Pricing Exotic Options in a Black-Scholes World

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Abstract

In this paper we present derivations of pricing formulae for Collars, Break Forwards, Range Forwards, Forward Start Options, Compound Options, Chooser Options, Asian Options, Lookback Options, Barrier Options, Binary Options, Asset Exchange Options, and Quanto Options. The purpose of this paper is to provide a set of tools with which to value exotic options. The formulae are made as general as possible and are derived in the Black-Scholes world (Black and Scholes (1973)). The results unify and generalise many published and unpublished results and provide a general notational structure.

1 Introduction

The purpose of this paper is to provide a set of tools with which to value exotic options. We do this by presenting derivations of exact and approximate analytical formulae for a range of exotic options[1]. We work throughout in the Black-Scholes world (Black and Scholes (1973)). That is we make the following assumptions; all assets follow correlated geometric Brownian motion processes with constant volatilities[2], assets pay only continuous dividends, costless trading takes place in continuous time, no restrictions on short selling, no restrictions on borrowing and lending at the riskless rate which is constant.

We also unify and generalise many published results and provide a general notational structure. Our basic notation is as follows:

S is a current asset price (S_T the value at time T in the future).

σ is the volatility of S.

r is the riskless rate.

d is a continuous yield on an asset.

T and τ are times to maturity of options (or to decision events).

F is a forward price.

K and k are strike prices of options.

$C_{TYPE}(S,K,T)$ is a call option of type *TYPE*, similarly with P for a put (where *TYPE* is missing it is a standard option).

$E[]$ denotes the risk neutral expectation.

ϕ is 1 for a call and -1 for a put.

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[1] We class any option which has a contract specification different from that of a standard European or American call or put as exotic.

[2] In most cases we can allow the volatility to be a deterministic function of time.

N(.) is the standard normal distribution function.

$N_2(.)$ is the standard bivariate normal distribution function.

For example the Black-Scholes formula is

$$\left.\begin{matrix} C(S,K,T) \\ P(S,K,T) \end{matrix}\right\} = \phi S e^{-dT} N(\phi x) - \phi K e^{-rT} N\left(\phi x - \phi\sigma\sqrt{T}\right) \begin{cases} \phi = 1 \\ \phi = -1 \end{cases}$$

$$x = \frac{\ln\left(\frac{Se^{-dT}}{Ke^{-rT}}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$

Other notation will be introduced as it is used.

1 Portfolios of Standard Options

The simplest extension of standard options is to instruments which can be broken down into portfolios of standard options. These can be categorised as instruments which have general piecewise linear payoffs at a discrete set of future dates. Simple examples are collars, break forwards, and range forwards.

1.1 Collars

The payoff from a collar is equivalent to

$$\min\left(\max(S_T, K_1), K_2\right), \quad 0 < K_1 < K_2$$

which can be rewritten as

$$K_1 + \max(0, S_T - K_1) - \max(0, S_T - K_2)$$

So, a collar is equivalent a portfolio of:

lending the present value of K_1.

long a call option with strike price K_1.

short a call option with strike price K_2.

1.2 Break forward

The payoff from a break forward is equivalent to

$$(F - K) + \max(0, S_T - F), \quad 0 < F < K$$

So, a break forward is equivalent to a portfolio of

lending the present value of $(F - K)$.

long a call option with strike price equal to the forward price (at *T*) F.

1.3 Range forward

The payoff from a range forward is equivalent to

$$(S_T - F) + \max(0, K_1 - S_T) - \max(0, S_T - K_2), \quad 0 < K_1 < F < K_2$$

So, a range forward is equivalent to

buying a forward contract with forward price F.

buying a put option at striking price $K_1 < F$.

selling a call option at striking price $K_2 > F$.

2 Forward Start Options

A forward start option is a standard European option whose strike price is set equal to the current asset price at some prespecified future date (the grant date) (Rubinstein (1991a)). Employee incentive stock options are basically forward start options.

The value of the forward-starting at the money call with time to expiration τ on the grant date t can be written as

$$C(S_t, S_t, \tau) = S_t C(1,1,\tau)$$

Therefore the current value of the forward-start option is therefore

$$Se^{-dt} C(1,1,T) = e^{-dt} C(S,S,T)$$

3 Compound Options

A compound option is a standard option whose underlying asset is a standard option (Geske (1979), Rubinstein (1991d)). The payoff of a compound option can be summarised by

$$\max\left(0,\psi C(S_\tau, K, T-\tau) - \psi k\right)$$

where the overlying option has strike price k and time to expiration τ (ψ is 1/-1 for call/put) and the underlying option has striking price K, time-to-expiration $T > \tau$ and S_τ is the value of the asset underlying the underlying option after time T.

The formula for the present value after time τ of the underlying option is

$$C(S_\tau, K, T-\tau) = \phi S_\tau e^{-d(T-\tau)} N(\phi z) - \phi K e^{-r(T-\tau)} N\left(\phi z - \phi\sigma\sqrt{T-\tau}\right)$$

$$z = \frac{\ln\left(\frac{S_\tau e^{-d(T-\tau)}}{K e^{-r(T-\tau)}}\right)}{\sigma\sqrt{T-\tau}} + \frac{1}{2}\sigma\sqrt{T-\tau}$$

This is just the Black-Scholes formula. Therefore the current value of the compound option is

$$C_{COM} = e^{-r\tau} E\left[\max\left(0,\psi C(S_\tau, K, T-\tau) - \psi k\right)\right]$$

$$C_{COM} = e^{-r\tau} \int_{-\infty}^{+\infty} \max\left(0,\psi C(Se^u, K, T-\tau) - \psi k\right) f(u)\,du$$

where

$$f(u) = \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-v^2/2}, \; u = \ln\left(\frac{S_\tau}{S}\right), \; v = \frac{u - \mu\tau}{\sigma\sqrt{\tau}}, \text{ and } \mu = (r-d) - \frac{\sigma^2}{2}.$$

To evaluate the integral we have

$$\psi S e^{-r\tau} \int_{\ln\left(X/S\right)}^{\psi\infty} e^{u} N(z) f(u) du = \psi\phi S e^{-dT} N_2(\psi\phi x, \phi y; \psi\rho)$$

$$\psi K e^{-r\tau} \int_{\ln\left(X/S\right)}^{\psi\infty} N\left(z - \sigma\sqrt{T-\tau}\right) f(u) du = \psi\phi K e^{-rT} N_2\left(\psi\phi x - \psi\phi\sigma\sqrt{\tau}, \phi y - \phi\sigma\sqrt{T}; \psi\rho\right)$$

$$\psi k e^{-r\tau} \int_{\ln\left(X/S\right)}^{\psi\infty} f(u) du = \psi k e^{-r\tau} N\left(\psi\phi x - \psi\phi\sigma\sqrt{\tau}\right)$$

where

$$x = \frac{\ln\left(Se^{-d\tau} / Xe^{-r\tau}\right)}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}, \quad y = \frac{\ln\left(Se^{-dT} / ke^{-rT}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}, \quad \rho = \sqrt{\tau/T}$$

and X solves the equation

$$\phi X e^{-d(T-\tau)} N(\phi z') - \phi K e^{-r(T-\tau)} N\left(\phi z' - \phi\sigma\sqrt{T-\tau}\right) - K = 0$$

where

$$z' = \frac{\ln\left(Xe^{-d(T-\tau)} / ke^{-r(T-\tau)}\right)}{\sigma\sqrt{T-\tau}} + \frac{1}{2}\sigma\sqrt{T-\tau}$$

That is we integrate the value of the underlying option over the probability density of the asset price at *t* from its value (*X*), which makes the underlying option equal to the overlying strike price, up to infinity.

4 Chooser Options

These are options which allow the holder to choose at some predetermined date whether the option is a standard call or put (Rubinstein (1991b)).

4.1 Standard Chooser

A standard chooser option allows the holder to choose at some predetermined date (τ in the future) whether the option is a standard call or put with the predetermined strike price (K). The payoff from a standard chooser option at the choice date can be written as

$$\max\left(C(S,K,T-\tau),P(S,K,T-\tau)\right)$$

Using put-call parity this can be written as

$$\max\left(C,\left(C-Se^{-d(T-\tau)}+Ke^{-r(T-\tau)}\right)\right)=C(S,K,T-\tau)+\max\left(0,Ke^{-r(T-\tau)}-Se^{-d(T-\tau)}\right)$$

Therefore the payoff from a standard chooser option today will be the same as the payoff from,

buying a call with underlying asset price S, striking price K and time-to-expiration T.

buying a put with underlying asset price $Se^{-d(T-\tau)}$, striking price $Ke^{-r(T-\tau)}$ and time-to-expiration τ.

Therefore the value of the standard chooser is

$$Se^{-dT}N(x)-Ke^{-rT}N\left(x-\sigma\sqrt{T}\right)-Se^{-dT}N(-y)+Ke^{-rT}N\left(-y+\sigma\sqrt{T}\right)$$

where

$$x=\frac{\ln\left(Se^{-dT}/Ke^{-rT}\right)}{\sigma\sqrt{T}}+\frac{1}{2}\sigma\sqrt{T},\quad y=\frac{\ln\left(Se^{-dT}/Ke^{-rT}\right)}{\sigma\sqrt{\tau}}+\frac{1}{2}\sigma\sqrt{\tau}$$

4.2 Complex chooser option

The complex chooser generalises the standard chooser by allowing the standard call and put which are chosen between to have different strikes and times to maturity. The payoff from a complex chooser option can be written as,

$$\max\left(C(S,K_1,T_1-\tau),P(S,K_2,T_2-\tau)\right)$$

where the chosen call (put) has striking price K_1 (K_2) and time-to-expiration $T_1-\tau$ ($T_2-\tau$) on the choice date.

The current value of a complex chooser option is therefore

$$C_{\text{CHOOSER}}=e^{-r\tau}E\left[\max\left(C(S,K_1,T_1-\tau),P(S,K_2,T_2-\tau)\right)\right]$$

$$C_{\text{CHOOSER}}=e^{-r\tau}\int_{-\infty}^{+\infty}\max\left(C(S,K_1,T_1-\tau),P(S,K_2,T_2-\tau)\right)f(u)du$$

where

$$f(u)=\frac{1}{\sigma\sqrt{2\pi\tau}}e^{-v^2/2},\ u=\log\left(\frac{S_\tau}{S}\right),\ v=\frac{u-\mu\tau}{\sigma\sqrt{\tau}}\ \text{and}\ \mu=(r-d)-\frac{\sigma^2}{2}$$

This can be evaluated in a similar way to the compound option to give

$$C_{CHOOSER}=Se^{-dT_1}N_2(x,y_1;\rho_1)-K_1e^{-rT_1}N_2\left(x-\sigma\sqrt{\tau},y_1-\sigma\sqrt{T_1};\rho_1\right)$$
$$-Se^{-dT_2}N_2(-x,-y_2;\rho_2)+K_2e^{-rT_2}N_2\left(-x+\sigma\sqrt{\tau},-y_2+\sigma\sqrt{T_2};\rho_2\right)$$

where

$$x = \frac{\ln\left(Se^{-d\tau} / Xe^{-r\tau}\right)}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}, \quad y_i = \frac{\ln\left(Se^{-dT_i} / K_i e^{-rT_i}\right)}{\sigma\sqrt{T_i}} + \frac{1}{2}\sigma\sqrt{T_i}, \; \rho_i = \sqrt{\tau / T_i}$$

and X solves the following equation

$$\begin{aligned} &Xe^{-d(T_1-\tau)}N(z_1) - K_1 e^{-r(T_1-\tau)}N\left(z_1 - \sigma\sqrt{T_1-\tau}\right) + \\ &Xe^{-d(T_2-\tau)}N(-z_2) - K_2 e^{-r(T_2-\tau)}N\left(-z_2 + \sigma\sqrt{T_2-\tau}\right) = 0 \end{aligned}$$

where

$$z_i = \frac{\ln\left(Xe^{-d(T_i-\tau)} / K_i e^{-r(T_i-\tau)}\right)}{\sigma\sqrt{T_i-\tau}} + \frac{1}{2}\sigma\sqrt{T_i-\tau}$$

5 Asian Options

Asian options are standard calls or puts on an arithmetic average of the asset price over the life of the option (Kemna and Vorst (1990), Levy (1992), Levy and Turnbull (1992)).

Average strike options are standard calls or puts with the strike price set equal an arithmetic average of the asset price.

They are typically used to reduce the sensitivity of an option to underlying price manipulation at expiration or hedge regular cash flows.

Definitions

If the average $A(t_m)$ at time t_m of the asset prices $S(t_i)$ at times t_i is given by

$$A(t_m) = \frac{1}{m+1}\sum_{i=0}^{m} S(t_i)$$

the pay-off of an Asian call at expiration date t_N is

$$\max(A(t_N) - K, 0)$$

Put-Call Parity

At the maturity date we know that

$$C(T) - P(T) = A(t_N) - K$$

Therefore at any earlier date we have

$$C(t) - P(t) = \tfrac{1}{N+1}\sum_{i=0}^{N} F(t_i) - Ke^{-r(T-t)}$$

Valuation

The difficulty with pricing Asian options arises because the sum of lognormally distributed variables has no simple closed form expression. However, the geometric average is itself lognormally distributed and therefore the valuation of geometric Asian options is straightforward leading to formulae very similar to Black-Scholes.

One approach for arithmetic Asian options is to approximate the distribution of the arithmetic average as lognormal. The first two moments of the lognormal distribution are set equal to the first two moments of the arithmetic average (which are straightforward to calculate). This again results in a Black-Scholes formula.

Consider an Asian call option. The pay-off at maturity is

$$\max(M(t) - K^*, 0)$$

where,

$$M(t) = A(t_N) - A_k, \quad K^* = K - A_k, \quad A_k = \tfrac{(m+1)}{N+1} A(t_m)$$

here we have separated out the known part of the average and adjusted the strike by this amount. This makes the following algebra slightly easier since we don't have to carry around the known part.

Now we assume that $M(t)$ is lognormally distributed, $\ln(M(t)) \sim N(\alpha, \beta)$

Now for a lognormally distributed asset we have

$$E[\max(M(t)-K^*)] = e^{\alpha+\frac{1}{2}\beta} N(d_1) - K^* N(d_2)$$
$$d_1 = \tfrac{\alpha - \ln(K^*)+\beta}{\sqrt{\beta}} \quad , d_2 = d_1 - \sqrt{\beta}$$

this is the basis of the Black-Scholes formula. In order to obtain the option value we simply discount this by the riskless rate. Now, the moments of a lognormal variable are given by $E[M^n] = \exp(n\alpha + n^2 \beta/2)$. So for the first two moments we have

$$\left.\begin{aligned} \ln(E[M]) &= \alpha + \tfrac{1}{2}\beta \\ \ln(E[M^2]) &= 2\alpha + 2\beta \end{aligned}\right\} \begin{aligned} \alpha &= 2\ln(E[M]) - \tfrac{1}{2}\ln(E[M^2]) \\ \beta &= \ln(E[M^2]) - 2\ln(E[M]) \end{aligned}$$

Now, we need the true moments of $M(t)$ which is made up of the unknown, future fixings

$$M = \tfrac{1}{N+1}(S_{m+1} + S_{m+2} + \ldots + S_N)$$

This can be rewritten as

$$M = \tfrac{1}{N+1}(S_m x_{m+1} + S_m x_{m+1} x_{m+2} + \ldots + S_m x_{m+1} x_{m+2} \ldots x_N)$$

where

$$x_i = \exp((r-d) - \tfrac{1}{2}\sigma^2)\Delta t_i + \sigma\sqrt{\Delta t_i}\tilde{z}_i$$

and are independent. Taking expectations we obtain

$$\mathrm{E}[M] = \tfrac{S_m}{N+1} \sum_{k=m+1}^{N} \mathrm{E}[x_{m+1}] \ldots \mathrm{E}[x_k]$$

$$\mathrm{E}[M] = \tfrac{S_m}{N+1} \sum_{k=m+1}^{N} \exp((r-d) \sum_{j=m+1}^{k} \Delta t_j)$$

For the second moment we have

$$M^2 = \left(\tfrac{S_m}{N+1}\right)^2 (x_{m+1})^2 (1 + x_{m+2} + \ldots + x_{m+2} \ldots x_N)^2$$

taking expectations we obtain

$$\mathrm{E}[M^2] = \left(\tfrac{S_m}{N+1}\right)^2 \mathrm{E}[(x_{m+1})^2] \mathrm{E}[(1 + x_{m+2}(1 + \ldots + x_{m+3} \ldots x_N))^2]$$

$$= \left(\tfrac{S_m}{N+1}\right)^2 \mathrm{E}[(x_{m+1})^2](1 + 2\mathrm{E}[x_{m+2}]\mathrm{E}[(1 + \ldots x_N)] + \mathrm{E}[(x_{m+2})^2]\mathrm{E}[(1 + \ldots x_N)^2])$$

This can be evaluated recursively or by fully expanding and summing the terms.

If the known average becomes large then $K^* = K - A_k$ can become negative and the Black-Scholes formula breaks down. However, this corresponds to the case of a negative strike price for a standard option. For a call this guarantees it will finish in the money and for a put that it will finish out of the money. So there is no longer any option and the value is simply

$$e^{-rT} \mathrm{E}[M(t) - K^*] = e^{-rT} (\mathrm{E}[M(t)] - K^*)$$

Alternatively we can assume $A(t_N) = M(t) + A_k$ is lognormally distributed so we have

$$E[A(t_N)] = E[M(t)] + A_k$$

$$E[A(t_N)^2] = E[M(t)^2] + 2A_k E[M(t)] + A_k^2$$

and we simply replace $E[M]$ and $E[M^2]$ with $E[A]$ and $E[A^2]$ in the expressions for α and β.

S_m can be viewed as the asset price at any time before the first fixing or between fixings with Δt_{m+1} the time to the first or next fixing. There can be non-uniform fixing intervals and also weighting factors on each of the fixings. Also, there can be different volatilities and interest rates between fixings.

The payoff of an **average strike** call option is $\max(S_{t_N} - A(t_N), 0)$.

This is just the payoff of an option to exchange the average for the asset at maturity and therefore we can use Margrabes formula to exchange one asset for another.

The covariance between these two assets can be computed in the same way as the first two moments of the average

$$A(t_N) = A_k + M(t)$$

$$A(t_N) = A_k + \tfrac{S_m}{N+1}(x_{m+1} + x_{m+1}x_{m+2} + \ldots + x_{m+1}x_{m+2}\cdots x_N)$$

$$S_N = S_m x_{m+1} x_{m+2} \ldots x_N$$

$$E[A(t_N)S_N] = A_k S_m E[x_{m+1}x_{m+2}\cdots x_N] + \tfrac{S_m^2}{N+1}(E[x_{m+1}^2 x_{m+2}\cdots x_N] + \ldots + E[x_{m+1}^2 x_{m+2}^2 \cdots x_N^2])$$

6 Lookback Options

Lookback options are standard calls or puts except that the strike price is set to the minimum price (for a call) or the maximum price (for a put) observed over the life of the option (Goldman, Sosin and Gatto (1979), Garman (1989)). The dates and times at which the price is observed are specified in advance and are called fixings. These options thus allow the holder to buy or sell the option for the best of the observed prices. If the maturity date is a fixing date then they are not really options since they will always be exercised. That is the worst payoff that can occur is zero if the price at maturity is the maximum or minimum of the observed prices. Simple analytical formulae exist for European style lookbacks which are fixed continuously. In reality these options are fixed at most on a daily basis and often much less frequently. This can substantially affect the value of the option since the more infrequently the price is observed the less extreme the maximum or minimum will be.

We consider first the continuously fixed lookback call. The pay-off of this option is

$$S_T - \min(M, M_T)$$

where M is the current minimum or prespecified strike and M_T is the minimum over the remaining time to expiry. The value of the option is given by the discounted expectation of this payoff

$$C_{LB} = e^{-rT}(E[S_T] - E[\min(M, M_T)])$$

This can be written as

$$C_{LB} = Se^{-dT} - Me^{-rT}\int_{\ln(M/S)}^{\infty} f(n)dn - Se^{-rT}\int_{-\infty}^{\ln(M/S)} e^n f(n)dn$$

where

$$f(n)=\frac{2}{\sigma\sqrt{T}}N'\left(\frac{n-\mu T}{\sigma\sqrt{T}}\right)+\frac{2\mu}{\sigma^2}e^{\frac{2\mu}{\sigma^2}n}N\left(\frac{n+\mu T}{\sigma\sqrt{T}}\right),\quad \mu=r-d-\frac{1}{2}\sigma^2$$

The first term is the discounted expectation of the terminal asset price ($Se^{(r-d)T}$). The second term is the discounted current strike (minimum) conditional on the minimum over the remaining time not setting a lower value. Finally the third term is the minimum over the remaining time conditional on this being lower than the current value. Evaluating the integrals leads to

$$\begin{aligned}
C_{LB} &= Se^{-dT}\\
&-Me^{-rT}\left\{1-\left[N\left(\frac{\ln(M/S)-\mu T}{\sigma\sqrt{T}}\right)+e^{\frac{2\mu}{\sigma^2}\ln\left(\frac{M}{S}\right)}N\left(\frac{\ln(M/S)+\mu T}{\sigma\sqrt{T}}\right)\right]\right\}\\
&-Se^{-rT}\left\{\begin{array}{l}\dfrac{\frac{2\mu}{\sigma^2}+2}{\frac{2\mu}{\sigma^2}+1}e^{\mu T+\frac{1}{2}\sigma^2 T}N\left(\dfrac{\ln(M/S)+\mu T+\sigma^2 T}{\sigma\sqrt{T}}\right)+\\[2ex] \dfrac{\frac{2\mu}{\sigma^2}}{\frac{2\mu}{\sigma^2}+1}e^{\left(\frac{2\mu}{\sigma^2}+1\right)\ln\left(\frac{M}{S}\right)}N\left(\dfrac{\ln(M/S)+\mu T}{\sigma\sqrt{T}}\right)\end{array}\right\}
\end{aligned}$$

This can be simplified and re-arranged to give the price of the continuously fixed lookback call as a standard call struck at the current minimum plus an adjustment

$$\begin{aligned}
C_{LB} &= Se^{-dT}N\left(x+\sigma\sqrt{T}\right)-Me^{-rT}N(x)\\
&+\tfrac{S}{B}\left(e^{-rT}\left(\tfrac{S}{M}\right)^{-B}N\left(y+B\sigma\sqrt{T}\right)-e^{-dT}N(y)\right)
\end{aligned}$$

where

$$B=\frac{2(r-d)}{\sigma^2},\quad x=\frac{\ln(\frac{S}{M})+((r-d)-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},\quad y=\frac{-\ln(\frac{S}{M})-((r-d)+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

For the continuously fixed lookback put we simply reverse the signs of the arguments to $N(.)$ and reverse the sign of the formula

$$\begin{aligned}
P_{LB} &= -Se^{-dT}N\left(-x-\sigma\sqrt{T}\right)+Me^{-rT}N(-x)\\
&-\tfrac{S}{B}\left(e^{-rT}\left(\tfrac{S}{M}\right)^{-B}N\left(-y-B\sigma\sqrt{T}\right)-e^{-dT}N(-y)\right)
\end{aligned}$$

M is now interpreted as the current maximum.

7 Barrier Options

Barriers options are also generically called knock-in or knock-out (or less commonly drop-in or drop-out)(Cox and Rubinstein (1985), Rubinstein and Reiner (1991a)). Specifically they are called down-and-in, down-and-out, up-and-in, and up-and-out. They are standard options which appear (knock-in) or disappears (knock-out) if the asset price hits a pre-determined barrier level from above or below. They may give a fixed pay-off (rebate) if the option disappears or never appears. They are cheaper than standard options because of the possibility of the option disappearing or never appearing. For example a down-and-out call option will disappear if the asset price hits the barrier which is below the current asset price and usually the strike price. As the asset price becomes very low relative to the strike price the chances of it finishing in the money are very low. With a standard option the buyer pays for this chance. The down-and-out option eliminates this allowing the buyer to only pay for higher probability payoffs. The dates and times at which the crossing of the barrier by the asset price is checked are specified in advance (the fixings). Analytical formulae can be derived for continuously fixed European style barrier options. In reality these options are fixed at most on a daily basis and often less frequently. This substantially affects the price of the option since the probability of the price having crossed the barrier decreases with the frequency of fixing.

In order to price the continuously fixed barrier options we need three probability density functions.

The density function of the natural logarithm of the risk-neutral underlying asset return

$$f(u)=\frac{1}{\sigma\sqrt{2\pi T}}e^{-v^2/2}, \quad v=\frac{u-\mu T}{\sigma\sqrt{T}}, \quad \mu=(r-d)-\tfrac{\sigma^2}{2}$$

The density function of the natural logarithm of the risk-neutral underlying asset return when the underlying asset price starts above/below the barrier (*H*), crosses the barrier but ends up above/below the barrier at expiration ($\psi = 1 / \psi = -1$)

$$g(u) = e^{2\mu\alpha/\sigma^2} \frac{1}{\sigma\sqrt{2\pi T}} e^{-v^2/2}, \quad v = \frac{(u - 2\psi\alpha - \psi\mu T)}{\sigma\sqrt{T}}, \quad \alpha = \ln\left(\tfrac{H}{S}\right)$$

The density function of the first passage time (τ) for the underlying asset price to cross the barrier

$$h(\tau) = \frac{-\psi\alpha}{\sigma\tau\sqrt{2\pi\tau}} e^{-v^2/2}, \quad v = \frac{(-\psi\alpha - \psi\mu\tau)}{\sigma\sqrt{\tau}}$$

The value of any European knock-out option is simply the integral of the payoff over the terminal probability density of the asset conditional on it not crossing the barrier.

The value of a European knock-in option is simply the integral of the payoff over the terminal probability density conditional on the barrier being crossed.

The integrals can always be split into two, one with the terminal asset value and one with the strike price where the integrals are over the region where the payoff is none zero. This is either defined by the strike price or the barrier level whichever is the higher for a call or lower for a put.

Finally barrier options often pay a rebate if they knock-out or never knock-in. Here we will just consider a fixed rebate. Knock-in options are easier since the rebate is paid at the expiry date if the option never knocks in. Thus to value this we simply integrate the rebate (which is constant) over the probability density conditional on the barrier being crossed.

For knock-out options the date on which the rebate will be paid is unknown. Thus the time over which we must discount the rebate is random and we must integrate over the probability density of the first passage time through the barrier.

Using ϕ to indicate a call or put and ψ to indicate whether the asset price starts from above or below the barrier we can represent every case with six integrals:

$$
\begin{aligned}
I_1 &= e^{-rT}\int_{\ln\left(K/S\right)}^{\phi\infty} \phi\left(Se^{u}-K\right)f(u)du \\
&= \phi Se^{-dT}N(\phi x)-\phi Ke^{-rT}N\left(\phi x-\phi\sigma\sqrt{T}\right)
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= e^{-rT}\int_{\ln\left(H/S\right)}^{\phi\infty} \phi\left(Se^{u}-K\right)f(u)du \\
&= \phi Se^{-dT}N(\phi x_1)-\phi Ke^{-rT}N\left(\phi x_1-\phi\sigma\sqrt{T}\right)
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= e^{-rT}\int_{\ln\left(K/S\right)}^{\psi\infty} \phi\left(Se^{u}-K\right)g(u)du \\
&= \phi Se^{-dT}\left(H/S\right)^{2\lambda}N(\psi y)-\phi Ke^{-rT}\left(H/S\right)^{2\lambda-2}N\left(\psi y-\psi\sigma\sqrt{T}\right)
\end{aligned}
$$

$$
\begin{aligned}
I_4 &= e^{-rT}\int_{\ln\left(H/S\right)}^{\psi\infty} \phi\left(Se^{u}-K\right)g(u)du \\
&= \phi Se^{-dT}\left(H/S\right)^{2\lambda}N(\psi y_1)-\phi Ke^{-rT}\left(H/S\right)^{2\lambda-2}N\left(\psi y_1-\psi\sigma\sqrt{T}\right)
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= \mathrm{Re}^{-rT}\int_{\ln\left(H/S\right)}^{\psi\infty} \left[f(u)-g(u)\right]du \\
&= \mathrm{Re}^{-rT}\left[N\left(\psi x_1-\psi\sigma\sqrt{T}\right)-\left(H/S\right)^{2\lambda-2}N\left(\psi y_1-\psi\sigma\sqrt{T}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
I_6 &= R\int_0^T e^{-r\tau}h(\tau)d\tau \\
&= R\left[\left(H/S\right)^{a+b}N(\psi z)-\left(H/S\right)^{a-b}N\left(\psi z-2\psi b\sigma\sqrt{T}\right)\right]
\end{aligned}
$$

where

$$
x=\frac{\ln(S/K)}{\sigma\sqrt{T}}+\lambda\sigma\sqrt{T},\quad x_1=\frac{\ln(S/H)}{\sigma\sqrt{T}}+\lambda\sigma\sqrt{T}
$$

$$y=\frac{\ln\left(H^2/SK\right)}{\sigma\sqrt{T}}+\lambda\sigma\sqrt{T},\ y_1=\frac{\ln(H/S)}{\sigma\sqrt{T}}+\lambda\sigma\sqrt{T}$$

$$\lambda=1+\left(\mu/\sigma^2\right),\ z=\frac{\ln(H/S)}{\sigma\sqrt{T}}+b\sigma\sqrt{T}$$

$$a=\mu/\sigma^2,\ b=\frac{\sqrt{\mu^2+2\sigma^2 r}}{\sigma^2}$$

The first is the standard Black-Scholes integral, the second is identical except the non-zero payoff region is defined by H (this is the same as for a gap option, (see section 8)).

The third and fourth is the integral of the payoff over the probability density when the barrier is crossed, where K and H define the non-zero payoff region respectively.

Finally, integrals five and six are the value of the rebate for knock-in and knock-out options respectively.

The In Barrier Option Family

We now give an example of constructing the price of knock-in barrier options from the six integrals derived above.

Example: Down-and-in call.

Firstly there are two distinct cases, one with the strike price greater than the barrier and one with the strike price less than the barrier.

The first case is the simplest since the barrier is below the strike price then the option will only payoff if the barrier was crossed and so we only need the conditional density *g(u)*. The value of the option is the sum of the call pay-off integrated over the probability density of the terminal asset price conditional on it crossing the barrier (I_3) and the rebate (I_5):

$$C_{DI(K>H)} = I_3 + \mathrm{I}_5 \quad \{\psi = 1, \phi = 1\}$$

The second case is more complicated. If the terminal asset price is between the barrier and the strike price then the option pays off with certainty and so we only need the unconditional density *f(u)*. But if the terminal asset price is above the strike price the option only pays off if the barrier was crossed and so we need the conditional density *g(u)*. The value of the option is the sum of the call pay-off integrated over *f(u)* between *K* and *H* ($I_1 - I_2$) and integrated over *g(u)* from *H* to infinity (I_4) plus the rebate (I_5):

$$C_{DI(K<H)} = I_1 - I_2 + I_4 + \mathrm{I}_5 \quad \{\psi = 1, \phi = 1\}$$

Summary of the In Barrier Option Family

Down-and-in call	$C_{DI(K>H)} = I_3 + \mathrm{I}_5$	$C_{DI(K<H)} = I_1 - I_2 + I_4$ $+\mathrm{I}_5$	$\{\psi = 1, \phi = 1\}$
Down-and-in put	$P_{DI(K>H)} = I_2 - I_3 + I_4$ $+\mathrm{I}_5$	$P_{DI(K<H)} = I_1 + \mathrm{I}_5$	$\{\psi = 1, \phi = \text{-}1\}$
Up-and-in call	$C_{UI(K>H)} = I_1 + \mathrm{I}_5$	$C_{UI(K<H)} = I_2 - I_3 + I_4$ $+\mathrm{I}_5$	$\{\psi = \text{-}1, \phi = 1\}$
Up-and-in put	$P_{UI(K>H)} = I_1 - I_2 + I_4$ $+\mathrm{I}_5$	$P_{UI(K<H)} = I_3 + \mathrm{I}_5$	$\{\psi = \text{-}1, \phi = \text{-}1\}$

The Out Barrier Option Family

Here we give an example of constructing the price of knock-out barrier options from the six integrals.

Example: The Down-and-out call.

Once again there are two distinct cases, one with the strike price greater than the barrier and one with the strike price less than the barrier.

In first case, since the barrier is below the strike price then the option will only payoff if the barrier was not crossed and so we only need the density conditional on the barrier not being crossed which is simply *(f(u)-g(u))* The value of the option is the sum of the call pay-off integrated over the probability density of the terminal asset price conditional on it not crossing the barrier ($I_1 - I_3$) and the rebate (I_6):

$$C_{DO(K>H)} = I_1 - I_3 + I_6 \quad \{\psi = 1, \phi = 1\}$$

In the second case, if the terminal asset price is below the barrier then the option has disappeared and the payoff is zero (we consider the rebate later) even though the asset price is above the strike price. If the terminal asset price is above the barrier price the option only pays off if the barrier was not crossed and so we need the conditional density *f(u)-g(u)*. The value of the option is the sum of the call pay-off integrated over *f(u)-g(u)* from *H* to infinity ($I_2 - I_4$) plus the rebate (I_6):

$$C_{DO(K<H)} = I_2 - I_4 + I_6 \quad \{\psi = 1, \phi = 1\}$$

Summary of the Out Barrier Option Family

Down-and-out call	$C_{DO(K>H)} = I_1 - I_3 + I_6$	$C_{DO(K<H)} = I_2 - I_4 + I_6$	$\{\psi = 1, \phi = 1\}$
Down-and-out put	$P_{DO(K>H)} = I_1 - I_2 + I_3 - I_4 + I_6$	$P_{DO(K<H)} = I_6$	$\{\psi = 1, \phi = -1\}$
Up-and-out call	$C_{UO(K>H)} = I_6$	$C_{UO(K<H)} = I_1 - I_2 + I_3 - I_4 + I_6$	$\{\psi = -1, \phi = 1\}$
Up-and-out put	$P_{UO(K>H)} = I_2 - I_4 + I_6$	$P_{UO(K<H)} = I_1 - I_3 + I_6$	$\{\psi = -1, \phi = -1\}$

8 Binary Options

These are options which have discontinuous payoffs also called Digital options (Rubinstein and Reiner (1991b)).

8.1 Cash or Nothing Options

The pay-off of these options can be written as

$$X \text{ if } \phi S_T > \phi K, 0 \text{ otherwise}$$

The present value of the expectation of this pay-off is exactly like the second term in the Black-Scholes equation. The value is therefore,

$$Xe^{-rT} N\left(\phi x - \phi\sigma\sqrt{T}\right)$$

8.2 Asset or Nothing Options

The pay-off of these options can be written as

$$S_T \text{ if } \phi S_T > \phi K, 0 \text{ otherwise}$$

The present value of the expectation of this pay-off is exactly like the first term in the Black-Scholes equation. The value is therefore,

$$Se^{-dT} N(\phi x)$$

8.3 Gap Options

The pay-off of these options can be written as

$$\phi S_T - \phi X \;\; \text{if} \;\; \phi S_T > \phi K, \; 0 \;\; \text{otherwise}$$

These are slightly more general than standard options since we have separate 'strikes' for determining whether exercise is optimal (K) and the size of the pay-off (X).

They can be valued by simply substituting X for K in the Black-Scholes equation where it determines the size of the pay-off.

$$\phi S e^{-dT} N(\phi x) - \phi X e^{-rT} N\left(\phi x - \phi \sigma \sqrt{T}\right)$$

$$\mathrm{x} = \frac{\ln\left(\frac{Se^{-dT}}{Ke^{-rT}}\right)}{\sigma\sqrt{\mathrm{T}}} + \frac{1}{2}\sigma\sqrt{T}$$

9 Asset Exchange Options

These are options to exchange one asset for another (Margrabe (1978), Rubinstein (1991c)). The payoff of these options can be written as

$$\max(0, S_1 - S_2)$$

which can be rewritten in units of asset 2's price

$$C/S_2 = \max(0, S_1/S_2 - 1)$$

The value of a call option on asset 1(in units of asset 2) with a strike price of 1 is

$$C = S_1 e^{-d_1 T} N(x) - S_2 e^{-d_2 T} N\left(x - \Sigma\sqrt{T}\right)$$

$$x = \frac{\ln\left(\dfrac{S_1 e^{-d_1 T}}{S_2 e^{-d_2 T}}\right)}{\Sigma\sqrt{T}} + \tfrac{1}{2}\Sigma\sqrt{T}\,, \quad \Sigma = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$$

10 Quanto Options

These are standard options where the payoff depends on an exchange rate which is assumed to be lognormally distributed (a prime indicates the value is in foreign currency units)(Reiner (1992)).

10.1 A Foreign Equity Call Struck in Foreign Currency

These give the holder investment in foreign equity, protection against the downside risk of the equity, but no protection against foreign exchange risk. The payoff is

$$X \max(0, S' - K')$$

An option writer in the foreign country is indifferent between this option which pays off in your currency and a standard option paying off in the local currency.

The value is a standard call option in the foreign currency multiplied by the current exchange rate.

10.2 A Foreign Equity Call Struck in Domestic Currency

These give the holder investment in foreign equity, protection against the downside risk of the equity, and some protection against foreign exchange risk. The payoff is

$$\max(0, S'X - K)$$

An option writer in the foreign country would see it as

$$\max(0, S' - KX')$$

This is an option to exchange K units of our currency for one unit of stock. Therefore we can use the Margrabe formula for an option to exchange one asset for another.

10.3 A Fixed Exchange Rate Foreign Equity Call

These give the holder investment in foreign equity, protection against the downside risk of the equity, and some protection against foreign exchange risk. The payoff is

$$X_o \max(0, S' - K') = \max(0, S'X_o - K)$$

An option writer in the foreign country would see it as

$$X_o X' \max(0, S' - K')$$

If the equity and the exchange rate are uncorrelated we can take expectations of the products separately otherwise we have to adjust for the exchange rate.

10.4 An Equity Linked Foreign Exchange Call

These give the holder investment in foreign equity, no protection against the downside risk of the equity, but protection against the downside risk of the exchange rate. The payoff is

$$S' \max(0, X - K)$$

An option writer in the foreign country would see it as

$$S'\max(0,1-KX') = KS'\max(0,{}^{1}\!/_{K}-X')$$

This is exactly analogous to a fixed exchange rate foreign equity put with the roles of the equity and exchange rate interchanged. So we can use the results from that example.

The value of these options can be described by the Black-Scholes formula but with adjusted parameters. The adjustments are summarised in the following table.

Quanto Options Summary Table

Option Type	**Corresponding Parameters**				
Black-Scholes (domestic equity)	S	K	r	d	σ_S
Garman-Kohlhagen (currency)	X	K	r	r_f	σ_X
Foreign equity/ foreign strike	$S'K$	$K'X$	r_f	d	$\sigma_{S'}$
Foreign equity/ domestic strike	$S'X$	K	r	d	$\sigma_{S'X}$
Fixed exchange rate foreign equity	$S'X_0$	$K'X_0$	r	$r+d-r_f+\rho_{S'X}\sigma_{S'}\sigma_X$	$\sigma_{S'}$
Equity-linked foreign exchange	$S'X$	KS'	$r+d-r_f+\rho_{S'X}\sigma_{S'}\sigma_X$	d	σ_X

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