Multi-Period Minimax Hedging Strategies

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Abstract

Option pricing and hedging under transaction costs are of major importance to marketmakers and investors. In this paper we present the basic minimax strategy which determines the optimum number of shares that minimizes the worst-case potential hedging error under transaction costs for the next period. We present two extensions of this strategy. The first extension is the two-period minimax where the worst case is defined over a two-period setting. The objective function of the basic minimax strategy is augmented to include the hedging error for the second period. The second extension is the variable minimax strategy where early rebalancing is triggered by the minimax hedging error. Simulation results suggest that the basic minimax strategy and its two extensions are superior in performance to delta hedging and that the variable minimax strategy is superior to both the basic and the two-period strategies. This result is due to the opportunity provided by the variable minimax strategy to rebalance early.

Keywords: options, minimax, hedging, transaction costs, nonsmooth optimization.

1 Introduction

In Howe, Rustem and Selby[9], we present minimax strategies to solve a hedging problem when there are transaction costs and discuss the performance of these strategies. We define the generic minimax formulation of a hedging strategy, hereafter referred to as **the basic minimax hedging strategy** or **Basic minimax**, and develop it into a number of specific strategies which we call **variants**. **Basic minimax** determines for an individual option the number of shares that minimizes the worst-case potential hedging error within a preset rebalancing interval. In this paper, we extend Basic minimax to accommodate situations where the hedger does not follow the policy of constant rebalancing. These strategies effectively address the problems associated with the term-structure of volatility.

The minimax strategy determines simultaneously the optimal policy and the (bounded) uncertain variables that define the worst state of the world. While inherently pessimistic in outlook, we believe a decision maker cannot afford to overlook the "worst case", at least for contingency planning. In addition, any excessive pessimism of the strategy may be avoided by refining the variability bounds of the uncertainties. As a further justification of minimax, we should point out that expected value optimization calculations have to some extent been reconciled with, or justified in view of, the worst case scenario.

The first extension is a two-period minimax strategy, hereafter called **Two-Period minimax,** where the worst case is defined over a two-period setting. In this extension, the objective function of Basic minimax is augmented to include the hedging error for the second period. In Basic minimax we have a one time period setting that is equal to the rebalancing interval. In Two-Period minimax we have a two time period setting, but the rebalancing interval remains equal to one period.

The second extension is a variable minimax strategy, hereafter called **Variable minimax,** where early rebalancing is triggered by the minimax hedging error. In Basic minimax, we preset a rebalancing interval that is constant throughout the life of the option and the hedger necessarily rebalances at the end of each interval, and may not rebalance within that interval. In Variable minimax we also preset a constant rebalancing interval, but the hedger may rebalance before the end of that interval. Under Variable minimax the hedger can monitor the actual hedging error within a rebalancing interval; if he finds the actual hedging error unacceptable, he can rebalance before the end of that interval.

We focus on European call options and the hedging problem that we describe below pertains to the writing or selling of this type of options. Our analysis also applies to American call options provided that the American call's underlying stock does not pay dividends. In this case, it is never optimal to exercise an American call before the expiration date.[2]

Our analysis does not apply to put options. Although there is a certain symmetry between puts and calls, it is not perfect. The potential profit or loss, respectively, from buying or selling a call is unlimited whereas that from buying or selling a put is limited. In particular, when transaction costs are introduced, separate analyses are necessary.[3]

In Sections 2 and 3, we give a general introduction to the hedging problem and to an option pricing model. In Section 2, we describe the hedging problem. In Section 3, we describe an option pricing model and delta hedging. In Section 4, we present the basic minimax hedging strategy with transaction costs. In Section 5, we present Two-Period minimax, and in Section 6, Variable minimax. In Section 7, we discuss the minimax solution. Because Basic minimax, the two multi-period extensions and delta hedging are very complex strategies, in order to compare and contrast them it is necessary to undertake a simulation study. We report the results in Section 8. In Section 9, we illustrate the suitability of the multi-period strategies.

The following notation will be used in this paper:

$B = B(S,t)$	call price
S	stock price
X	exercise price
r	risk-free interest rate
t	current date
T	expiration date

[2]This result is discussed in Merton[11].

[3]This is illustrated in Neuhaus[12] where separate option pricing models were designed based on whether the call option is bought or sold.

$T-t$	time to maturity
σ	volatility
$\Theta(d)$	the cumulative normal distribution function
Δt	hedging interval
N	the contracted number of shares of stock
n	number of shares to hold
k	the number of written call options
K	transaction cost as percent of transaction volume

The following *subscripts* will be used:

0	time 0, the initiation date of the contract
t	time t, any time such that $0 < t < T$
T	time T, the expiration date

The following *superscript* will be used:

i	refers to stock i or option i

2 Introduction to the hedging problem

A writer of a European call option receives the call premium but incurs a potential liability in the case of exercise by the buyer at the expiration date. If the writer of the option does not own the contracted amount of stock, the potential liability is unlimited. The writer, hereafter also called the **hedger**, wishes to modify his exposure to risk: he would like to avoid a potentially large loss in the case when the final stock price is above the exercise price. He would hedge this risk by holding part or all of the contracted number of shares. In practice, hedgers can choose from a variety of strategies ranging from ad hoc strategies to sophisticated ones based on option pricing theories. As hedging strategies with theoretical foundations are widely studied and generally acknowledged to be efficient, we assume that hedgers use them. In particular, we assume that delta hedging is also used.

We address this hedging problem in Sections 4, 5 and 6, respectively, where we present Basic minimax, Two-Period minimax and Variable minimax: these strategies are based on the notion of a **"minimax hedging error"**.[4]

3 The Black and Scholes option pricing model and delta hedging

In a dynamic strategy the hedger modifies his position in response to movements in the stock price. In Section 3.1, we present the Black and Scholes[1] (BS) option pricing model[5], which is the basis

[4]The minimax hedging error for Basic minimax and Variable minimax is defined and discussed in Section 4; for Two-Period minimax, in Section 5.

of delta hedging. The BS model is also used in the minimax hedging strategy to be described in Section 4. In Section 3.2, we present a method of adjusting the BS option pricing model when transaction costs are included in the option valuation. In Section 3.3, we discuss delta hedging.

3.1 Black and Scholes Option Pricing Model

We give below the Black and Scholes formula for the value of an option $B = B(S,t)$.

$$B = S\Theta(d_1) - Xe^{-r(T-t)}\Theta(d_2) \quad (3.1)$$

$$d_1 = \frac{\ln(S/X) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \quad (3.2)$$

$$d_2 = \frac{\ln(S/X) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \quad (3.3)$$

where

$\Theta(d)$ the cumulative normal distribution function , i.e.

$$\Theta(d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \quad (3.4)$$

3.2 The Leland Model : introduction of transaction costs

The BS option pricing model does not include transaction costs. Leland[10] developed an option pricing model that includes transaction costs[6]. In his model, the hedging errors, including transaction costs, will almost surely approach zero as $\Delta t \to 0$. His model was extended by Neuhaus[12]. Leland and Neuhaus both incorporated transaction costs in their models by modifying the volatility. We present below Leland's modification, which is more closely related to discrete delta hedging than is the Neuhaus modification. The revised volatility $\hat{\sigma}$ is given by

$$\hat{\sigma} = \sqrt{\sigma^2\left[1 + \sqrt{\frac{2}{\pi}}\frac{K}{\sigma\sqrt{\Delta t}}\right]} \quad (3.5)$$

where K[7] is the roundtrip transaction cost expressed as a proportion of trading volume.

We replace σ by $\hat{\sigma}$ in the BS option pricing model when we include transaction costs in the analysis. We define $K = 2\hat{K}$, where $\hat{K}$ is half the roundtrip transaction cost.

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[5]Other option pricing models, such as the model developed by Cox and Ross[3], may be used but this possibility is not explored here.

[6]Other option pricing models with transaction costs have been developed by other authors. We use Leland's[10] model because it is applicable to discrete rebalancing and it fits in the framework of the minimax hedging strategy. See Boyle and Emanuel[2], Gilster and Lee[6], Panas[13], Neuhaus[12], and Davis and Norman[5].

[7]In Leland[10], transaction costs varies from $K = 0.0$ to $K = 0.04$.

3.3 The delta of a call option and delta hedging

Delta and delta hedging are closely related to the BS option pricing model. Delta, D, is the change in option price per unit change in stock price, that is

$$D = \frac{\partial B}{\partial S}. \qquad (3.6)$$

Using the BS option pricing model Eqn (3.1), delta is given analytically by

$$D = \Theta(d_1). \qquad (3.7)$$

Black and Scholes[1] argue that the writer can realize a riskless portfolio by delta hedging; he finds the delta of an option and bases the number of shares to hold on it. He sets the hedge ratio, defined as the number of shares that he must hold divided by the number of shares per option at any time, equal to delta. For N, the contracted number of shares per option, and n_t, the number of shares to hold at any time t, under delta hedging,

$$n_t = D_t N . \qquad (3.8)$$

Because delta changes with time, the hedge portfolio is only instantaneously riskless. He must rebalance continuously to keep the portfolio riskless. Such a portfolio strategy is called a "**delta-neutral**" strategy. However, because he cannot rebalance continuously, he uses discrete delta hedging where he rebalances at discrete intervals of time. With discrete delta hedging, he incurs a **hedging error** which, for an interval of time, is the net position of a hedge portfolio brought about by changes in S_t. The hedging error (HE) for a portfolio of a **written call option and stock held long**, for the interval t to $t+1$ is:

$$HE = N(B_t - B_{t+1}) + n_t(S_{t+1} - S_t). \qquad (3.9)$$

Large hedging errors tend to increase the cost of rebalancing the hedge.

4 The basic minimax hedging strategy

In this section, we develop a strategy to solve the hedging problem which was introduced in Section 2.3. This strategy is based on the concept of a **worst case scenario**, which the hedger specifies in terms of movements in stock price, and it finds a hedge that minimizes the effect of such a scenario[8]. In Section 4.1, we formulate the minimax problem. In Section 4.2, we present the worst case

[8] In practice, the Black and Scholes assumption of constant instantaneous standard deviation of returns is rarely satisfied. Indeed, the effect of volatility variations is frequently as bad, if not worse, than that of the variations of the underlying variable and neither delta hedging nor the basic minimax hedging strategy can hedge against these variations. We discuss this problem, in the context of Variable minimax, in Section 6.

scenario. In Section 4.3, we present the hedging error which is the underlying cost to be minimized. In Section 4.4, we present the objective function. In Section 4.5, we define the minimax hedging error for Basic minimax. In Section 4.6, we discuss the treatment of transaction costs. In Section 4.7, we present the variant of the basic minimax hedging strategy used in this paper[9].

4.1 Minimax problem formulation

The problem is to minimize an objective function under a worst case scenario. The minimizing vector is $n_t \in \Re^k$, and the maximizing vector is $S_{t+1} \in \Re^k$, which is allowed to take any value within predefined bounds.

The minimax problem is

$$\min_{n_t} \max_{S_{t+1}} \quad f(n_t, S_{t+1}) \qquad (4.1)$$

$$\text{subject to} \qquad S_l \le S_{t+1} \le S_u \qquad (4.2)$$

where

$$n_t = \begin{bmatrix} n_t^1 \\ \vdots \\ n_t^k \end{bmatrix} \qquad \text{and} \qquad S_{t+1} = \begin{bmatrix} S_{t+1}^1 \\ \vdots \\ S_{t+1}^k \end{bmatrix} \qquad (4.3)$$

and $f(n_t, S_{t+1})$ is the objective function, presented in Section 4.4, and $S_l \le S_{t+1} \le S_u$ is the range defined in Section 4.2. There are no constraints on n_t, the number of shares to hold at time *t*: non-negative n_t implies a long position in shares; negative n_t implies a short position in shares.

We note that Eqn (4.1) is a continuous minimax problem where the worst case value of the objective function may correspond to a mid-range solution; in this case, the continuous minimax problem can not be replaced by a discrete minimax formulation. For a discussion on mid-range solutions, see Howe[9] and Howe, Rustem and Selby[9].

4.2 The worst case scenario under Basic minimax[10]

The hedger defines the worst case in terms of extreme movements in stock price. The range of S_{t+1} has upper and lower bounds that delimit the 95% confidence interval[11] of all possible values of the

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[9]In this paper, we use two of the high performing variants presented in Howe, Rustem and Selby[9] where several variants of the basic minimax strategy were studied.

[10]Other worst case scenarios can be defined (see Howe, Rustem and Selby[9]), however, we limit the definition of the worst case scenario to the one given in this section. The reason for this is discussed in Section 4.7.

future stock price, i.e. **two standard deviations** about the expected value of the stock price at time $t+1$. This 95% confidence interval is based on an estimate of the volatility of the stock price and on the assumption that the stock price follows a lognormal distribution function. The worst case scenario is hereafter referred to as the **95%-Level**.

4.3 The hedging error

From Section 3.3, under delta hedging, the hedging error is given by Eqn (3.9). When **actual** values of B_t, S_t, B_{t+1} and S_{t+1} are substituted into Eqn (3.9), we have the actual hedging error under delta hedging. When **actual** values of B_t and S_t and **potential** values of B_{t+1} and S_{t+1} are substituted into Eqn (3.9), we have the potential hedging error under delta hedging. The potential hedging error under delta hedging is the basis of the objective function in the minimax hedging strategy. In minimax, potential S_{t+1} is taken from a predefined range that maximizes the objective function; potential B_{t+1} is the value of the call option based on the pricing model[12] given potential S_{t+1}, i.e. potential $B_{t+1} = B_{t+1}(S_{t+1})$. The minimax strategy minimizes the maximum potential hedging error plus interest payments on borrowed money[13]. In Section 4.5, we define the minimax hedging error and give the definition of actual hedging error and potential hedging error in the context of minimax.

4.4 The objective function

4.4.1 The objective of minimax hedging

In any dynamic hedging strategy, hedging errors are incurred; in order to correct for these errors, the hedge is rebalanced, with the cost of rebalancing being added to the cost of hedging. At time t, the hedger can attempt to minimize the potential hedging error between t and $t+1$. His decision at time t on n_t, the number of shares to hold, affects the actual hedging error between t and $t+1$. The minimax hedging strategy aims to minimize the maximum potential hedging error between t and $t+1$.

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[11]We consider the 95% confidence interval as a reasonable range to consider under normal market conditions. Under abnormal market conditions, the 99% confidence interval may be more appropriate.
[12]We use the Black and Scholes[1] option pricing model, Eqn (3.1), with a modified volatility, Eqn (3.5).
[13]This is discussed in detail in Section 4.4.3.

4.4.2 The objective function defined

The minimax strategy aims to minimize the potential hedging error by using it as the objective function. In discrete delta hedging, where rebalancing is done at discrete intervals, we expect that the desirable properties of delta hedging given in Section 3.3 will not be observed consistently in time. By minimizing the maximum potential hedging error plus interest payments on borrowed money[14], should the worst case occur, the hedger adopts a cautious strategy. If the worst case occurs, he has effectively minimized its worst effect; if it does not occur, he may incur a hedging error higher than that using delta hedging.

This direct way of minimizing the potential hedging error is based on the no-arbitrage argument of Merton[11] where he considers a portfolio containing an option, the underlying stock and a riskless bond (i.e. riskless in the sense of default) that is suitably chosen such that the aggregate investment in the portfolio is zero. He demonstrates that there is a strategy for finding the combination of the option, stock and bond that would ensure that the return on the portfolio would be nonstochastic. Because of the condition of zero aggregate investment, he argues that in order to avoid arbitrage profits, the return on this portfolio must be zero. In the case of a portfolio of written call options, underlying stock and bonds, given Merton's assumptions, the return on this particular portfolio must be zero. We shall work with such a portfolio and we shall call it the "**ideal portfolio**". This "ideal portfolio" is the benchmark we used in defining the objective function. We derive basic properties of the minimax hedging strategy on the basis of a self-financing portfolio; conditional on these results, we add the effect of costs. We will return to this when we discuss Eqn (4.7a) below.

We define $U_1:\Re^k x \Re^k \to \Re^1$, $U_2:\Re^k \to \Re^k$, $U:\Re^k x \Re^k \to \Re^{k+1}$ and Q is a $(k+1)*(k+1)$ positive definite diagonal weighting matrix. $U^d \in \Re^{k+1}$ is the vector of desired values for the potential hedging error and the transaction cost terms: we use a desired value of zero, i.e. the desired hedging error is zero[15] and the desired transaction cost is zero.

[14] Refer to Eqn (4.6a) below.

[15] In minimax, we have the opportunity to adopt any desired value. We adopt a desired value of zero because in delta hedging, the expected value of the hedging error is zero.

The objective function is given by

$$f(n_t, S_{t+1}) \;=\; \tfrac{1}{2} < U - U^d, Q(U - U^d) > \qquad (4.4)$$

where

$$U(n_t, S_{t+1}) = \begin{bmatrix} U_1(n_t, S_{t+1}) \\ \dots\dots\dots\dots \\ U_2(n_t) \end{bmatrix} \quad \text{and} \quad U^d = \begin{bmatrix} U^d{}_1 \\ \dots\dots \\ U^d{}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \dots\dots \\ 0 \end{bmatrix} \qquad (4.5)$$

$$U_1(n_t, S_{t+1}) = \sum_{i=1}^{k} n_t^i (S_{t+1}^i - S_t^i) + \sum_{i=1}^{k} N^i (B_t^i - B_{t+1}^i(S_{t+1}^i)) + \sum_{i=1}^{k} \left(-(n_t^i - n_{t-1}^i) S_t^i + C_{t-1}^i (1 + r\Delta t) \right) r\Delta t$$

$$(4.6a)$$

where $C_{t-1}^i = C_{t-2}^i(1 + r\Delta t) - (n_{t-1}^i - n_{t-2}^i) S_{t-1}^i - \hat{K} | (n_{t-1}^i - n_{t-2}^i) S_{t-1}^i |.$ (4.6b)

$$U_2(n_t) = \begin{bmatrix} U_2^1(n_t^1) \\ \vdots \\ U_2^k(n_t^k) \end{bmatrix} \qquad (4.7)$$

where[16] $U_2^i(n_t^i) = \hat{K}\left(n_t^i - n_{t-1}^i\right) S_t^i.$ (4.8)

We first identify all the variables in Eqns (4.4) to (4.8) and then give the economic interpretation of Eqn (4.6a).

C_{t-1}^i is the cumulative value of **cash inflow minus cash outflow** at time $t-1$. $C_{t-1}^i(1+r\Delta t)$ is C_{t-1}^i with interest payments. The first term of (4.6b) is the cumulative value of cash inflow minus cash outflow from the previous period with interest payments. The second term is a cash outflow if the $n_{t-1}^i > n_{t-2}^i$; otherwise, it is a cash inflow. The third term is always a cash outflow. We note that C_{t-1}^i will normally be a negative number. At time t, C_{t-1}^i is a constant: all the variables in (4.6b) have actual values.

Because transaction costs introduce nondifferentiability into the equation, they do not come into the objective function directly as part of U_1. Instead, we introduce a penalty term, U_2^i to represent a

[16]This is discussed in detail in Section 4.6.

penalty for transaction costs for each option i at time t. The treatment of transaction costs is discussed in detail Section 4.6.

In the weighting matrix Q, the weights q_i, $i = 1,..,k+1$, which are specified by the hedger, represent his preferences: with a high q_1 he prefers to minimize the potential hedging error that may be incurred from time t to time $t+1$; with a high q_i, $i = 2,..,k+1$, he prefers to minimize the penalty term.

For each option i, $i = 1,...,k$, $B^i_{t+1}(S^i_{t+1})$ is determined using Eqns (3.1), (3.2) and (3.3) with the modified volatility (3.5).

We now give an economic interpretation of (4.6a). U_1 represents the potential hedging error, inclusive of interest payments on borrowed money, between time t and time $t+1$: it comprises the potential shift in the stock position, the potential shift in the option position and the potential interest payment. The first two terms of (4.6a) give the return on a portfolio of written call options and underlying stocks. The third term represents the opportunity cost of money, i.e. the interest payments on borrowed money, because the portfolio is not self-financing. We wish to find the mix of options and stocks that minimizes the deviation of the return on the portfolio, including opportunity cost, from the return on the "ideal portfolio", the value of which is zero, based on Merton's conditions of zero aggregate investment and no-arbitrage.

4.5 The minimax hedging error

In minimax, we distinguish actual from potential hedging error. **Actual hedging error, inclusive of interest payments on borrowed money,** is calculated when **actual** B_t, S_t, B_{t+1} and S_{t+1} are substituted into Eqn (4.6a). **Potential hedging error, inclusive of interest payments on borrowed money,** is calculated when **actual** values of B_t and S_t and **potential** values of B_{t+1} and S_{t+1} are substituted into Eqn (4.6a). Potential S_{t+1} is taken from a predefined range that maximizes the objective function; potential B_{t+1} is the value of the call option based on the pricing model[17]

[17]We use the Black and Scholes[1] option pricing model, Eqn (3.1), with a modified volatility, Eqn (3.5).

given potential S_{t+1}, i.e. potential $B_{t+1} = B_{t+1}(S_{t+1})$. The minimax strategy minimizes the maximum potential hedging error, inclusive of interest payments on borrowed money.

We define the **minimax hedging error** at time t as

$$\textbf{minimax hedging error} = U_1(n_{t*}, S_{t+1*}). \quad (4.9)$$

The minimax hedging error is the potential hedging error, inclusive of interest payments on borrowed money, given the solution n_t* and $S_{t+1}*$.

4.6 Transaction Costs

In this section, we discuss the treatment of transaction costs in the minimax hedging strategy. The roundtrip transaction cost K is used in valuing the option[18] and $\hat{K}$, with $\hat{K} = \frac{1}{2}K$, is used as part of the cumulative value of cash inflow minus cash outflow and as part of the penalty term in the objective function. In Section 4.6.1, we discuss transaction costs as part of the cumulative value of cash inflow minus cash outflow; in Section 4.6.2, we discuss transaction costs as part of the objective function.

4.6.1 Transaction costs in the cost of hedging

The performance of delta hedging and the variants of minimax is measured by the final cumulative value of cash inflow minus cash outflow at the maturity of the option. After finding n_t by solving the minimax problem using Eqns (4.4) to (4.8), we can evaluate the **actual** cumulative value of cash inflow minus cash outflow at time t. This is given by

$$C_t^i = C_{t-1}^i(1 + r\Delta t) - (n_t^i - n_{t-1}^i)S_t^i - \hat{K}|(n_t^i - n_{t-1}^i)S_t^i| \quad (4.10)$$

The last term is the transaction cost at time t: **this is always incurred and it is always a cash outflow**. At time $t = 0$, the actual cumulative value of cash inflow minus cash outflow includes the option premium which is a cash inflow. This is given by

$$C_0^i = -n_0^i S_0^i + NB_0^i - \hat{K}|n_0^i S_0|. \quad (4.11)$$

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[18]See Section 3.2.

All variants of the minimax hedging strategy will use Eqns (4.10) and (4.11) to compute the actual cumulative value of cash inflow minus cash outflow.

4.6.2 The transaction cost term in the objective function

From Eqns (4.4), (4.7) and (4.8), the transaction cost term (TC) of the objective function can be expressed as

$$TC = \sum_{i=1}^{k} q_i (U_2^i - U^{d_2^i})^2 = \sum_{i=1}^{k} q_i (\hat{K}(n_t^i - n_{t-1}^i) S_t^i)^2 . \qquad (4.12)$$

Eqn (4.12) should be interpreted as a penalty term. Because we chose a desired value of zero for U_2^d, the right hand side equality holds. The effect of this term on the solution is dependent on the level of transaction cost $\hat{K}$ and on the weights q_i. We use a uniform weighting system for this study: $q_i = q$, $i = 1,..,k$, where q is held constant.

For low values of $\hat{K}$, we need a high value of q so that the transaction cost term TC is not dominated by the other terms in the objective function. Conversely, for high values of $\hat{K}$, we need a low value of q to ensure that TC does not dominate the objective function. For the simulation and empirical illustration, we set[19] the roundtrip transaction cost to $K = 0.02$ and the weighting coefficient to $q = 100$.

4.7 The variant of the basic minimax hedging strategy

There are a number of variants for Basic minimax; in this paper, we consider the 95%-Level variant[20]. In Sections 5 and 6, we also consider extensions to this variant because, first, it includes transaction costs in the objective function, and second, this variant computes the minimax hedging error based on the most likely future values of the stock price, i.e. the 95%-Level worst case scenario[21].

[19] This value is based on simulation results, reported in Howe[8], showing the variation of K with q.

[20] This is variant C in the Minimax Hedging Strategy paper (see Howe, Rustem and Selby[9]).

[21] Unlike the other robust variant, variant D in Howe, Rustem and Selby[9], which may give rise to very extreme hedging errors in a multi-period setting.

5 Two-Period Minimax Strategy

This strategy provides the hedger with a tool for computing the minimax hedging error in two time periods. We design the strategy for the hedger who wishes to have a constant rebalancing interval, Δt. He decides on the number of shares to hold on the basis of the calculated minimax hedging error for the following two time periods, and he rebalances at the end of the first of the two periods. If he cannot rebalance at the end of a period because, say, there is a shortage of stock, by having taken into account the potential hedging error in the second period in deciding on the number of shares to hold at the start of the first period, he ensures that the actual hedging error in the second period will not be worse than the component of the Two-Period minimax hedging error[22] for that period.

Two-Period minimax considers a worst case scenario for two time periods. The difference between this two-period worst case and the one-period worst case is that the range for the future stock price S_{t+2} is wider for the two-period case. This results in a higher minimax hedging error for the two periods compared to the summation of the errors over two periods when using Basic minimax. In this sense, Two-Period minimax is a more cautious strategy than Basic minimax.

5.1 Minimax problem formulation

In Two-Period minimax, the minimizing vector is $n_t \in \Re^k$, and the maximizing vectors are $S_{t+1} \in \Re^k$, $S_{t+2} \in \Re^k$.

The minimax problem is given by

$$\min_{n_t} \max_{(S_{t+1}, S_{t+2})} \quad f(n_t, S_{t+1}, S_{t+2}) \tag{5.1}$$

$$\text{subject to} \quad S_{t+1_l} \le S_{t+1} \le S_{t+1_u} \tag{5.2}$$

$$S_{t+2_l} \le S_{t+2} \le S_{t+2_u}$$

where

$$n_t = \begin{bmatrix} n_t^1 \\ \vdots \\ n_t^k \end{bmatrix} \qquad S_{t+1} = \begin{bmatrix} S_{t+1}^1 \\ \vdots \\ S_{t+1}^k \end{bmatrix} \qquad S_{t+2} = \begin{bmatrix} S_{t+2}^1 \\ \vdots \\ S_{t+2}^k \end{bmatrix} \tag{5.3}$$

[22]The Two-Period minimax hedging error is given by Eqn (5.11).

where $f(n_t, S_{t+1}, S_{t+2})$ is the objective function, presented in Section 5.2, $S_{t+1l} \le S_{t+1} \le S_{t+1u}$ and $S_{t+2l} \le S_{t+2} \le S_{t+2u}$ are ranges defined as the 95%-Level[23]. There are no constraints on n_t, the number of shares to hold at time t: non-negative n_t implies a long position in shares; negative n_t implies a short position in shares.

5.2 The objective function

We define $U_1 : \Re^k x \Re^k \to \Re^1$, $U_2 : \Re^k x \Re^k x \Re^k \to \Re^1$, $U_3 : \Re^k \to \Re^k$, $U : \Re^k x \Re^k x \Re^k \to \Re^{k+2}$ and Q as a $(k+2)x(k+2)$ positive definite diagonal weighting matrix. U_1 refers to the potential hedging error for the first time period and U_2 refers to that for the second time period. U_3 refers to a penalty term for transaction costs associated with buying or selling of stocks. $U^d \in \Re^{k+2}$ is the vector of desired values for the potential hedging error for the two periods and the transaction cost terms: we use a desired value of zero, i.e. the desired hedging error is zero and the desired transaction cost is zero.

The objective function is given by

$$f(n_t, S_{t+1}, S_{t+2}) \quad = \quad \tfrac{1}{2} < U - U^d, Q(U - U^d) > \tag{5.4}$$

where

$$U(n_t, S_{t+1}, S_{t+2}) = \begin{bmatrix} U_1(n_t, S_{t+1}) \\ U_2\left(n_t, \begin{bmatrix} S_{t+1} \\ S_{t+2} \end{bmatrix}\right) \\ U_3(n_t) \end{bmatrix} \quad \text{and} \quad U^d = \begin{bmatrix} U_1^d \\ U_2^1 \\ U_3^d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.5}$$

$$U_1(n_t, S_{t+1}) = \sum_{i=1}^{k} n_t^i (S_{t+1}^i - S_t^i) + \sum_{i=1}^{k} N^i (B_t^i - B_{t+1}^i(S_{t+1}^i)) + \sum_{i=1}^{k} \left(-(n_t^i - n_{t-1}^i) S_t^i + C_{t-1}^i (1 + r\Delta t)\right) r\Delta t \tag{5.6}$$

$$U_2\left(n_t, \begin{bmatrix} S_{t+1} \\ S_{t+2} \end{bmatrix}\right) = \sum_{i=1}^{k} n_t^i (S_{t+2}^i - S_{t+1}^i) + \sum_{i=1}^{k} N^i (B_{t+1}^i(S_{t+1}^i) - B_{t+2}^i(S_{t+2}^i))$$
$$+ \sum_{i=1}^{k} \left(-(n_t^i - n_{t-1}^i) S_t^i + C_{t-1}^i (1 + r)\Delta t\right)(1 + r\Delta t)(r\Delta t) \tag{5.7}$$

where

$$C_{t-1}^i = C_{t-2}^i (1 + r\Delta t) - (n_{t-1}^i - n_{t-2}^i) S_{t-1}^i - \hat{K} | (n_{t-1}^i - n_{t-2}^i) S_{t-1}^i |. \tag{5.8}$$

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[23]See Section 4.2.

$$U_3(n_t) = \begin{bmatrix} U_3^1(n_t^1) \\ \vdots \\ U_3^k(n_t^k) \end{bmatrix} \tag{5.9}$$

where[24]

$$U_3^i(n_t^i) = \hat{K}\left(n_t^i - n_{t-1}^i\right)S_t^i \tag{5.10}$$

We first identify all the variables in Eqns (5.4) to (5.10) and then give an economic interpretation of Eqns (5.6) and (5.7).

C_{t-1}^i is the cumulative value of cash inflow minus cash outflow[25] at time $t-1$. $C_{t-1}^i(1+r\Delta t)$ is C_{t-1}^i with interest payments. The first term of (5.8) is the cumulative value of cash inflow minus cash outflow from the previous period with interest payments. The second term is a cash outflow if the $n_{t-1}^i > n_{t-2}^i$; otherwise, it is a cash inflow. The third term is always a cash outflow. We note that C_{t-1}^i will normally be a negative number. At time t, C_{t-1}^i is a constant: all the variables in (5.8) have actual values.

Because transaction costs[26] introduce nondifferentiability into the equation, they do not come into the objective function as part of U_1 nor U_2. Instead, we introduce U_3^i to represent a penalty for transaction costs for each option i at time t.

In the weighting matrix Q, the weights q_i, $i = 1,..,k+2$, which are specified by the hedger, represent his preferences: a high q_1 represents an emphasis on minimizing the potential hedging error that may be incurred from time t to time $t+1$; a high q_2 represents an emphasis on minimizing the potential hedging error that may be incurred from time $t+1$ to time $t+2$. High q_i, $i = 3,..,k+2$, represents an emphasis on minimizing the corresponding transaction cost term.

For each option i, $i = 1,\ldots,k$, $B_{t+1}^i(S_{t+1}^i)$ and $B_{t+2}^i(S_{t+2}^i)$ is determined using the Black and Scholes[1] option pricing model and using Leland's[10] modified volatility.

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[24]Eqn (5.10) is discussed in detail in Section 4.6.
[25]This is first introduced in Section 4.4.2.
[26]The treatment of transaction costs is discussed in detail Section 4.6.

We now give an economic interpretation of Eqns (5.6) and (5.7). U_1 represents the potential hedging error between time t and time $t+1$; it comprises the potential shift in the stock position, the potential shift in the option position and the potential interest payment. U_2 represents the potential hedging error between time $t+1$ and time $t+2$. U_3 refers to a penalty term for transaction costs associated with buying or selling of stocks. We wish to minimize the potential hedging error, including interest payments on borrowed money, for two time periods. At the same time, we wish to constrain transaction costs.

5.3 The Two-Period minimax hedging error

In contrast to the minimax hedging error for one time period given by Eqn (4.9), the minimax hedging error for two time periods is given by,

$$\textbf{Two-Period minimax hedging error} = U_1(n_{t*}, S_{t+1*}) + U_2(n_{t*}, S_{t+1*}, S_{t+2*}) \quad (5.11)$$

6 Variable Minimax Strategy

This strategy is the same as Basic minimax in all respects except that under Variable minimax the hedger can rebalance within the preset interval, and in deciding when to rebalance he takes account of the actual hedging error. If the actual hedging error is unacceptable to him, he may wish to rebalance before the end of the preset interval; if the actual hedging error is not unacceptable, he would rebalance at the end of the preset interval.

At the start of each time period, the hedger specifies his worst case scenario for that time period. One such scenario may be a large movement in the price of the underlying stock that results in an actual hedging error that is unacceptable to him. He uses Basic minimax to minimize the potential hedging error that corresponds to such a scenario. If the stock price moves in the direction that makes the hedging error unacceptable to him, he may wish to rebalance early. For example, if the preset rebalancing interval is 1 week, i.e. 5 trading days, he may rebalance on day 1, 2, 3 or 4, and the next preset interval will start on the following day.

Under Variable minimax, the hedger uses the minimax hedging error calculated[27] by Basic minimax as a criterion in deciding whether to rebalance before the end of the preset interval. Because the minimax hedging error corresponds to the worst case scenario for the next period, if the stock price is within the preset range, he knows that the absolute value of the actual hedging error would not be higher than the minimax hedging error. Despite this knowledge, he may consider such an actual hedging error to be unacceptable to him. If he does, he can avoid the accumulation of unacceptable hedging errors by rebalancing early should the actual hedging error be worse than the **threshold error**, which is defined as the **proportion of the minimax hedging error** that is acceptable to him. This threshold error serves as a **trigger for early rebalancing**.

If, at any time within the preset interval, the hedger finds that the actual hedging error is worse than his threshold error, he may rebalance at that time. However, if, at any time within the preset interval, the actual error is not worse than his threshold error, he would not rebalance before the end of the preset interval. He can use the minimax hedging error as a criterion for deciding when to rebalance. The time at which he rebalances becomes the start of the next preset interval.

Because Variable minimax uses a system to monitor the actual hedging error and allows the hedger to rebalance early when the actual hedging error becomes unacceptable to him, Variable minimax is more responsive to unfavorable stock price movements than Basic minimax. In this sense, Variable minimax is a more aggressive strategy than Basic minimax. We note that this strategy indirectly provides the hedger with a means of dealing with changes in the volatility of stock returns, particularly if there is a significant volatility term-structure. It should be noted, however, that we do not expressly address stochastic volatility.

6.1 Minimax problem formulation

Variable minimax is Basic minimax augmented by a **system to monitor the actual hedging error.** The discussions, including the definition of the minimax hedging error, in Section 4 apply. In particular, the minimax problem is

[27]This is given by Eqn (4.10).

$$\min_{n_t} \max_{S_{t+1}} \quad f(n_t, S_{t+1}) \qquad (4.1)$$

$$\text{subject to} \qquad S_l \le S_{t+1} \le S_u \qquad (4.2)$$

where $f(n_t, S_{t+1})$ is the objective function, presented in Section 4.4, and $S_l \le S_{t+1} \le S_u$ is the range defined in Section 4.2. There are no constraints on n_t, the number of shares to hold at time t: non-negative n_t implies a long position in shares; negative n_t implies a short position in shares.

6.2 The monitoring system

The minimax potential hedging error, with the corresponding minimizing variable n_t, applies to the time period between t and $t+1$; hereafter we refer to this time period as τ. We define a smaller interval Δt such that $m\Delta t = \tau$ where m is the number of small intervals of length Δt. Here, we consider multi-periods within the maximum period, τ. On solving the minimax problem, we find the value of n_t that minimizes the maximum hedging error that could occur within the time period t to $t+1$, given the preset range of S_{t+1}, i.e. the maximum hedging error is the value that we have insured against, when using n_t. It should be noted that there may be combinations of time $t + m_0\Delta t$, $m_0 = 0,1,..,m$, and corresponding stock price $S_{t+m_0\Delta t}$, that give the same hedging error as the one defined by the minimax solution.

For each small interval Δt, we define a **certain percentage x%** of the absolute value of the minimax hedging error M as the threshold V, i.e.

$$V = \frac{x}{100} M. \qquad (6.1)$$

For each time period t to $t + m_0\Delta t$, $m_0 = 0,1,..,m$, we calculate the absolute value of the actual hedging error, A, and compare this with the threshold V. If the actual hedging error is negative and

$$A_{t+m_0\Delta t} \ge V \qquad (6.2)$$

at time $t + m_0\Delta t$, with actual stock price $S_{t+m_0\Delta t}$, then we rebalance and solve the minimax problem again, and update t , i.e. set to $t + m_0\Delta t$ (the current time). If condition (6.2) is not satisfied for any time $t + m_0\Delta t$, $m_0 = 0,1,..,m$, then we rebalance at time $t + m\Delta t$.

7 The minimax solution

The solution under minimax, Basic, Two-Period and Variable, is obtained using an algorithm[28] to solve problem (4.1)-(4.2). In this section, we discuss the minimax solution under Basic minimax. The following discussion also applies to Variable minimax; it also applies to Two-Period minimax when the one-period maximizing variable S is replaced by its two-period counterpart.

The algorithm is based on generating successive directions of descent for $f(n,S)$ in n, while ensuring that the direction chosen maximizes $f(n,S)$ with respect to S. The direction chosen is therefore one that iteratively progresses towards the minimax solution.

Because the hedge recommendation under minimax is different from the hedge recommendation under delta hedging, in the Black and Scholes[1] world, the hedge recommendation under minimax is suboptimal. In the minimax hedging strategy, for any fixed n_t, we determine S_{t+1}, from the predefined range $S_l \le S_{t+1} \le S_u$, that maximizes the hedging error. We therefore can identify theoretically all the maxima corresponding to all the possible values of n_t. The strategy calculates the n_t that minimizes over these maxima. Although the number of shares n_t introduces some risk into the portfolio because it is not the same as the hedge recommendation under delta hedging, n_t ensures that if the *actual* S_{t+1}, as opposed to the minimax value, falls within the range $S_l \le S_{t+1} \le S_u$, the absolute value of the *actual* hedging error, inclusive of interest payments on borrowed money, will not be worse (higher) than the absolute value of the minimax hedging error. This is the **minimax robustness property**. The n_t value thus computed results in a robust strategy that is non-inferior in performance for any stock price within the predefined range.

8 Simulation study of the performance of the different strategies

We give the results of the simulation mainly in terms of the performance and the relative performance of the minimax strategies. We define the **performance** of a strategy[29] as the final

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[28] Further details of the algorithm and numerical experiments are discussed in Howe[8].
[29] A strategy may be either a minimax variant or delta hedging.

cumulative value[30] of cash inflow minus cash outflow in using that strategy on an option, standardized[31] as a percentage of the notional[32] contract value of that option. We define **relative performance** of a minimax variant as the performance of that variant minus the performance of delta hedging (DH). In the following commentary on the simulation, we say that a variant **outperforms** DH if its performance is higher than that of DH and the difference is significant at the 5% level. The simulation consists of 1250 replications, subdivided into 5 levels of sigma.

8.1 Objective of the simulation

This simulation is intended to serve as a feasibility study on potential extensions to Basic minimax. Towards this, the simulation is used to ascertain whether the two multi-period extensions of Basic minimax outperform delta hedging (DH), to ascertain whether Basic minimax outperforms Two-Period minimax, which is designed to be a more cautious strategy than Basic minimax, and to ascertain whether Variable minimax, which is designed to be a more aggressive strategy than Basic minimax, outperforms Basic minimax.

8.2 From set-up to wind-down

In this section, we discuss the mechanics of hedging, from setting up to the winding down of the hedge. We consider delta hedging, Basic minimax, Two-Period minimax and Variable minimax. Delta hedging, Basic minimax, and Two-Period minimax involve rebalancing the hedge at uniform intervals of time; in this simulation, the interval is one week. Variable minimax involves the monitoring of the actual hedging error on a daily basis. Weekly and daily data include the price of the stock, the price of the option, the risk-free interest rate, the time to maturity and the annualized volatility of returns on the stock, given as one of five preset levels: 0.2, 0.3, 0.4, 0.5, and 0.6. The annualized risk-free interest rate is preset at 0.10. Dividends are excluded from the analysis. In this simulation, N, the number of contracted shares, is 100. In addition, for Variable minimax, we set the threshold level at 10% of the minimax hedging error.

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[30]This is calculated after winding down the hedge.

[31]Following Samuelson[14], all cells in Table 8.4a have been standardized by dividing the original profit by the exercise price. In the simulation study, we set the exercise price at X=1000; this makes the original profit effectively standardized.

[32]The notional value of the contract is the number of contracted shares multiplied by the exercise price. The total is the summation over all notional values.

8.3.1 Setting up the hedge

At time 0 each strategy holds the same number of shares (n_0) based on delta, and the same initial cumulative value of cash inflow minus cash outflow given by Eqn (4.11).

8.3.2 Rebalancing the hedge

Every week through to the maturity date T, the hedge is rebalanced according to the strategies' recommendation. The trajectory of the number of shares held at time t , n_t, varies with the hedging strategy used. The actual cumulative value of cash inflow minus cash outflow at time t is given by Eqn (4.12).

8.3.3 Winding down the hedge

At time T, if the holder does not exercise his option to buy the shares, each strategy disposes of its portfolio in the same way: selling any shares held, or buying any shares sold short, at time $T-1$ at S_T.

8.4 Results of the simulation study

All four strategies are used to hedge the risk of writing each of the 1250 European call options. The average performance in using a strategy is calculated for all options with a constant volatility level. In Table 8.4a, we summarize the relative performance of Basic minimax and the two multi-period extensions. In Table 8.4b, we summarize the difference in relative performance between minimax strategies. These tables also contain, in italics in small font, the absolute value of the t-statistics, followed by ** if the difference is significant at the 2% level, and * if it is significant at the 10% level[33].

[33]We use significance levels of 2% and 10% to express the results of a 2-tailed test; however, we use a 1-tailed test for the sign of the difference at the 5% level.

Table 8.4a Relative performance of three minimax strategies.*(t-values in italics)*

Units: Percentage points of final cumulative value of cash inflow minus cash outflow divided by the notional value of the contract

STRATEGY / Volatility	Basic	Two-Period	Variable
0.2	3.3 *20.1***	3.3 *26.4***	3.6 *30.6***
0.3	2.9 *19.7***	2.9 *23.0***	3.4 *14.4***
0.4	2.9 *12.3***	2.4 *10.4***	3.2 *13.5***
0.5	2.8 *11.9***	1.5 *6.8***	2.9 *13.2***
0.6	2.8 *7.0***	1.7 *10.1***	2.8 *11.7***
Average	**2.9**	**2.3**	**3.1**

Table 8.4b Difference in relative performance between minimax strategies.
(t-values in italics)

Units: Percentage points of final cumulative value of cash inflow minus cash outflow divided by the notional value of the contract

STRATEGY / Volatility	Basic minus Two-Period	Variable minus Basic	Variable minus Two-Period
0.2	0.0 *0.1*	0.3 *1.7**	0.3 *2.5***
0.3	0.0 *0.2*	0.5 *3.4***	0.5 *3.6***
0.4	0.5 *2.2**	0.3 *1.7**	0.8 *3.7***
0.5	1.3 *4.7***	0.1 *0.2*	1.4 *5.5***
0.6	1.1 *2.7***	0.0 *0.0*	1.1 *4.4***
Average	**0.6**	**0.2**	**0.8**

8.4.1 Main results from Table 8.4a and Table 8.4b

In this section we give the main results from Table 8.4a and Table 8.4b.

From Table 8.4a, each strategy outperforms DH by about 3 percentage points, with Variable Minimax being slightly the better performer. Two-Period minimax is the worst performer, outperforming DH by just over 2 percentage points. For all three strategies, their relative performances fall with increasing volatility; the fall is most marked in Two-Period minimax.

From Table 8.4b, for low levels of volatility, Variable minimax outperforms Basic minimax, but for high levels of volatility, Variable minimax performs much the same as Basic minimax. For low levels of volatility, Two-Period minimax performs much the same as Basic minimax, but for high levels of volatility, Basic minimax outperforms Two-Period minimax.

8.5 Statistical testing of hypotheses

In this section we test a number of hypotheses on the difference in relative performance of the three minimax strategies: Basic, Two-Period and Variable. In Table 8.5, we list the hypotheses and give their accept/reject state at the 5% level of significance[34].

In Howe, Rustem and Selby[9], we have established that the basic minimax hedging strategy outperforms DH for a number of variants. Further, we have established that the 95%-Level variant, the variant that we use as Basic minimax, outperforms DH. Because Two-Period minimax has the same formulation as Basic minimax except for the inclusion of a second-period variable that determines the contribution of the second period to the worst-case scenario (see Section 5), we hypothesize (H1) that Two-Period minimax will outperform DH. Similarly, because Variable minimax has exactly the same formulation as Basic minimax except for the inclusion of a monitoring system that determines an option to rebalance early, we hypothesize (H2) that Variable minimax will outperform DH.

Because Two-Period minimax considers the worst case scenario for two time periods, it computes the minimax hedging error for two periods (Eqn (5.11)) based on a wider range for the second period compared to the range in Basic minimax when Basic minimax is used for the second period. Because Two-Period minimax computes its hedge recommendation based on the contribution of the second period to the minimax hedging error, the hedge recommendation is not necessarily consistent with the hedge recommendation of Basic minimax, and the hedge recommendation of Two-Period minimax may result in a worse hedging error if the worst case occurs during the first period. Given the reason above and because the hedger follows a policy of rebalancing after one period, we hypothesize (H3) that Basic minimax outperforms Two-Period minimax.

[34]We use the 5% level of significance to reflect a less stringent criterion on the acceptance or rejection of a hypothesis because the simulation data exhibit a wide range of moneyness and the hypotheses we tested are not sufficiently refined to accommodate variations with respect to moneyness.

Because in Variable minimax the hedger can avoid incurring large negative hedging errors, we hypothesize (H4) that Variable minimax outperforms Basic minimax.

If Hypotheses H3 and H4 are accepted, then by transitivity, Variable minimax outperforms Two-Period minimax. In case H3 and H4 are not accepted for some levels of volatility, we hypothesize (H5) that Variable minimax outperforms Two-Period minimax.

Table 8.5 The accept/reject state of the hypotheses[35], for each level of volatility. (*R*=Reject)

Hypothesis:	Volatility: 0.2	0.3	0.4	0.5	0.6
H1 Two-Period outperforms DH	*Accept*	*Accept*	*Accept*	*Accept*	*Accept*
H2 Variable outperforms DH	*Accept*	*Accept*	*Accept*	*Accept*	*Accept*
H3 Basic outperforms Two-Period	*R*	*R*	*Accept*	*Accept*	*Accept*
H4 Variable outperforms Basic	*Accept*	*Accept*	*Accept*	*R*	*R*
H5 Variable outperforms Two-Period	*Accept*	*Accept*	*Accept*	*Accept*	*Accept*

8.5.1 Comment on the rejected hypotheses

H3 is rejected for low levels of volatility. A possible reason for this may be that when the volatility is low the ranges defined by the 95%-Level are relatively small. For Two-Period minimax, the range for the second period may not be large compared to the range for Basic minimax when it is used for the second period. This may have resulted in similar minimax hedging errors under Two-Period minimax and Basic minimax.

H4 is accepted for low levels of volatility but rejected for higher levels. This may suggest that the relatively large 95%-Level ranges at higher levels of volatility resulted in very high minimax hedging errors that gave high threshold errors that were rarely exceeded. If the threshold errors are not

[35]Except for Hypothesis 5, we tested the hypotheses by defining the Null Hypothesis as no significant difference between means; the Alternative Hypothesis corresponds to the hypothesis outlined above. In Table 8.5, an accept state means an acceptance of the Alternative Hypothesis.

exceeded, Variable minimax performs much the same as Basic minimax. This suggests that the percentage to use in calculating the threshold value is critical to the performance of Variable minimax and should be made a function of volatility.

8.6 Rank ordering

The simulation results suggest the following rank order of positive differences in performance. Table 8.6 gives the rank ordering of the strategies.

Table 8.6 Rank order of positive significant differences in performance for each level of volatility.

Volatility level / Strategy	0.2	0.3	0.4	0.5	0.6	Average
Variable Minimax	1	1	1	1	1.5	1.1
Basic Minimax	2.5	2.5	2	2	1.5	2.1
Two -period minimax	2.5	2.5	3	3	3	2.8
Delta Hedging	4	4	4	4	4	4.0

All minimax strategies for the 5 levels of volatility outperform delta hedging. For low levels of volatility (volatility=0.2 or 0.3), Basic minimax has the same rank as Two-Period minimax while for a high level of volatility (volatility=0.6), Basic minimax has the same rank as Variable minimax.

For the average rank order, Variable minimax outperforms Basic minimax, which is consistent with the view that Variable minimax is more responsive to the development of unacceptable hedging errors. Basic minimax outperforms Two-Period minimax, which is consistent with the view that Two-Period minimax is less suitable when the hedger rebalances at the end of one period.

9 Illustrations

In this section, we present illustrations showing the performance of the strategies. In Section 9.1, we illustrate the performance of Variable minimax and in Section 9.2, that of Two-Period minimax.

9.1 Variable minimax: an illustration

An investigation of the performance of the Variable minimax was undertaken for deep-in, deep-out and at-the-money options.

Where the option is progressively deep-in-the-money, in general, the minimax strategies, Basic and Variable, become increasingly unsuitable over time. Because the worst case scenario, defined by the 95% confidence interval about the expected future stock price, is not likely to cover the exercise price, there is a very high probability that the option will remain in-the-money for the next time period. When the option is sufficiently deep-in-the-money, a marginal increase in the stock price is equal to a marginal increase in the value of the option, and as time evolves, the choice of n_t is driven towards N, the number of contracted shares. This implies that minimax is increasingly insensitive to stock price movements and that the minimax hedging error is increasingly a poor measure of the bound on the actual hedging error.

Where the option is progressively deep-out-of-the-money, in general, the minimax strategies become increasingly unsuitable over time. Because the worst case scenario, defined by the 95% confidence interval about the expected future stock price, is not likely to cover the exercise price, there is a very high probability that the option will remain out-of-the-money for the next time period. Because the value of the option reduces to zero over the life of the option, the choice of n_t is driven towards zero. This implies that minimax is increasingly insensitive to stock price movements and that the minimax hedging error is increasingly a poor measure of the bound on the actual hedging error.

Where the option remains close to at-the-money, in general, the minimax strategies are suitable. The worst case scenario, defined by the 95% confidence interval about the expected future stock price, is likely to cover the exercise price. The possible oscillation of the stock price about the exercise price implies that there may be a high variation in potential hedging error corresponding to the 95%-Level. As minimax is designed to be sensitive to this variation in potential hedging error, the minimax hedging error provides a good measure of the bound on actual hedging error. This can be seen in Figure 9.1b (below) where the minimax hedging error (shaded area) increases in time as the option gets closer to maturity, anticipating an increase in the actual hedging error. We note that this increase in the minimax hedging error does not imply that minimax is increasingly unsuitable as a hedging strategy. It implies that minimax is increasingly sensitive to increasing variation in actual hedging error as time gets closer to maturity and the option remains close to at-the-money.

We illustrate the performance of Variable minimax when an option remains close to at-the-money. The time series of stock prices corresponding to the at-the-money option is shown in Figure 9.1a. The exercise price of the option is represented by the horizontal line (Xprice = 1000). The figure is accompanied by Figure 9.1b showing a graph of the minimax hedging errors and actual hedging errors. The shaded area gives the minimax hedging error at every time period, i.e. 5 trading days. The line graph gives the actual hedging error on a daily basis. The arrow in the figure points to the event that early rebalancing has been triggered in Variable minimax.

Figure 9.1a. The stock price series of an option that remains close-to-at-the-money.

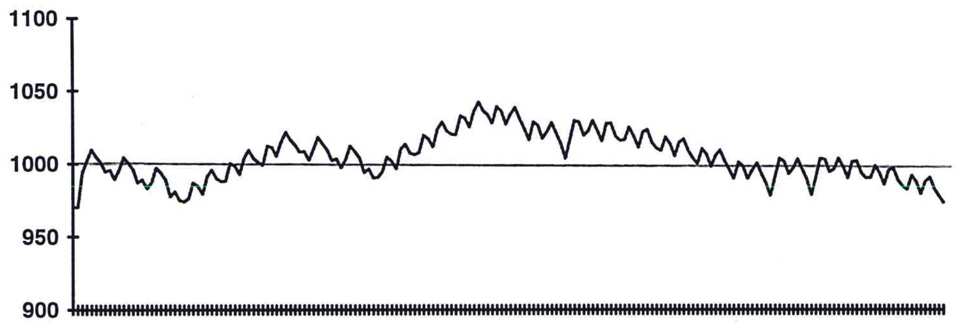
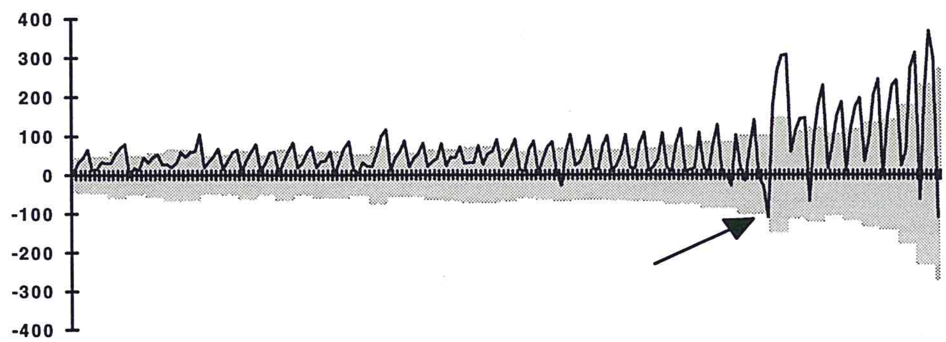


Figure 9.1b. The minimax hedging errors and actual hedging errors, where the arrow points to the actual hedging error that triggered an early rebalancing.



9.2 Two-Period minimax: an illustration

As noted in Section 9.1, because Two-Period minimax has the same formulation as Basic minimax except for the inclusion of a second-period variable, the performance of Two-Period minimax over time is essentially similar to that of Basic minimax. However, for the same reason, Two-Period minimax is different from Basic minimax in the sense that its hedge recommendation falls somewhere in between that of Basic minimax and that of delta hedging.

We present a simple illustration where Two-Period minimax performs better than Basic minimax when the hedger fails to rebalance after the first time period. We consider an option written on the Cadbury Schweppes stock[36], expiring in August 1992 with exercise price of 460p.

At time 0, May 20, 1992 , we have the following data, given in Table 9.2a, with N=100.

Table 9.2a Market prices at time 0, including the hedge recommendation under each strategy.

Strategy	Stock S_0	Exercise X	Option B_0	Hedge Recommendation
Delta hedging	489	460	45	83
Basic minimax	489	460	45	66
Two-Period minimax	489	460	45	72

The absolute value of the minimax hedging error under Basic minimax for one time period is 852, based on $465 \leq S_{t+1} \leq 517$, and that for Two-Period minimax (for two time periods) is 1024, based on $465 \leq S_{t+1} \leq 517$ and $457 \leq S_{t+2} \leq 531$.

From Table 9.2a, we see that the option is in-the-money. The hedge recommendation under Basic minimax suggests that the strategy found a solution consistent with a fall in the future stock price. The hedge recommendation under Two-Period minimax suggests that the strategy is more cautious. It anticipated that a fall in the stock price will result in the worst case hedging error for the first period, just like Basic minimax, but, in its calculation, it also considered a countervailing rise in the stock price for the second period. This way of searching for a solution is reflected in the hedge recommendation which is somewhat in between the hedge recommendation of delta hedging and that of Basic minimax.

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[36]Data source: Datastream International.

At time 1, if the hedger was able to rebalance, his hedging error, under each strategy, is given Table 9.2b.

Table 9.2b Market prices at time 1, including the actual hedging error under each strategy.

Strategy	Stock S_1	Exercise X	Option B_1	Actual hedging error
Delta hedging	484	460	41	-15
Basic minimax	484	460	41	70
Two-Period minimax	484	460	41	40

If the hedger fails to rebalance at time 1, and at time 2 the stock price does rise to a very high level, say the upper bound on S_2 (=531), as anticipated by Two-Period minimax, the actual hedging error for two time periods, under each strategy, is given Table 9.2c.

Table 9.2c Stock price (market price), option price (valued using the option pricing model) and the actual hedging error corresponding to S_2=531.

Strategy	Stock S_2	Exercise X	Option B_2	Actual hedging error
Delta hedging	531	460	83	-562
Basic minimax	531	460	83	-1276
Two-Period minimax	531	460	83	-1024

The actual hedging error under Basic minimax exceeded its minimax hedging error. The cautious hedge recommendation under Two-Period minimax resulted in a better hedging error than that under Basic minimax. As it turns out, delta hedging has the most acceptable hedging error in absolute terms for the second period; this is due to a high hedge recommendation at time 0.

10 Conclusion

In this paper we have presented two multi-period extensions of the basic minimax hedging strategy that the hedger can use under specific situations. Two-Period minimax is designed for the hedger who wishes to consider the possibility that he may fail to rebalance at the end of a preset, one-period, interval. Variable minimax is designed for the hedger who wishes to actively avoid negative hedging errors. The results of the simulation suggest the following rank ordering of the strategies: Variable minimax, Basic minimax, Two-Period minimax, delta hedging.

Variable minimax is a more aggressive strategy in the sense that the hedger regularly monitors the development of the actual hedging error and rebalances early in response to an undesirable event. The strength of Variable minimax is that it gives the hedger a criterion that provides him with the opportunity to rebalance early in order to limit his accumulated hedging errors to some acceptable level. This feature is useful when the volatility of the returns on the underlying stock changes during the coming period. However to the extent that more frequent rebalancing increases transaction costs, any benefits from early rebalancing may be offset by such an increase, even though the Variable minimax constrains these costs.

Two-Period minimax is a more cautious strategy and, as such, it is less suitable than Basic minimax when the hedger can rebalance at the end of the first period. The strength of the Two-Period minimax is that it provides the hedger with a buffer that can, to some extent, absorb the negative effects of an undesirable event that may occur during the second period, if he failed to rebalance at the end of the first period. As with Variable minimax, this feature is useful when the volatility of the returns on the underlying stock changes, especially during the second period.

The two multi-period extensions studied in this paper have been shown to perform well in the circumstances for which they have been designed. The results of the simulation are such that further advances would be expected through studies of Variable minimax under different degrees of moneyness, and studies of Two-Period minimax under changing levels of volatility. However, we do not consider that Variable minimax could usefully be extended to an n-period setting because the system to monitor the actual hedging error is designed to make the strategy responsive to immediate unfavorable events, i.e. in the current period, and not in succeeding periods.

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