On the Simulation of Contingent Claims

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For many complex option valuation problems analytical solutions are not possible. In these cases Monte Carlo simulation is an important numerical solution tool. Furthermore, simulation can reveal important insights into the hedging of these complex options. In its basic form, however, Monte Carlo simulation is computationally inefficient. In this article we address this problem. We describe a new approach to constructing control variates, based on the "Greek letter" risk exposures of the option, which can improve the efficiency of the simulation dramatically. We illustrate the use of the technique for a standard European call option, and for a realistic example involving valuing and hedging a look-back call with discrete fixing of the minimum price under stochastic volatility.

In today's rapidly developing derivatives markets, new and more complex instruments are continuing to appear. For many of these instruments, analytical valuation formulas are not possible, or are possible only in an idealized world. In these cases Monte Carlo simulation is an important tool. Simulation can also give important insights into the hedging of these instruments. Yet in its basic form Monte Carlo simulation is not computationally efficient.

The Monte Carlo simulation technique for option valuation was first introduced by Boyle [1977]. It is a numerical valuation technique that is typically used to value complex path-dependent European-style options. The basis of the technique is the observation by Cox and Ross [1976] that if a riskless hedge can be formed the option value can be expressed as the discounted expectation of the payoff it would produce in a risk-neutral world. The technique is thus to simulate a large number of risk-neutral price paths, and the payoff on the option for each path. The estimate of the option price is the discounted mean of these simulated payoffs.

The technique has been used extensively in the literature to obtain option prices for models that do not have analytical formulas, including models with stochastic volatility such as Johnson and Shanno [1987], Hull and White [1987], and Scott [1987]. But little work has been done on the efficiency of the technique since Boyle [1977].

One way to increase efficiency is to use a "control variate" in the simulation. A control variate is a known value that is closely related to the unknown quantity to be simulated. For example, a good control variate for an option problem might be the price of a similar type of option for which an analytic valuation equation exists. The simulation then estimates the *difference* between the unknown value and the control variate, which may be determined with much greater accuracy than the unknown value itself for any given number of simulation runs.

Hull and White [1988] describe how control variates can be applied in a naive way to lattice valuation techniques such as the binomial method. More recently, Kemna and Vorst [1990] propose Monte Carlo simulation as a valuation method for arithmetic average rate (Asian) options. They use the analytical formula for the geometric average option as a naive control variate in order to increase the efficiency of the method to an acceptable level.

The idea of control variates is closely related to the idea of hedging risk. The variance or risk of the payoff of a delta-hedged option is much smaller than that of an unhedged payoff. So by simulating a delta hedge for each price path and subtracting this from the option payoff we can reduce the variability of the resultant payoff. Therefore the estimate of the mean payoff will be more accurate.

In this article we describe a new, general approach to constructing and combining control variates to improve the efficiency of Monte Carlo simulation of contingent claims. The control variates are intimately related to the hedging of the option. They therefore not only lead to improved efficiency but can also provide insights into the hedging or replication of the option. We illustrate these ideas with a realistic example of the valuation of a foreign exchange lookback call for which the minimum price is fixed at discrete time intervals and for which the volatility of the underlying exchange rate is stochastic.

MONTE CARLO SIMULATION AND VARIANCE REDUCTION: AN ILLUSTRATIVE EXAMPLE

We describe the procedure for Monte Carlo valuation of a European call option using control variates in order to give an elementary presentation of the implementation of the control variate techniques. This example also illustrates the power of these techniques.

The value of an option can be regarded as the mean payoff taken over many price paths in a risk-neutral world and discounted back to today at the riskless interest rate. Suppose that we wish to value a European call with strike price K and time to maturity T, on a non-dividend-paying stock S with volatility σ in a Black-Scholes world. The risk-neutral stochastic process for the asset price is

$$dS = rSdt + \sigma Sdz \tag{1}$$

or

$$d \ln (S) = (r - 1/2\, \sigma^2)\, dt + \sigma dz \tag{2}$$

We first divide the lifetime [0, T] of the option into N equally spaced intervals of length $\Delta t \equiv T/N$. For each h = 1, ..., N we have $t_h - t_{h-1} = \Delta t$. The logarithmic increments $\Delta s_t \equiv \log(S_t) - \log(S_{t-\Delta t})$ of the stock price over each of these time steps are iid normal, with mean $(r - 1/2\sigma^2)\Delta t$ and variance $\sigma^2 \Delta t$. We have

$$\log(S_T) = \log(S_0) + \sum_{h=1}^{N} \Delta s_{t_h} \tag{3}$$

$$\Delta s_{t_h} = \left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\,\tilde{z} \tag{4}$$

where $\tilde{z}$ is a standard normal random variable, and lower case s denotes the log of the stock price. Thus, with N draws of $\tilde{z}$ from a standard normal distribution we can simulate the final stock price S_T, and determine the option payoff value, $\max\{(S_T - K), 0\}$.

A simple way to implement the control variate principle is to construct a second price path based on the negatives of the standard normal random variables ($\tilde{z}$) that were drawn in constructing the price path (3). The average payoff from the two price paths is then used as the sample payoff. Since the two price paths are negatively correlated, the variance of their average is lower than if they were independent. We do not alter the distribution of the asset price because for each value of $\tilde{z}$ the negative is equally likely. This is called the antithetic variates method.

A more sophisticated approach is to base the control variates on a hedged portfolio. Consider delta-hedging a European call option. For the hedge we have a continuously rebalanced holding $\partial C(t_h)/\partial S$ of the stock. The changes in the value of this hedge as the stock price evolves randomly exactly offset the changes in the option value due to the random changes in the stock price. If we rebalance only at the discrete time intervals of the simulation, the hedge will not be perfect, but for reasonably frequent rebalancing we expect it to be very good.

For example, the hedging procedure for writing a call consists of selling the call, putting the premium in the bank, and rebalancing our holding in the stock at each time step (with resultant cash flows into and out of our bank account, which earns the riskless rate). At the maturity date the hedge, consisting of the cash account plus the value of the stock, will have closely replicated the payoff of the option. This is expressed as

$$C(t_0)e^{rT} - \left[\sum_{h=0}^{h=N-1}\left(\frac{\partial C(t_h)}{\partial S} - \frac{\partial C(t_{h-1})}{\partial S}\right) \times S_{t_h} e^{T-t_h}\right] + \frac{\partial C(t_{N-1})}{\partial S} S_T = y_T \quad (5)$$

where $\partial C(t_{-1})/\partial S = 0$. The first term is the premium received, inflated at the riskless rate to the maturity date; the second term is the cash flows due to rebalancing inflated to the maturity date; the third term is the value of the stock held at the maturity date; and y_T is the payoff of the option. The negative of the expression in brackets, which we will call the hedge, can be rewritten as (see the Appendix):

$$x^1 = \sum_{h=0}^{N-1} \frac{\partial C(t_h)}{\partial S}(S_{t_h} - E[\Delta S_{t_h}])\, e^{r(t_N - t_{h+1})} \quad (6)$$

where

$$E[\Delta S_{t_h}] = S_{t_h}(e^{r\Delta t_h} - 1) \quad (7)$$

Similarly, we can add a gamma hedge into (5). The gamma hedge is equal to

$$x^2 = \sum_{h=0}^{N-1} \frac{\partial^2 C(t_h)}{\partial S^2} \times ((\Delta S_{t_h})^2 - E[(\Delta S_{t_h})^2])\, e^{r(t_N - t_{h+1})} \quad (8)$$

where

$$E[(\Delta S_{t_h})^2] = S_{t_h}{}^2 (e^{(2r+\sigma^2)\Delta t_h} - 2(e^{r\Delta t_h} - 1) + 1) \quad (9)$$

In general we will have

$$P = y_T - \beta_1 x^1 - \beta_2 x^2 - \ldots \quad (10)$$

where P is the expected future value of the option, and x^1, x^2, etc., are the control variates. The hedge is not perfect, because we are rebalancing at discrete intervals (and possibly our hedge ratios, delta, gamma, etc., are not exactly correct. In particular, for short-maturity options we may omit the inflation factor to save on computation time). The β factors are included to account for these errors. Although the hedge is not perfect, the variance of P will be much smaller than that of y_T.

Note that the control variates have zero mean, so they do not alter the mean that we estimate. Therefore if we estimate the mean of P by Monte Carlo simulation we obtain a much more accurate estimate of the expected future value of the option.

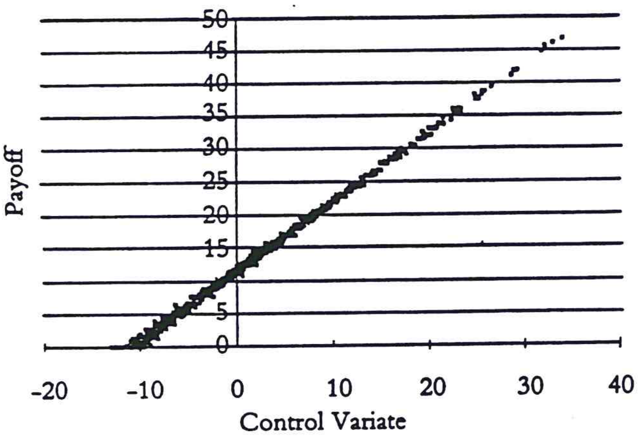
If we simulate M paths and the payoff (y_i) and the control variates (x_i^1 and x_i^2) for each path i, the estimate of the expected future option value is

$$P = \frac{1}{M}\sum_{i=1}^{M}(y_i - \beta_1 x_i^1 - \beta_2 x_i^2) \quad (11)$$

The problem now is to estimate P and the βs.

Note that we do not have to estimate the βs;

EXHIBIT 1
MONTE CARLO SIMULATION OF BLACK-SCHOLES WITH DELTA-BASED CONTROL VARIATE
X = 100.0, T = 1.0, S100.0, SIGMA = 0.1, r = 0.1, N = 52



we can assume they are equal to one. We would still have a hedged portfolio, and the means of the control variates would still be zero. Therefore we would still obtain a more accurate estimate. However, it is straightforward to estimate both P and a set of βs simultaneously. This is a least squares estimation problem, which is illustrated in Exhibit 1 for a single delta-based control variate.

Simulated paths of the stock price that increase on average will result in higher positive values of the control variate (delta will tend to one, and the differences between the stock price change and its mean will be positive). Paths that decrease on average will result in lower values of the control variate (delta will tend to zero, and the differences between the stock price change and its mean will be negative).

Since the control variate is highly correlated with the payoff, the points lie very close to a straight line. The intercept of this line with the y-axis, where the control variate is zero, gives the mean payoff. When the control variate is zero, the hedge value is zero, and the payoff is the mean payoff.[1]

The least squares estimation can be performed easily as

$$\beta = (X'X)^{-1} X'Y \tag{12}$$

where $\beta = (P, \beta_1, \beta_2)$, X is a matrix whose M rows correspond to the M simulations and are made up of the control variates $(1, x_i^1, x_i^2)$, and Y is the vector of the simulated payoffs (the prime denotes transpose).

The matrixes X′ X and X′ Y can be accumulated as the simulation proceeds as follows

$$(X'X)^{n+1}_{j,k} = (X'X)^{n}_{j,k} + x^j_{n+1}\, x^k_{n+1} \tag{13}$$

$$(X'Y)^{n+1}_{j} = (X'Y)^{n}_{j} + x^j_{n+1}\, y_{n+1} \tag{14}$$

where j and k indicate the number of the control variate, 0, 1, or 2 in this case; $x_n^0 = 1$; and x_n^i and y_n are the values returned by the nth simulation. Therefore all we need to do is accumulate these matrixes as we simulate the paths, invert the matrix (X′ X), and multiply it by (X′ Y) to obtain the vector β. The first element of the vector is the mean payoff, which we discount back at the riskless rate to obtain the option value. The other elements are the correlations of the control variates with the payoff.

A schematic computer program of the full procedure with delta- and gamma-based control variates can be written as follows:

```
for j = 0 to 2
        XY[j] = 0
        for k = 0 to 2
                XX[j][k] = 0
        next k
next j

cv[0] = 1

for i = 1 to M

S = S0
t = 0

cv[1] = 0
cv[2] = 0

for h = 0 to N – 1

        t = hdt
```

```
        compute delta(t, S) and gamma(t, S)

        evolve the stock price

        cv1 = cv1 + delta(dS – E[dS])
        cv2 = cv2 + gamma(dSdS – E[dSdS])

next h

y = payoff of option; e.g., max(0, S – K) for a call

for j = 0 to 2
        XY[j] = XY[j] + cv[j]y
        for k = 0 to 2
                XX[j][k] = XX[j][k] + cv[j]cv[k]
        next k
next j
next i

IXX = inverse of XX

β = IXX × XY

call value = β[0]exp(–rT)
```

Exhibit 2 shows the results for a typical European call. The simple Monte Carlo estimate with no variance reduction has an unacceptably large error (standard deviation of 0.42) for the 1,000 simulations we have used.

Application of the hedge-based control variates alone produces a variance reduction of 300 times. We can combine the antithetic and control variate techniques. The control variates become the sum of the appropriate control variates for the two price processes generated by the antithetic procedure, and the samples are the means of the payoffs for the two price processes.

The combined procedure reduces the variance of the estimate by 12,000 times. The simulation of each path is now more time-consuming because of the need to compute the hedge ratios, but this is small compared with the reduction in the variance (we give some actual computation times later). The standard deviation of the estimate has been reduced so that the estimate is now accurate to two decimal places. To obtain this reduction for the simple Monte Carlo estimate, we would have to perform a factor of 12,000 more simulations, that is, 12 million.

EXHIBIT 2
APPLICATION OF VARIANCE REDUCTION TO MONTE CARLO VALUATION OF A STANDARD CALL

European Call Option Value		18.58
Variance Reduction Method	Variance	SD
None	0.18	0.42
Antithetic Variate	0.0073	0.085
Control Variates	0.0006	0.024
Antithetic and Control Variates	0.000015	0.0039

Exercise price = 100
Time to maturity = 2 years
Initial stock price = 100
Volatility = 10%
Riskless rate = 10%
Number of time steps (N) = 104
Number of simulations (M) = 1,000

One reason for the very large reduction in the variance in this example is the fact that the partial differentials or hedge ratios used in the control variates are exact. That is, we have the Black-Scholes formula for the value of a standard European call from which we can calculate the partial derivatives exactly. The existence of the Black-Scholes formula obviates the need to value the option by Monte Carlo simulation. For this example, then, the variance can be reduced to zero in the limit as the time step in the simulation tends to zero. This corresponds to the Black-Scholes model of exact replication of the option.[2]

A REALISTIC APPLICATION

The same techniques can be applied in a straightforward way to more complex and realistic problems and give similarly dramatic results. Consider the problem of valuing a foreign exchange look-back call option under stochastic volatility. The payoff of a look-back call option is the difference between the terminal value of the underlying asset and the minimum this value attained during the life of the option.

There is an analytical formula for the value of a continuously fixed look-back option with constant volatility (see Goldman, Sosin, and Gatto [1979]):

$$L(t, S, \sigma) = Se^{-r_f(T-t)} N\left(x + \sigma\sqrt{T - t}\right) - m_{0,t}e^{-r_d(T-t)} N(x) + \frac{S}{B}\left(e^{-r_d(T-t)} \times \left(\frac{S}{m_{0,t}}\right)^{-B} N\left(y + B\sigma\sqrt{T - t}\right) - e^{-r_f(T-t)} N(y)\right) \quad (15)$$

where

$$B = \frac{2(r_d - r_f)}{\sigma^2}$$

$$x = \frac{\ln\left(\frac{S}{m_{0,t}}\right) + \left(r_d - r_f - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$y = \frac{-\ln\left(\frac{S}{m_{0,t}}\right) - \left(r_d - r_f + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

where t is the current date; S is the exchange rate; $m_{0,t}$ is the current minimum exchange rate observed up to time t; and r_d and r_f are the domestic and foreign interest rates. But if the fixings (i.e., the recording of the exchange rate) occur at discrete times (for example, once daily at the closing price), and the volatility is stochastic, the value of the option will be significantly different from that given by this formula.

We assume the risk-neutral processes for the underlying foreign exchange rate S and the volatility σ of the foreign exchange rate are

$$dS = (r_d - r_f)\, Sdt + \sigma Sdz \quad (16)$$

$$d\sigma = \alpha(\bar{\sigma} - \sigma)\, dt + \theta\sigma dw \quad (17)$$

where dz and dw are independent Wiener processes. The process for the volatility given by Equation (17) is a mean-reverting proportional process. That is, the volatility tends to drift back to the long-term level $\bar{\sigma}$ at a rate determined by α, and the proportional volatility of the volatility is given by θ.

The following control variates are used

$$x^1 = \sum_{h=1}^{N} \frac{\partial L}{\partial S}(\Delta S_{t_h} - E[\Delta S_{t_h}]) \quad (18)$$

$$x^2 = \sum_{h=1}^{N} \frac{\partial L}{\partial \sigma}(\Delta \sigma_{t_h} - E[\Delta \sigma_{t_h}]) \quad (19)$$

$$x^3 = \sum_{h=1}^{N} \frac{\partial^2 L}{\partial S^2}((\Delta S_{t_h})^2 - E[(\Delta S_{t_h})^2]) \quad (20)$$

where the expectations in (18) and (20) are given by Equations (7) and (9) with r replaced by $r_d - r_f$. The expectation ($E[\Delta\sigma_{t_h}]$) for the process (17) is given by

$$E[\Delta\sigma_{t_h}] = (\sigma_{t_h} - \bar{\sigma})(e^{-\alpha\Delta t_h} - 1) \quad (21)$$

These control variates correspond to the delta ($\partial L/\partial S$), vega ($\partial L/\partial\sigma$), and gamma ($\partial^2 L/\partial S^2$) of the option, respectively. The vega-based control variate is introduced to reduce the variance due to the stochastic volatility, or, put another way, to hedge the stochastic volatility risk. The time step is taken to correspond to daily fixing of the minimum.

Exhibit 3 shows the results of applying the variance reduction techniques to a typical look-back call. The example is taken from Dolbear [1992] and represents typical values observed in the yen/DM market, where the rate of mean reversion corresponds to a half-life for shocks to volatility of six months.

The results are similar to those for the European call. For this example the total variance reduction is by a factor of 1,200. The error in the estimate is reduced so that the value is accurate to two decimal places with only 1,000 simulations. The computation times are for a 486DX PC running at 33MHz.

Note that if we use only 100 simulations, we would still get an answer accurate to two decimal

EXHIBIT 3
APPLICATION OF VARIANCE REDUCTION TO MONTE CARLO VALUATION OF A LOOK-BACK CALL

Continuous Fixing Analytical Formula Value		2.79	
Foreign Exchange Look-Back Call Option Value		2.59	
Variance Reduction Method	Variance	SD	Time(s)
None	0.0096	0.098	3
Antithetic Variate	0.00092	0.030	15
Control Variates	0.000046	0.0068	90
Antithetic and Control Variates	0.000008	0.0028	78

Time to maturity = 0.5 years	Volatility = 6%
Initial foreign exchange rate = 100	Foreign interest rate = 7%
Domestic interest rate = 5%	Volatility of the volatility (θ) = 0.085
Rate of mean reversion (α) = 1.35	
Number of time steps (N) = 128	Number of simulations (M) = 1,000

places with computation time of under ten seconds. Without these variance reduction techniques, it would take a factor of 1,200 more simulations to achieve the same accuracy or on the order of hours of computation time on the same computer.

Finally, in implementing this technique we are forced to consider suitable hedge ratios for the option, and the simulation will indicate the accuracy and robustness of the choice of hedge ratios.

CONCLUSIONS

We have described a new approach to constructing control variates, intimately related to the hedging of the option, which can improve the efficiency of a Monte Carlo simulation dramatically. This approach improves in several ways on that introduced by Boyle [1977] and subsequently used by Kemna and Vorst [1990] to value Asian options.

First, we simultaneously estimate the correlation of the hedge control variate with the payoff. This improves the efficiency of the method and provides information on the accuracy of the hedge. Second, because the control variates are applied step by step along each path, they can improve the accuracy of a simulation with stochastic volatility and interest rates.

The careful use of these variance reduction techniques can make Monte Carlo simulation an efficient and powerful method for the analysis of complex contingent claims. The number of simulations required to obtain acceptable levels of error can be reduced so much that the valuations can be performed in almost real-time on a powerful desktop computer.

Finally, the techniques we have described have implications for the rapidly expanding field of parallel processing. The number of simulations needed to obtain accurate valuations has been reduced to much the same level as the number of processing units that typical parallel hardware contains. This greatly simplifies the implementation of Monte Carlo simulations on this type of hardware, and simultaneously we obtain a very general and flexible real-time valuation technique.

APPENDIX

PROOF OF THE EQUIVALENCE BETWEEN A HEDGED PORTFOLIO AND A MARTINGALE CONTROL VARIATE

The hedging procedure for writing a call option consists of selling the call, putting the premium in the bank, and rebalancing our holding in the stock at each time step (with resultant cash flows into and out of our bank account, which earns the riskless rate). At the maturity date the hedge, consisting of the cash account plus the value of the stock, will have exactly replicated the payoff of the option.

This can be expressed as follows:

$$C(t_0)e^{rT} - \left[\sum_{h=0}^{h=N-1}\left(\frac{\partial C(t_h)}{\partial S} - \frac{\partial C(t_{h-1})}{\partial S}\right) \times S_{t_h}e^{T-t_h}\right] + \frac{\partial C(t_{N-1})}{\partial S}S_T = y_T \quad \text{(A-1)}$$

where $\partial C(t_{-1})/\partial S = 0$.

The first term is the premium received, inflated at the riskless rate to the maturity date; the second term is the cash flows due to rebalancing inflated to the maturity date; the third term is the value of the stock held at the maturity date; and y_T is the payoff of the option.

Now consider an expansion of the summation term

$$\frac{\partial C(t_0)}{\partial S} S_{t_0} e^{rT} + \frac{\partial C(t_1)}{\partial S} S_{t_1} e^{r(T-t_1)} + \ldots +$$

$$\frac{\partial C(t_{N-1})}{\partial S} S_{t_{N-1}} e^{r(T-t_{N-1})} + \frac{\partial C(t_{N-1})}{\partial S} S_T -$$

$$\frac{\partial C(t_0)}{\partial S} S_{t_1} e^{r(T-t_1)} - \frac{\partial C(t_1)}{\partial S} S_{t_2} e^{r(T-t_2)} - \ldots -$$

$$\frac{\partial C(t_{N-2})}{\partial S} S_{t_{N-1}} e^{r(T-t_{N-1})} \qquad \text{(A-2)}$$

Now rewrite, grouping terms with $\partial C(t_h)/\partial S$ at the same time step

$$-\frac{\partial C(t_0)}{\partial S}(S_{t_1} - S_{t_0} e^{r\Delta t})e^{r(T-t_1)} - \frac{\partial C(t_1)}{\partial S} \times$$

$$(S_{t_2} - S_{t_1} e^{r\Delta t})e^{r(T-t_2)} - \ldots - \frac{\partial C(t_{N-1})}{\partial S} \times$$

$$(S_{t_{N-1}} - S_{t_{N-2}} e^{r\Delta t})e^{r(T-t_{N-1})} + \frac{\partial C(t_{N-1})}{\partial S} S_T \qquad \text{(A-3)}$$

Therefore Equation (A-1) becomes

$$C(t_0)e^{rT} + \left[\sum_{h=0}^{h=N-1} \frac{\partial C(t_h)}{\partial S}\left(S_{t_{h+1}} - S_{t_h} e^{r\Delta t}\right)\right.$$

$$\left. \times\ e^{r(T-t_{h+1})}\right] = y_T \qquad \text{(A-4)}$$

The expression in brackets, which is the hedge and a martingale control variate, can be rewritten as

$$x^1 = \sum_{h=0}^{N-1} \frac{\partial C(t_h)}{\partial S}\left(\Delta S_{t_h} - E[\Delta S_{t_h}]\right) \times e^{r(t_N - t_{h+1})} \qquad \text{(A-5)}$$

ENDNOTES

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[1]Note that the variables involved in the regression are not jointly normal, so the estimation of the regression coefficients will be biased. However, as Exhibit 1 demonstrates, the bias is insignificant for any reasonable choice of control variates.

[2]Note that we should strictly calculate the variance about the true value of the option for these variance reduction statistics to be meaningful. For example, if the simulation produces the value 3.14 every time, the variance of this number about its own mean and would be zero, but the variance about the true value would be very large. However, since the bias in the estimate is usually insignificant, the variance of the simulated value about its own mean gives a good indication of the improvement in accuracy of the Monte Carlo estimate.

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10. *Owner (if owned by a corporation, its name and address must be stated and also immediately thereunder the names and addresses of stockholders owning or holding 1% or more of the total amount of stock. If not owned by a corporation, the names and addresses of the individual owners must be given. If owned by a partnership or other unincorporated firm, its name and address, as well as that of each individual must be given. If the publication is published by a non-profit organization, its name and address must be stated.):*
Capital Cities/ABC, Inc.
77 West 66th Street
New York, NY 10023-6298.
11. *Known bondholders, mortgagees, and other securityholders owning or holding 1% or more of total amount of bonds, mortgages, or other securities (if there are none, so state):*
Cede Fast
c/o The Depository Trust Company
P.O. Box 863
Bowling Green Station
New York, NY 10274-0863

As of August 31, 1994

OTHER INDIVIDUALS:

None

OFFICERS & DIRECTORS:

*Mr. Warren E. Buffett
1440 Kiewit Plaza
Omaha, NE 68131

*Shares are owned directly by five subsidiaries of Berkshire Hathaway Inc., of which Mr. Buffett is Chairman of the Board and in which he has controlling interest.

INSTITUTIONAL:

Cede & Co.
c/o Depository Trust Company
Post Office Box 222
New York, NY 10274

INCE & Co.
c/o Morgan Guaranty Trust Co. of New York
Box 1479
Church Street Station
New York, NY 10008

Kray & Co.
One Financial Place
440 South LaSalle Street
Chicago, IL 60605

12. *For completion by non-profit organizations authorized to mail at special rates (DMM Section 423, 12 only):* Not applicable.
13. *Publication Name:* Journal of Derivatives.
14. *Issue Date for Circulation Data Below:* Fall.
15. *Nature and extent of circulation:*

		Average number of copies of each issue during preceding twelve months	*Actual number of copies of the single issue nearest to filing date*
A.	*Total number of copies printed (net press run)*	4,025	4,000
B.	*Paid circulation:*		
1.	*Sales through dealers and carriers, street vendors, and counter sales*	0	0
2.	*Paid or requested mail subscriptions*	2,260	2,044
C.	*Total paid and/or requested circulation*	2,260	2,044
D.	*Free distribution by mail, carrier, or other means, samples, complimentary, and other free copies*	71	93
E.	*Free distribution outside the mail*	0	0
F.	*Total free distribution (sum of 15D and 15E)*	71	93
G.	*Total distribution (sum of 15E and 15F)*	2,331	2,137
H.	*Copies not distributed*		
1.	*Office use, leftovers, spoiled*	1,694	1,863
2.	*Return from News Agents*	0	0
I.	*Total (sum of 15G, 15H(1), and 15H(2)*	4,025	4,000
	Percent Paid and/or Requested circulation (15C/15G × 100)	96.95	95.64

16. *I certify that the statements made by me above are correct and complete.*

John W. Phipps
Controller