications were done in the simulation.

In Table 5.3b, very roughly the allocation has generated actual groups of time series of the underlying stock with the following characteristics.

Table 5.3b The actual groups.

	Crossovers	Abrupt Changes
Group 1:	*Very few*	*Very few*
Group 2:	*Several*	*Very few*
Group 3:	*Very few*	*Several*
Group 4:	*Several*	*Several*
Group 5:	*Few*	*Few*

5.4 From set-up to wind-down

In this section, we discuss the mechanics of hedging, from setting up to the winding down of the hedge. We consider delta and five minimax hedging strategies. These strategies involve rebalancing the hedge at uniform intervals of time; in the simulation, the interval is one day. Daily data include the stock price, the option price, the risk-free interest rate, the time to maturity and the volatility. The risk-free interest rate[44] is preset to 0.10. Dividends are excluded from the analysis and N, the number of contracted shares, is 100.

5.4.1 Setting up the hedge

At time 0 each strategy holds the same number of shares (n_0) based on delta , and the same initial cumulative value of cash inflow minus cash outflow given by Eqn (4.12).

5.4.2 Rebalancing the hedge

Every day through to the maturity date T, the hedge is rebalanced according to the strategies' recommendation. The trajectory of the number of shares held at time t , n_t , varies with the hedging strategy used. The actual cumulative value of cash inflow minus cash outflow at time t is given by Eqn (4.11).

5.4.3 Winding down the hedge

At time T, if the holder does not exercise his option to buy the shares, each strategy disposes of its portfolio in the same way: selling any shares held, or buying any shares sold short, at time $T-1$ at S_T.

5.5 Summary of simulation results

We give the results of this simulation mainly in terms of the performance and the relative performance of the minimax strategy. We define the **performance** of a strategy[45]as the final cumulative value[46] of cash inflow minus cash outflow in using that strategy on an option, standardized[47] as a percentage of the notional[48] contract value of that option. We define

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[44]This is the continuous rate.
[45] A strategy may be either a minimax variant or delta hedging.
[46]This is calculated after winding down the hedge.
[47]Following Samuelson[14], all cells in the following table, Table 5.5.1, have been standardized by dividing the original profit by the exercise price. In the simulation study, we set the exercise price at X=1000; this makes the original profit effectively standardized. In the empirical study, because of the differences in exercise prices, the standardization becomes relevant.

relative performance of a minimax variant as the performance of that variant minus the performance of delta hedging (DH). In the following commentary on this simulation, we say that a variant **outperforms** DH if its performance is higher than that of DH and, for any group of options, the difference for that group is significant at the 1% level.

In this section we discuss the results of the simulation: Section 5.5.1 compares delta hedging with the variants of minimax; Section 5.5.2 presents the high performing variants. In this paper, we limit the presentation to the major properties of the minimax variants.

5.5.1 Performance of delta hedging and minimax

In Table 5.5.1, we summarize the simulation results. The performance of a strategy is averaged over all options in a group. Each cell in the table gives the performance of a particular strategy, and the relative performance of that strategy.

Table 5.5.1 is accompanied by Figure 5.5.1 which gives a graphical representation of the relative performance of the minimax variants. The x-axis gives the average distance of a particular group from the exercise price. Each group consists of 250 options or stock price series, corresponding to 250 replications; each stock price series consists of 190 daily prices, corresponding to 38 weeks, with 5 trading days per week, for a 9-month option. The average squared deviation from the exercise price over 47500 (=190*250) prices was calculated; the square root gives the "average distance from the exercise price" for that group.

$$\text{average distance from the exercise price} = \sqrt{\frac{\sum_{i=1}^{250}\sum_{l=1}^{190}(S_l^i - X)^2}{190*250}} \qquad (5.1)$$

where i refers to a particular time series in a group and l refers to a particular price. The "average distance from the exercise price" represents the degree of **moneyness**[49] of the options in that group.

[48]The notional value of the contract is the number of contracted shares multiplied by the exercise price. The total is the summation over all notional values.

[49]The amount that the option is in-the-money or out-of-the-money.

In Figure 5.5.1, the two groups closest to the origin consist of options which are at-the-money. The group farthest from the origin has the largest distance from the exercise price which indicates that it consists of options which are either deep-in-the-money or deep-out-of-the-money.

Figure 5.5.1. Relationship between relative performance and average distance from the exercise price.

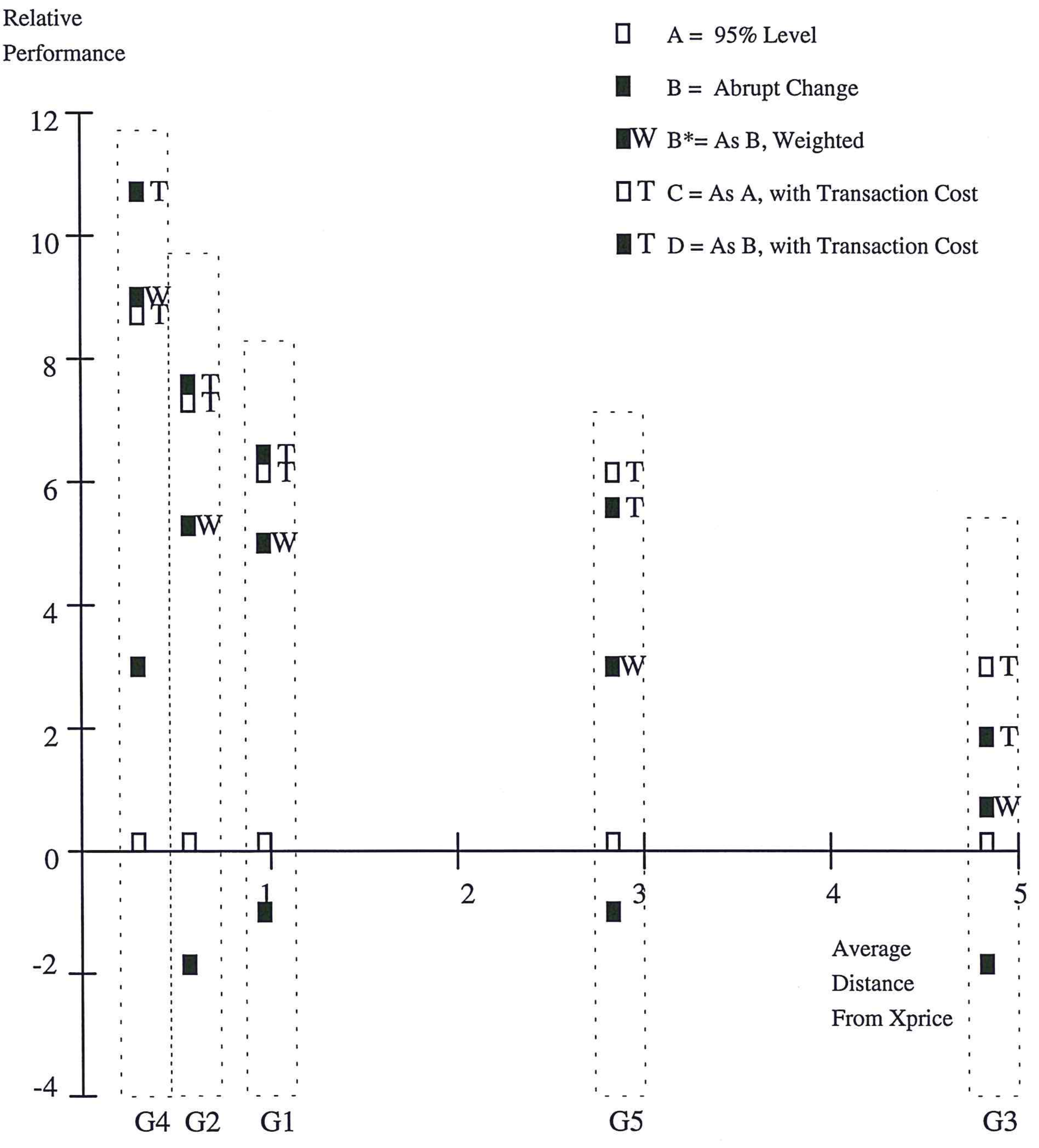

Table 5.5.1. Performance and relative performance of the different strategies.
Units: Percentage points of final cumulative value of cash inflow minus cash outflow divided by the notional value of the contract.

	Delta Hedging	A Potential hedge error 95%level	B A.C.	B* As B Weighted	C As A with TC	D As B with TC
Group 1 *(no crossovers, no abrupt changes)*						
Performance	0.7	0.8	-0.3	5.4	6.8	6.9
Relative Performance	0.0	0.1	-1.0	4.7	6.1	6.2
Group 2 *(crossovers only)*						
Performance	0.7	0.7	-1.2	6.2	7.7	8.1
Relative Performance	0.0	0.0	-1.9	5.5	7.0	7.4
Group 3 *(abrupt changes only)*						
Performance	-3.3	-3.3	-5.4	-2.6	-0.5	-1.4
Relative Performance	0.0	0.0	-2.1	0.7	2.8	1.9
Group 4 *(crossovers and abrupt changes)*						
Performance	-0.8	-0.6	2.4	8.1	7.7	9.9
Relative Performance	0.0	0.2	3.2	8.9	8.5	10.7
Group 5 *(general case)*						
Performance	-4.0	-4.0	-5.1	-0.7	2.0	1.4
Relative Performance	0.0	0.0	-1.1	3.3	6.0	5.4

Hypothesis testing has shown that variants B*, C and D are robust: they perform better than delta hedging for the group for which they are designed to perform best and in general they do not perform worse than delta hedging. Variants B*, C and D are strategies designed to constrain transaction costs: Variants C and D both have a transaction cost term in their objective function which is then minimized; Variant B* uses a heuristic method of constraining transaction costs based on short-term trends and distance from the exercise price.

Statistical hypothesis testing has also shown that:

1. Variants B*, C and D perform better than delta hedging for at-the-money options; in particular, when there are crossovers as well as abrupt changes.
2. The Abrupt-Change variants, B and D, are not suitable for deeply-in-the-money or deeply-out-of-the-money options even if the corresponding price series are characterized by a large number of abrupt changes.
3. The Abrupt-Change variants, B and D, are more suitable than their 95%-Level counterparts, A and C, for at-the-money options when the price series are characterized by a large number of abrupt changes.

4. The transaction-costs variants[50], C and D, perform better than their no-transaction-costs counterparts, A and B. The inclusion of a transaction cost term in the objective functions of variants C and D provides more cautious hedge recommendations and this helps constrain trading costs.

5. The weighted minimax variant B* performed better than its non-weighted variant, B. This implies that the proposed weighting system helps provide hedge recommendations that are conditioned by recent stock price levels. This conditioning leads to cautious trading and helps constrain trading costs.

In the next section, we present the high performing variants of minimax. These are C, D and B*, the variants that explicitly constrained transaction costs.

5.5.2 The high performing variants of minimax

In Figures 5.5.2.1, 5.5.2.2 and 5.5.2.3, respectively, we show the variation with volatility in the relative performance of Variants C, D and B*. Variant C uses the 95%-Level to define the worst case scenario. Variant D uses the Abrupt-Change to define its worst case scenario. Variant B* uses the Abrupt-Change to define its worst case scenario; it also uses a heuristic weighting system to give weighted hedge recommendations. In the figures, for each of the five volatility levels, a regression line has been fitted showing the relationship between relative performance and degree of moneyness which is represented by the average distance from the exercise price. All five regression lines have a negative slope. The analysis also

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[50]We repeat Table 4.7 for ease of reference.

Table 4.7. The hedging strategies to be used in the empirical and simulation studies.

Code	Strategy	Objective Function	Condition of S_{t+1}	Transaction Costs in Obj. Function
	Delta	delta neutrality	n.a	No
A	Minimax	potential hedging error	95% Level	No
B	Minimax	potential hedging error	Abrupt Change	No
B*	(A weighted version of B)			
C	As **A**	As **A**	As **A**	Yes
D	As **B**	As **B**	As **B**	Yes

show that the higher the volatility, the more robust the performance of the minimax strategy. This is exhibited in the figures by the decreasing slope with increasing volatility. For volatility levels 0.2 and 0.3, the performances of the three variants are significantly better than that of delta hedging for distances relatively close to the exercise price. For volatility levels 0.5 and 0.6, the performances are significantly better than that of delta hedging for a wider range of distances from the exercise price.

Comparing the figures, we see a trend towards increasing parallelism between the regression lines from Figure 5.5.2.1 to Figure 5.5.2.3. This can be interpreted as follows. In Figure 5.5.2.3, the lines have a relatively high degree of parallelism which shows that the higher the volatility, the better the performance. However, variant B* starts to perform badly when the lines cross the x-axis. The point of intersection with the x-axis for each of the 5 volatility levels are closer to zero when compared with the other figures (Figure 5.5.2.1 and Figure 5.5.2.2). This implies that variant B* performs best for a limited degree of moneyness.

In Figure 5.5.2.1, the lines have a relatively low degree of parallelism which shows that there is a trade-off between degree of moneyness and volatility. When the average distance is close to zero, variant C performed well for all volatility levels but there is a slightly better performance for lower volatilities. The inverse relationship between relative performance and volatility can be observed when the average distance is large. This implies that variant C performs for a wide degree of moneyness.

In Figure 5.5.2.2, the parallelism is intermediate between those in Figures 5.5.2.1 and 5.5.2.3. This strategy is the best performer of the three when the distance is close to zero. This implies that variant D is the most suitable strategy for at-the-money options. When the distance is large, variant D performs worse than variant C. This implies that variant D is not suitable for deeply-out or deeply-in-the money.

Figure 5.5.2.1

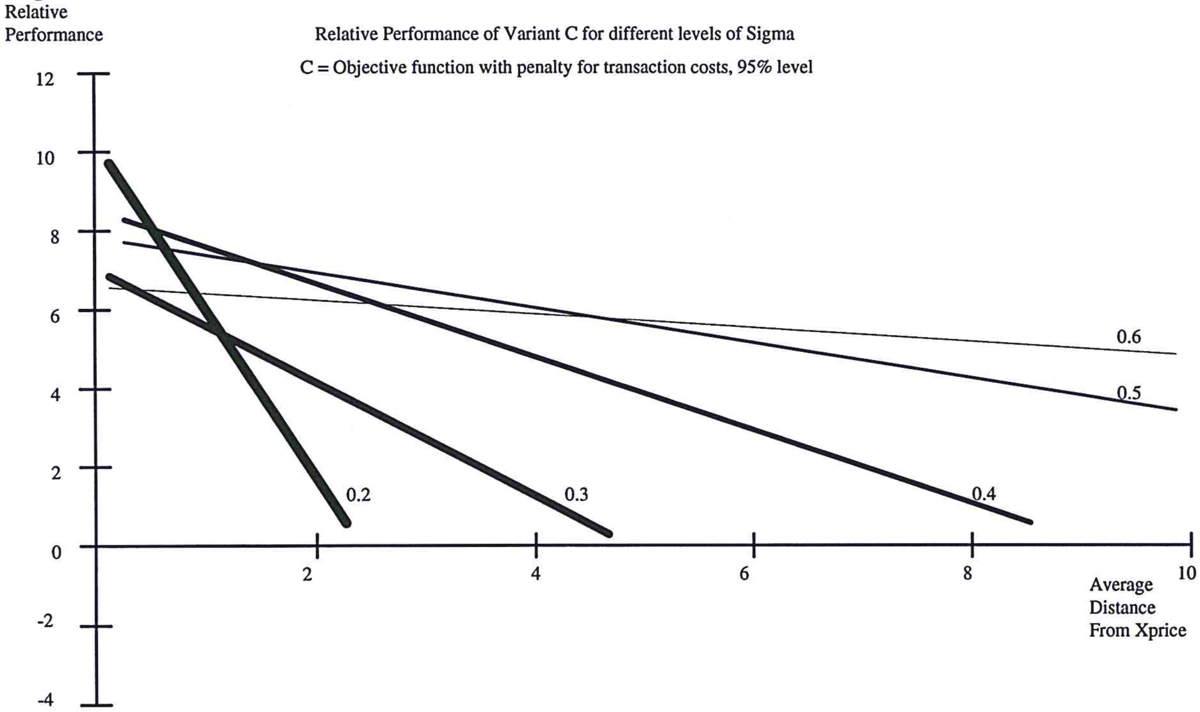


Figure 5.5.2.2

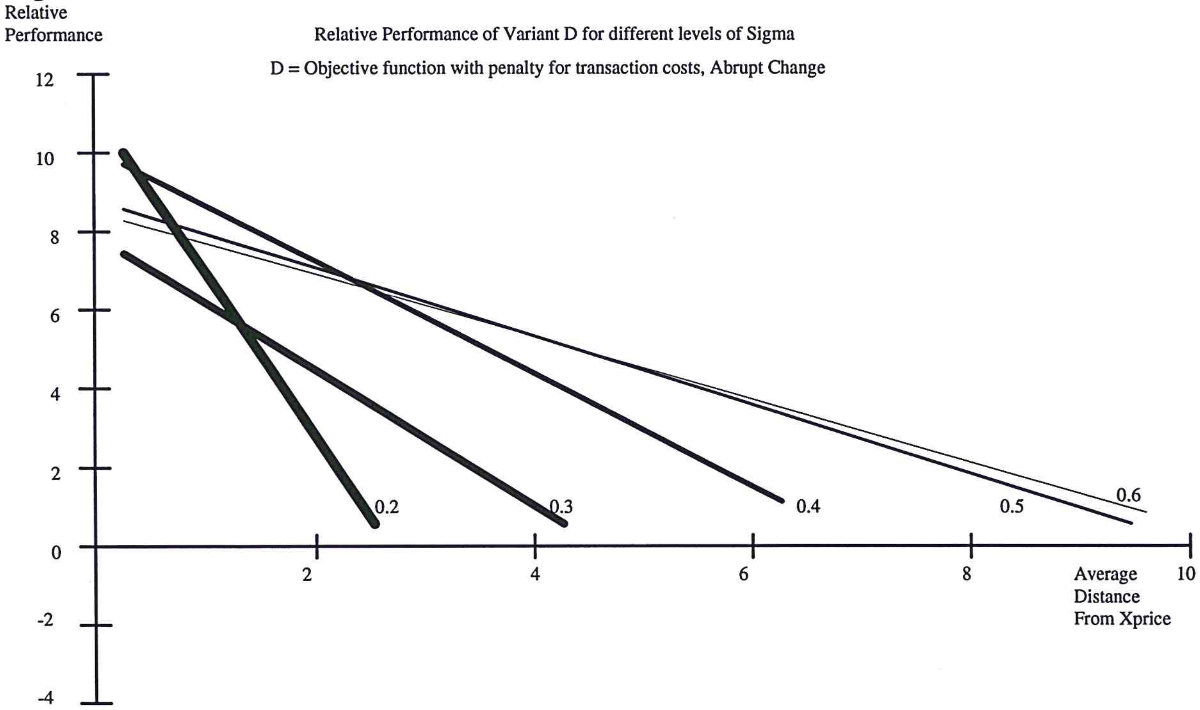
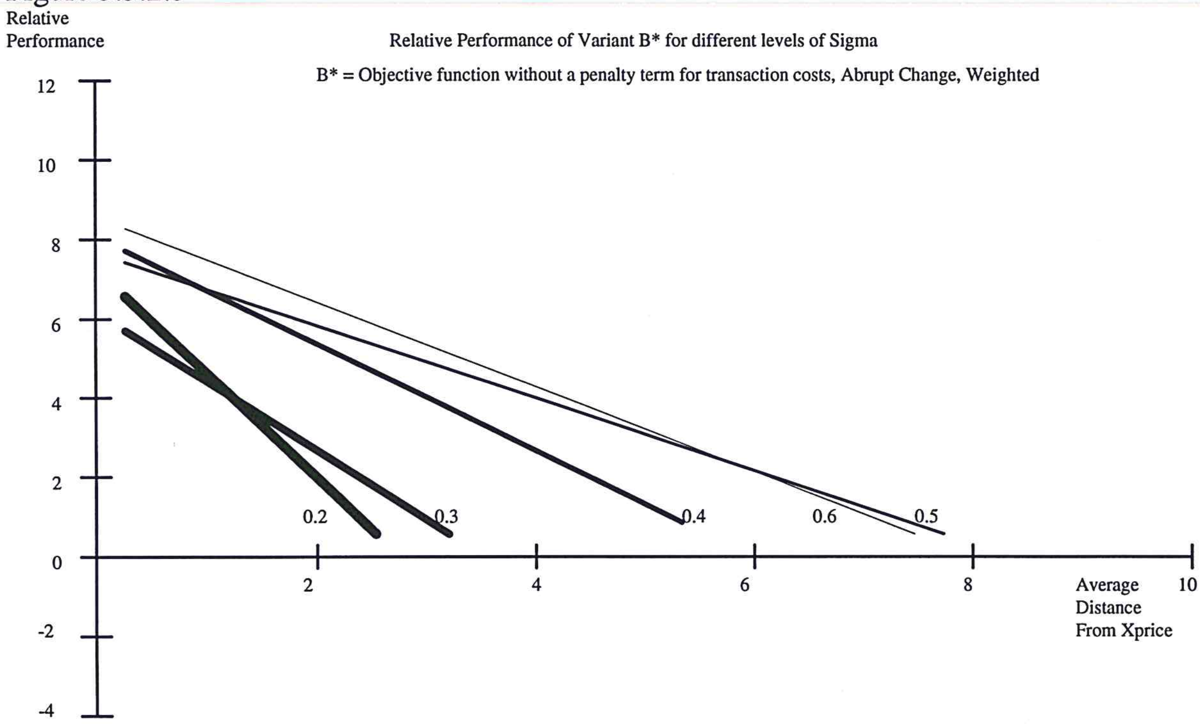


Figure 5.5.2.3



6 Illustrative hedging problem: a limited empirical study

In this section, we consider the problem of hedging the risk of writing a European call option[51] for a number of options available in the UK options market. We present a limited empirical study of 30 options: this is done as an illustration of the performance of the minimax hedging strategy when applied to real data. Actual market prices were used for both the stock and the option. The writer of the call incurs a potential liability in the case of exercise of the option by the buyer, and receives a premium; he is obliged to offer the buyer N shares of the stock at the exercise price, X, at the exercise date. In Section 6.1, we describe the mechanics of the hedging from set-up to wind-down. In Section 6.2, we present a summary of the results of the limited empirical study. In Section 6.3, we discuss the case where a particular option's worst case hedging error corresponds to a mid-range solution.

[51]Since $k = 1$, the superscript i will be dropped.

6.1 From set-up to wind-down

In Section 5.4, we discussed the mechanics of hedging used in the simulation, from setting up to winding down of the hedge. For the illustration, we use the same mechanics with a few changes in the parameters. The strategies involve rebalancing the hedge at uniform intervals of time; in the illustration, the interval is one week, i.e. 5 trading days. Weekly data[52] include the market price of the stock, the market price of the option, the risk-free interest rate, the time to maturity and an estimate of the volatility of the stock price. The estimates of volatility used in the illustration are based on the most recent 100 days of stock price movement and on implied volatility[53]. The risk-free interest rate is based on the discount rate[54] provided by a Treasury Bill that expires at about the same time as the option. In the illustration, N, the number of contracted shares, is 1000.

6.2 The hedging strategies applied to 30 options : Summary of results

The strategies defined in Section 4.7 were used to hedge the risk of writing one call option; 30 options were used in the study. Table 6.1 presents the performance of the different strategies. Table 6.2 presents the relative performance. The final row of Table 6.2 indicates that the relative performance of four of the five variants of minimax averaged over 30 options is better than that of delta hedging, and that for variant **B**, for which the relative performance is worse, the weighted version **B*** performs better than delta hedging. Under variant **B**, the recommended change in the number of shares to hold is highly volatile, and when transaction costs are included, the cumulative cost of this variant can get very high. The effect of the weighting is to dampen this volatility, and so reduce transaction costs. The final row of Table 6.2 shows marked differences in the relative performance of different minimax variants. The columns of Table 6.2 are generally highly variable. The variants of minimax are essentially specific to different classes of options. This implies that as options change

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[52]Data were supplied by Datastream, International. All prices are mid-prices. The stock price series used were Datastream's Adjusted Stock Price Series : these are the original stock price series with dividend adjustments.
[53]Implied volatility is the volatility implied by an option price observed in the market.
[54]The discount rate was used to find the current value of the Treasury Bill based on a face value of £1 and the time to maturity. The risk-free interest rate is the continuous rate that discounts the face value to the computed current value.

their moneyness, then the minimax tactic must also change. In other words, the minimax tactic to be implemented is a function of moneyness.

Because we have not established that the universe of calls from which the sample of 30 was selected is representative of the universe of all calls, the result that minimax performs better than delta hedging is not necessarily generalizable. However, because the selection criteria for the universe from which we selected the 30 calls do not include requirements on the number or size of **abrupt changes**[55], nor the number of **crossovers**[56], it is possible that minimax would perform (slightly) better than delta hedging for the universe of all calls written for UK stocks.

[55]An abrupt change is a fairly large movement in stock price. See Section 5.3.
[56]A crossover is a switch in the state of the option. See Section 5.3.

Table 6.1. Performance of the different strategies[57].

Units: Percentage points of final cumulative value of cash inflow minus cash outflow divided by the notional value of the contract.

	Xprice	Delta Hedging	**A**	**B**	**B***	**C**	**D**
Tesco	220	-4.0	-4.4	-15.6	-9.7	-2.6	-6.6
Boots	280	-1.6	-1.6	-8.2	-2.3	**0.1**	**0.0**
Sainsbury	360	-5.1	-5.2	-18.2	-8.4	-2.8	-8.3
BAir	220	-6.2	-6.4	-10.2	-7.0	-4.7	-3.6
GEC	180	-5.0	-5.2	-18.9	-14.2	-4.8	-9.8
Allied	650	-3.7	-3.8	-11.8	-2.6	-0.2	**0.0**
Ladbroke 1	240	-10.7	-10.8	-17.3	-6.5	-13.1	-3.5
Ladbroke 2	260	-7.5	-7.6	-15.5	-13.8	-9.8	-12.3
Cadbury 1	420	-0.3	0.0	-5.2	**0.5**	**2.5**	**2.8**
Cadbury 2	460	-3.9	-3.2	-5.7	**2.1**	**3.9**	**5.4**
Hanson 1	200	-6.1	-5.5	-5.1	**2.9**	-3.7	**3.8**
Hanson 2	180	-4.9	-5.1	-11.2	-9.2	-3.8	-5.3
P & O	600	-6.1	-5.4	-6.1	**0.6**	-1.6	**4.1**
Vodafone 1	390	-5.7	-5.8	-15.8	-11.4	-4.3	-9.0
Vodafone 2	360	-6.7	-6.7	-8.6	-3.8	-4.8	-0.9
Prudential 1	240	-10.0	-8.5	-7.4	**3.1**	-2.4	**4.2**
Prudential 2	220	-10.4	-9.9	-14.5	-0.7	-5.1	**0.9**
Marks&Spencer	300	-4.9	-4.4	-1.8	**4.0**	-3.1	**7.0**
Shell 1	460	-4.8	-4.6	-2.0	**2.1**	-2.5	**2.8**
Shell 2	500	-6.6	-6.6	-18.1	-9.6	-3.3	-5.0
Eurotunnel	390	-4.4	-3.7	-3.4	**0.4**	**0.0**	**3.3**
Glaxo	800	-3.0	-2.4	-0.3	**2.8**	-0.2	**4.2**
Guinness 1	500	-8.1	-7.4	-17.9	-7.4	-3.8	-2.5
Guinness 2	550	-8.5	-7.7	-2.4	**2.7**	-3.3	**2.2**
Forte	200	-8.4	-8.6	-8.8	-7.0	-11.3	-6.7
Thames Water	330	-11.6	-11.8	-8.6	-10.6	-10.2	-9.6
U. Biscuits	390	-3.4	-3.7	-5.4	-2.0	-6.2	-0.7
BTelecom 1	330	-5.8	-5.8	-10.2	-4.1	-0.6	**0.4**
BTelecom 2	360	-6.5	-5.9	-19.1	-9.3	**1.4**	-1.3
Wellcome	900	-6.4	-6.1	-8.4	-0.2	-2.8	**1.2**
Average		-6.0	-5.8	-10.1	-3.9	-3.4	-1.5

[57] We repeat Table 4.7 for ease of reference.

Table 4.7. The hedging strategies to be used in the empirical and simulation studies.

Code	**Strategy**	**Objective Function**	**Condition of S_{t+1}**	**Transaction Costs in Obj. Function**
	Delta	delta neutrality	n.a	No
A	Minimax	potential hedging error	95% Level	No
B	Minimax	potential hedging error	Abrupt Change	No
B*	(A weighted version of B.)			
C	As **A**	As **A**	As **A**	Yes
D	As **B**	As **B**	As **B**	Yes

Table 6.2. Relative performance of the different strategies[58]. (Derived from Table 6.1)

Units: Percentage points of final cumulative value of cash inflow minus cash outflow divided by the notional value of the contract

	Xprice	A	B	B*	C	D
Tesco	220	-0.4	-11.6	-5.7	**1.4**	-2.6
Boots	280	-0.1	-6.7	-0.7	**1.7**	**1.5**
Sainsbury	360	-0.1	-13.1	-3.3	**2.4**	-3.2
BAir	220	-0.1	-3.9	-0.8	**1.5**	**2.6**
GEC	180	-0.2	-14.0	-9.2	**0.2**	-4.8
Allied	650	-0.1	-8.1	**1.1**	**3.5**	**3.7**
Ladbroke 1	240	-0.1	-6.5	**4.2**	-2.4	**7.2**
Ladbroke 2	260	-0.2	-8.0	-6.3	-2.3	-4.9
Cadbury 1	420	-0.3	-5.6	**0.2**	**2.2**	**2.5**
Cadbury 2	460	**0.8**	-1.7	**6.0**	**7.8**	**9.3**
Hanson 1	200	**0.6**	**1.0**	**9.0**	**2.5**	**9.9**
Hanson 2	180	-0.2	-6.2	-4.3	**1.2**	-0.4
P & O	600	**0.7**	**0.0**	**6.6**	**4.5**	**5.3**
Vodafone 1	390	-0.1	-10.1	-5.7	**1.5**	-3.3
Vodafone 2	360	0.0	-1.9	**2.8**	**1.8**	**5.8**
Prudential 1	240	**1.5**	**2.6**	**13.1**	**7.6**	**14.2**
Prudential 2	220	**0.6**	-4.1	**9.7**	**5.3**	**11.3**
Marks&Spencer	300	**0.4**	**3.1**	**8.9**	**1.7**	**11.9**
Shell 1	460	**0.2**	**2.8**	**6.9**	**2.3**	**7.6**
Shell 2	500	**0.1**	-11.5	-3.0	**3.3**	**1.6**
Eurotunnel	390	**0.7**	**1.0**	**4.8**	**4.4**	**6.2**
Glaxo	800	**0.6**	**2.7**	**8.4**	**2.8**	**7.2**
Guinness 1	500	**0.7**	-9.8	**0.6**	**4.2**	**5.5**
Guinness 2	550	**0.9**	**6.2**	**11.3**	**5.2**	**10.7**
Forte	200	-0.2	-0.4	**1.4**	-2.9	**1.7**
Thames Water	330	-0.1	**3.0**	**1.0**	**1.4**	**2.0**
U. Biscuits	390	-0.2	-2.0	**1.5**	-2.8	**2.7**
BTelecom 1	330	**0.1**	-4.4	**1.8**	**5.3**	**6.3**
BTelecom 2	360	**0.5**	-12.6	-2.8	**7.8**	**5.2**
Wellcome	900	**0.4**	-1.9	**6.2**	**3.7**	**7.6**
Average		**0.2**	-4.1	**2.1**	**2.6**	**4.5**

[58]We repeat Table 4.7 for ease of reference.

Table 4.7. The hedging strategies to be used in the empirical and simulation studies.

Code	**Strategy**	**Objective Function**	**Condition of S_{t+1}**	**Transaction Costs in Obj. Function**
	Delta	delta neutrality	n.a	No
A	Minimax	potential hedging error	95% Level	No
B	Minimax	potential hedging error	Abrupt Change	No
B*	(A weighted version of B.)			
C	As **A**	As **A**	As **A**	Yes
D	As **B**	As **B**	As **B**	Yes

6.3 An example of a mid-range solution

In the hedging problem, the objective is to protect the value of the portfolio. There are two ways of doing this: buy put options or sell call options. We consider selling calls. In the process of hedging, through the selling of call options, the profit potential of the portfolio is limited. In delta hedging, the expected value of the hedging error is zero. Therefore, in minimax, we adopted[59] a desired value of zero. Because of this, the minimax strategy, just like delta hedging, gives hedge recommendations aimed at maintaining zero hedging error: a negative hedging error is highly undesirable and a positive hedging error, although desirable, is inconsistent with the philosophy of hedging.

In solving the minimax problem (4.1), we find that in the simple case of a single option, a mid-range solution corresponds to a positive hedging error. In the more general case of portfolios of options with both long and short positions, a mid-range solution does not necessarily correspond to a positive hedging error. In fact, it is more common to find a mid-range solution, for these types of portfolios, that corresponds to a negative hedging error.

For the single option case, an example is given by hedging a written call option on Cadbury Schweppes, the August 1992 option with an exercise price of 460 pence; the contracted number of shares is 1000. At the rebalancing date, March 11 1992, the stock price was 457 pence and the option price was 38 pence. The minimax hedging error found by the minimax strategy (Variant A) from the range defined by the 95%-Level (434 to 481 pence) corresponds to a worst-case price of 458 pence.

For the lower limit, $S_{t+1} = 434$, the objective function value was 32800 and the potential hedging error was -1780 pence. For the upper limit, $S_{t+1} = 481$, the objective function value was 44300 and the potential hedging error was -2072 pence. For the mid-range solution, $S_{t+1} = 458$, the objective function value was 104300 and the potential hedging error was 3197 pence (a positive number), which is also the minimax hedging error. The minimax

[59]In minimax, we have the opportunity to adopt any desired value.

strategy guarantees that, provided that the actual S_{t+1} is within the range (434 to 481 pence), the absolute value of the actual hedging error will be less than the absolute value of the minimax hedging error. At time $t+1$, March 18, 1992, the option price was 24 pence and the stock price was 433 pence, which is just below the lower limit. The absolute value of the actual hedging error was 1920; this is less than the absolute value of the minimax hedging error calculated at time t.

This example shows a mid-range solution where, as noted above, the minimax hedging error is positive. In the more general case of portfolios of options, with both long and short positions, a mid-range solution corresponding to a negative minimax hedging error is more common.

7 Conclusion

We have developed a dynamic hedging strategy that minimizes the effect of the worst case scenario. It requires a range for the future stock price to be specified. Minimax differs from delta hedging in that it allows the hedger to incorporate information or his beliefs about the future level of prices. The results of the simulation study suggest that minimax is robust in the sense that it performs better than delta hedging for the set of options for which it is explicitly designed, and in general, it does not perform worse than delta hedging. The simulation results also suggest that three variants of minimax are suitable for hedging the risk of writing an option when the price of the underlying stock is both highly volatile and crosses over the exercise price frequently (at-the-money options). This problem is of particular interest to market makers, investors, as well as speculators. The results of the limited empirical study are mainly consistent with the results of the simulation study. Finally, the results show that the minimax strategy is robust and that the worst case is not always characterized by the upper or lower bound of the stock price, as initial intuition might suggest. Potential future extensions include applications to portfolios and to a multi-period setting.

Appendix

Weighting hedge recommendations, Variant B*

Changes in stock price often contain an element of noise; the hedger may wish to react only to the signal element in the change. The probability that the hedger will consider that a given ΔS_t has a large noise element is higher the higher the standard deviation of S_t. Some crossovers leave S_t close to X; others leave S_t far from X. The probability of another crossover occurring is higher the closer S_t is to X, and the recommendation on n_t, the number of shares to hold, has a higher probability of being reversed, the closer S_t is to X. The hedger is less likely to accept the transaction costs in realizing the recommended n_t when he considers that another crossover is likely to occur, involving another possibly large and countervailing change in recommended n_t with corresponding large transaction costs. We give below an expression that the hedger can use to weight the number of shares recommended by a strategy to reflect his perception of the amount of noise in ΔS_t and the potential reversibility of the crossover. The hedger can give any weights to a and b to reflect the importance he attached to the standard deviation, denoted by sd, of S_t. The general expression is:

$$k1 = \left[w1 \frac{|\Delta S_t|}{a * sd} + w2 \frac{|S_t - X|}{b * sd} \right] \Big/ (w1 + w2) \qquad \text{(A1)}$$

He may give a non-zero value to $w2$ when there is a crossover from $t-1$ to t; there are no restrictions on the value he gives to $w2$.

In Section 4.5, we apply the expression to minimax variant **B**, to give a weighted version of that variant, **B***.

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