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1 Introduction

In this chapter we discuss the application of mathematical programming techniques such as linear programming, dynamic programming, and goal programming to the problem of the risk management of derivative securities (otherwise known as contingent *claims* or *options*). Derivative securities are those who value depends on the value of fundamental securities or assets such as stocks or bonds. In this chapter we will be concerned in particular with complex or exotic options. For example path dependent options whose value depends on the path the underlying asset price tool over the life of the option rather than just its final value as is the case with standard European options.

In a perfect market the Black-Scholes model (Black and Scholes (1973)) provides the recipe for the risk management of standard European options. A perfect market is one in which there are no transaction costs and no taxes, the market operates continuously, and the price of the underlying asset is continuous (that is there are no jumps in the asset price). In addition Black and Scholes assumed that the asset price follows a geometric Brownian motion (GBM) stochastic process with constant volatility and the risk free interest rate is constant[1]. The behaviour of the asset price under GBM can be characterised by its stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma S(t)dz(t) \tag{1.1}$$

where μ is the expected return on the asset and σ is the volatility of returns on the asset. Black and Scholes showed that options could be priced by constructing a perfectly riskless portfolio with an option, the underlying asset and cash. Itos lemma allows us to write down the stochastic differential equation govening the price of an option $c(S(t),t)$ which only depends on the asset $S(t)$ and time t

$$\begin{aligned} dc(S(t),t) = {} & \frac{\partial c(S(t),t)}{\partial t}dt + \frac{\partial c(S(t),t)}{\partial S(t)}\left(\mu S(t)dt + \sigma S(t)dz(t)\right) \\ & + \frac{1}{2}\frac{\partial^2 c(S(t),t)}{\partial S(t)^2}\sigma^2 S(t)^2 dt \end{aligned} \tag{1.2}$$

If we form a portfolio P in which we are short the option and long an amount $\frac{\partial c(S(t),t)}{\partial S(t)}$ of the asset, the equation governing the price of the portfolio is

[1] This was generalised by Merton (1973) to allow the volatility to be a deterministic function of time and the interest rate to be stochastic.

$$dP(S(t),t) = -dc(S(t),t) + \frac{\partial c(S(t),t)}{\partial S(t)} dS(t) \qquad (1.3)$$

Substituting into equation (1.3) using equations (1.1) and (1.2) gives

$$dP(S(t),t) = \frac{\partial c(S(t),t)}{\partial t} dt + \frac{1}{2} \frac{\partial^2 c(S(t),t)}{\partial S(t)^2} \sigma^2 S(t)^2 dt \qquad (1.4)$$

The portfolio P is riskless, that is it has no random component, and must therefore earn the riskless rate of interest

$$\frac{dP(S(t),t)}{P(S(t),t)} = rdt \qquad (1.5)$$

Substituting into equation (1.5) for dP using equation (1.3) and for P leads to the Black-Scholes partial differential equation

$$\frac{\partial c(S(t),t)}{\partial t} + rS(t)\frac{\partial c(S(t),t)}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 c(S(t),t)}{\partial S(t)^2} = rc(S(t),t) \qquad (1.6)$$

The solution to this partial differential equation subject to the boundary condition of a standard European call option $c(S(T),T) = \max(0, S(T) - K)$ is the Black-Scholes equation

$$c(S(t),t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \qquad (1.7)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

where K is the strike price and T is the maturity date of the option. The important point to note about equation (1.6) is that it does not depend on any parameters, in particular the expected return of the asset μ, which depend on investors risk preferences. This is because the option can be perfectly hedged, that is the risk due to the underlying asset completely eliminated, by a continuously rebalanced position in the underlying asset. However, this result relies critically on the assumptions of continuous trading and no transaction costs.

The quantity of the underlying asset which must be held in the hedge portfolio of a short option position, called the *delta* of the option, is the partial differential of the option price with respect to the underlying asset. This generalises to the case of multiple risky state variables, the quantity of the risky state variable which must be held is the partial differential of the option price with respect to the state variable. The partial derivatives of the Black-Scholes formula with respect to all the parameters have standard names which are defined in Table 1.

Table 1: Black-Scholes Risk Measures

Delta	$\dfrac{\partial c(S(t),t)}{\partial S(t)}$
Gamma	$\dfrac{\partial^2 c(S(t),t)}{\partial S(t)^2}$
Vega or Lambda	$\dfrac{\partial c(S(t),t)}{\partial \sigma}$
Theta	$\dfrac{\partial c(S(t),t)}{\partial t}$
Rho	$\dfrac{\partial c(S(t),t)}{\partial r}$

Note that for the case of Vega and Rho these are derivatives with respect to constant parameters in the model. Practitioners routinely do this because they find themselves using models which assume parameters are constant which they know are risky.

The Black-Scholes riskless hedge idea extends to more complex or exotic options as long as we remain in the perfect market world. For example, consider the case of a down and out call option, which is a type of barrier option. This is a standard European option except that if the asset price falls below a pre-determined level, H, called the barrier then the option disappears. The price of this option is also governed by the Black-Scholes partial differential equation (equation (6)) but with the additional boundary condition $c(H,t) = 0$.

Consider hedging this down-and-out call option. When the asset price is far from the barrier the option behaves like a standard call option, both in price and sensitivities. However, as the barrier is approached the option value begins to fall more and more rapidly. Consequently the delta rises near to the barrier and can be greater than one. If then the barrier is hit, the option instantly disappears and the delta changes discontinuously from a large value to zero. With no transaction costs this is of course not a problem, we simply rebalance our holding in the underlying asset to zero. With transaction costs, having to sell a large holding in the underlying asset is a significant problem. However the real problem is more subtle, it is that the delta changes rapidly with changes in the underlying asset price near the barrier (this is called gamma risk, see Table 1). Therefore, in the presence of transaction costs the delta hedging strategy will incur large transaction costs even if the barrier is not hit which can lead to the hedging strategy costing far more than the Black-Scholes price of the option. The reason for the failure of the Black-Scholes hedging strategy is that it should be taking into account the expected future hedging costs inclusive of transaction costs.

Thus, when we introduce market imperfections, in particular non-continuous trading and transaction costs the whole nature of the problem and solution change. The Black-Scholes delta hedging approach relies critically on being able to continuously rebalance the delta hedge without incurring transaction costs. In a real market we cannot trade continuously so we cannot make the hedge portfolio riskless and every trade incurs transaction costs so that if we try and trade close to continuously we will lose a large amount of money. For a typical market even daily rebalancing of the hedge can lead to costs which exceed the Black-Scholes value the option. It is possible to measure the performance of this hedging strategy, for example expected profit against variance of profit. But this approach is extremely sub-optimal (Clewlow and Hodges (1996)).

The problem of hedging under transaction costs was first tackled by Leland (1985) for standard European options. His solution was an adjustment to the volatility in the Black-Scholes formula which accounted for the presence of proportional transaction costs The intuition behind his approach is helpful in understanding the nature of the problem. Imagine we have written a standard European call on non-dividend paying stock and are delta hedging this liability. We will therefore have a long position in the underlying stock. Imagine now that the stock price goes up. The delta of the option will increase and we will therefore buy more stock. Now in the presence of transaction costs it will cost us more than the value of the stock to obtain the required amount of stock. That is, it will be as if the stock price had increased slightly more than it did. Now imagine that the stock prices goes down. The delta of the option will decrease and we will sell some stock. Again, the transaction costs will mean that we get slightly less than the actual value of the stock. That is, it will be as if the stock price decreased slightly more than it actually did. Overall the effect of transaction costs on our delta hedge will be similar to the stock having a slightly higher volatility than it actually has. Leland's adjustment of the Black-Scholes model has since been extended by Whalley and Wilmott (1993); and Hoggard, Whalley and Wilmott (1994). Boyle and Vorst (1992) proposed a model similar to Leland (1985)'s, although making different assumptions about the distribution of the changes in the underlying assets. They use a binomial tree to represent the stochastic behaviour of the underlying asset. Note that none of the above models are in any sense optimal. Approaches based on optimisation will be reviewed in the following sections.

2 One Period Hedges - The Linear Programming Approach

Since the problem with the Black-Scholes delta hedging strategy in the presence of transaction costs stems from the continuous trading, one solution is to move to single period or static hedging strategies. This approach leads to a linear programming framework. There are essentially two ways in which the problem can be formulated in this framework. The first is to keep the idea of the Black-Scholes risk measures but to only consider the next rebalancing period. We imagine we have a position x_T (usually -1) in a target option c_T, and we wish to solve for the set of holdings $\{x_i ; i = 1,\dots,n\}$ in a set of more basic securities $\{c_i ; i = 1,\dots,n\}$ (which may also be options) in order to neutralise a set of risk measures $\{R_j(c); j = 1,\dots,m\}$. We therefore obtain the following constraints:

$$x_1 R_j(c_1) + x_2 R_j(c_2) + \dots + x_n R_j(c_n) = x_T R_j(c_T) \;; \quad j = 1,\dots,m \tag{2.1}$$

$$x_i \geq 0 \; ; \; i = 1,\ldots,n^2 \tag{2.2}$$

We would then, for example, minimise the cost inclusive of transaction costs $(x_1c_1 + x_2c_2 + \ldots + x_nc_n)$, subject these constraints:

$$\begin{array}{l} \text{Min} \\ \{x_i; i = 1,\ldots,n\} \end{array} (x_1c_1 + x_2c_2 + \ldots + x_nc_n) \tag{2.3}$$
$$\text{subject to (2.1) and (2.2)}$$

The form of the optimisation problem is an expression of the risk preferences of the hedger. This is an important point to bear in mind when formulating these problems. The solution may be very different for different choices of the objective function. With this formulation we obtain a solution which is only optimal over the next period so if it is applied repetitively over multiple periods it may be seriously sub-optimal.

Alternatively, we may only be concerned with the liability at a single future date (the maturity date of the target assuming it generates no intermediate liabilities). In this case we imagine we have a set of scenarios $\{s_j; j = 1,\ldots,m\}$ for the underlying state variables at the future date and we wish to solve for the set of holdings $\{x_i; i = 1,\ldots,n\}$ in the set of basic securities $\{c_i; i = 1,\ldots,n\}$ in order to meet the target in all scenarios. We therefore obtain the following constraints:

$$\begin{array}{l} x_1c_1(s_j) + x_2c_2(s_j) + \ldots + x_nc_n(s_j) \geq x_Tc_T(s_j) \; ; \\ j = 1,\ldots,m \end{array} \tag{2.4}$$

$$x_i \geq 0 \; ; \; i = 1,\ldots,n \tag{2.5}$$

Note that equation (2.4) is now an inequality so that the hedge portfolio super-replicates the target. A strict equality could be used but this usually leads to less robust and more expensive hedges. We would then, for example, minimise the initial cost inclusive of transaction costs $(x_1c_1(0) + x_2c_2(0) + \ldots + x_nc_n(0))$, subject these constraints (see for example Aparicio and Hodges (1996)):

$$\begin{array}{l} \text{Min} \\ \{x_i; i = 1,\ldots,n\} \end{array} (x_1c_1(0) + x_2c_2(0) + \ldots + x_nc_n(0)) \tag{2.6}$$
$$\text{subject to (2.4) and (2.5)}$$

Dembo (1991) introduces stochasticity into these types of models by using a scenario optimisation approach in two stages. First he computes solutions to the deterministic problems under all scenarios and then he solve a co-ordinating or tracking model to find a single, feasible solution. The tracking model satisfies all the constraints and minimises the overall difference from the optimal solutions to the deterministic problems.

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[2] It is also possible to allow negative positions with a lower bound.

Another approach is the minimax hedging strategy of Howe, Rustem and Selby (1996). They aim to minimise the maximum potential hedging error between time periods. That is, at the rebalancing date, they finds the worst-case scenario of the underlying asset price giving the worst hedging error and then solve for the holding in the asset which minimises this. This is therefore most relevant where the underlying asset is highly volatile and crosses the exercise price frequently. If the worst possible scenario does not occur, it usually would have been better to use Black-Scholes delta hedging. Merton's 'ideal portfolio' is the benchmark to the hedging error in the objective function. They show how the rebalancing strategy can be used at the end of a time interval, at the beginning and end of the time interval (two-period minimax) or a variable minimax where the hedger monitors the hedging error and rebalances it whenever finds it unacceptable.

3 Multi Period Hedges - The Dynamic Programming Approach

It is possible to solve the problem of delta hedging options in the presence of transaction costs using a dynamic programming approach. Since it is not possible to form a riskless delta hedge when transaction costs are incurred in trading the underlying asset the solution now depends on the risk preferences we assume for the hedger. As we saw in section 2, these risk preferences manifest themselves in the form of the objective function over which we optimise. For example we could choose to maximise expected utility[3] of wealth at a future date or minimise the initial cost of the hedge subject to super-replication (i.e. guaranteeing a pay-off at least as large as the liability at a future date). This approach makes the very important assumption that we can specify very accurately the probability distribution of future states of the world. Given this distribution, the optimal solution we obtain is valid whatever future state of the world occurs. But, there is a significant computational effort involved in obtaining this kind of solution and if our probabilities are not correct then the solution may be severely sub-optimal (more so as we look over longer time horizons). The approaches in section 2 and 4 tend to be more robust to imperfectly specified probability distributions. Furthermore, for more sophisticated models and hedging strategies it may be computationally impractical to solve the problem directly in a dynamic programming framework.

Hodges and Neuberger (1989) (see also Davis and Panas (1991), Davis *et al* (1993), and Clewlow and Hodges (1996)) were the first to formulate this problem with proportional transaction costs[4] as one of stochastic optimal control and show how to solve it using dynamic programming. By careful choice of the utility function Hodges and Neuberger were able to obtain a formulation in which the only state variables were the asset price $S(t)$, the holding in the asset $x(S(t),t)$, and time t. The solution method is based on constructing a binomial tree approximation for the asset price. At each node in the tree a vector of possible holdings in the asset is held together with an associated vector of values of the portfolio. The solution method consists of working backwards from the option maturity date boundary condition, computing the portfolio value and applying the optimal control boundary conditions. The optimal control strategy consists of upper and lower limits on $x(S(t),t)$ within which $x(S(t),t)$ must be maintained. Figure 1 illustrates the typical control limits which are

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[3] A utility function of wealth $U(w(t))$ expresses individuals preferences for levels of wealth $w(t)$ in dimensionless units.

[4] By proportional transaction costs we mean the costs are proportional to the value of the asset traded.

obtained. The optimal delta hedging strategy consists of doing nothing while $x(S(t),t)$ remains between the control limits. But as soon as $x(S(t),t)$ reaches either the upper or lower limit the asset is traded continuously to control $x(S(t),t)$ never to be outside the limits. Also shown in Figure 1 is the Black-Scholes delta, the control limits lie roughly either side of the Black-Scholes delta, although not always.

Since the solution is obtained by a numerical procedure this model can be used for hedging mixed portfolios of long and short positions, mixed maturity dates and with general transaction cost structures (see Clewlow and Hodges (1996))

Edirisinghe *et al* (1993) took the alternative approach of minimising the initial cost obtaining a payoff at least as large as the liability at a future date (super-replication). The authors claim that this approach is independent of investor preferences but, although investor preferences are not explicitly modelled, the chosen formulation implicitly defines investor preferences. The important difference between this two approaches is that typical utility functions allow profits in good future states of the world to be traded off against losses in bad future states. The approach of minimising the initial cost of super-replication is applicable to situations where the cost of not meeting the liability are unacceptable under any circumstances. This result can be approximated by using a utility function which assigns negative wealth a relatively very low utility.

Dempster (1995) uses stochastic, multiperiod models to maximise expected utility of terminal wealth. He represents the stochasticity of the financial world through huge trees with thousands of paths. Dempster has developed techniques for shortening these tree. He shows that in most problems where we have huge trees of possible scenarios, one can find a sub-tree that represents the bulk of stochasticity. It is then feasible to solve these problem using advanced computational systems.

4 Hedging With Multiple Objectives - The Goal Programming Approach

Consider an investment bank which is writing complex options and thus has a large book of derivative instruments which it needs to hedge. This involves many conflicting objectives. The primary objective is to minimise the risk of the book and maximise profits. However, transaction costs, non-continuous trading, discrete lot sizes, etc. means that the risk can not be reduced to zero and the greater the risk reduction the greater the cost. Furthermore real assets do not follow GBM, at the very least they have jumps and their volatility is also stochastic. There are also sometimes restrictions on short selling and the interest rate at which borrowing can be obtained.

Jumps and stochastic volatility are sources of risk, in addition to that from the Weiner process driving the underlying asset, which cannot be hedged with a position in the underlying asset. This situation is referred to as an incomplete market and options must be introduced into the market to complete it and allow these additional sources of risk to be hedged. However, options markets have higher transaction costs and are less continuous than the underlying markets (for example the maturities and strike prices of exchange traded options, see Clewlow and Hodges (1994)) and so managing the transaction costs is very important.

In principle we could extend the optimal delta hedging approach to gamma/vega hedging. However, we would need to solve for the optimal holdings in all the available options. Solving this in a dynamic programming framework is very difficult if not impossible in practice.

The market imperfections lead to conflicting goals. These can be stated generally as:

1) Risk minimisation.
2) Transaction cost minimisation.
3) Minimisation of the opportunity costs of capital tied up in hedging.
4) Cash flow minimisation (hedge management cost minimisation).

2, 3, and 4 corresponding to profit maximisation.

These conflicting goals constitute a multi-objective problem, motivating the use of goal programming which allows the formulation and solution of multi-objective problems. Recently Clewlow and Pascoa (1996) used this approach to hedge barrier options in incomplete markets under transaction costs. They used the market prices of standard European options to obtain an implied discrete time, discrete state evolution of the underlying asset. This discrete time and state structure approximates the jumps and stochastic volatility of the real market and allowed them to obtain approximate prices for the standard options and barrier option in the future. They also use this implied evolution to generate a set of scenarios at the boundaries of the standard and barrier options by Monte Carlo simulation. Using these scenarios they solve the following LP problem; The goals are to minimise the hedge error and to minimise transaction costs. These goals are represented by the following constraints:

$$\sum_{i=1}^{n}\left(x_l(i)-x_s(i)\right)*V_i(s)=V_T(s)+e_p(s)-e_m(s) \qquad (4.1)$$

$$\left(\sum_{i=1}^{n} x_l(i)+x_s(i)\right)*\text{tcost}=\text{total\_tcost} \qquad (4.2)$$

$$0\le x_l(i)\le \text{large\_number}*dummy(i)$$
$$0\le x_s(i)\le \text{large\_number}*(1-dummy(i)) \qquad (4.3)$$

where

$V_T(s)$ is the price of target in scenario $s=1,\ldots,ns$;
$V_i(s)$ is the price of standard security $i=1,\ldots,n$ in scenario s.
$V0_i$ is the current price of standard security $i=1,\ldots,n$.
tcost is the cost per unit of option bought or sold.
$x_l(i)$ is the quantity of option i to buy (≥ 0).
$x_s(i)$ is the quantity of option i to sell (≥ 0).

$dummy(i)$ = integer variable $\{0,1\}$.
$e_p(s), e_m(s)$ is the positive and negative hedge error respectively.
total_tcost is the total transaction costs.

Equation (4.1) is the hedge error minimisation constraint, (4.2) is the transaction costs minimisation constraint and (4.3) prevents the simultaneous buying and selling of the same security.

The objective function is

$$\sum_{s=1}^{ns} e_p(s) + e_m(s) + \text{total\_tcost} \tag{4.4}$$

Clewlow and Pascoa show that this approach can provide accurate and robust hedges for complex options under realistic conditions including jumps, random volatility and transaction costs. Figure 2 gives an example of the hedge obtained for an up-and-out call option. The advantage of this approach is that it is very flexible many realistic constraints can be added such as bounds on the short and long positions, no short fall in the hedge, upper or lower bounds on the cost of the hedge whilst maintaining a realistic model of the market.

5 Conclusions

In this chapter we have reviewed the use of mathematical programming and optimisation in the risk management of derivatives securities. In section 1 we introduced the traditional Black-Scholes approach and discussed its flaws. The Black-Scholes approach assumes perfect markets whereas in reality there are transaction costs, non-continuous trading, jumps in asset prices and other risk sources. Failure to deal with these factors in a rigorous way can lead to large losses. In section 2 we introduced simple static replication alternatives to the Black-Scholes dynamic replication approach. The dynamic programming approach was considered in section 3, which gives us insights into the nature of the problem of risk management in an imperfect market. However, it was pointed out that this approach becomes impractical for realistic problems where the number of state variables becomes large very quickly. We then went on, in section 4, to describe a new approach we are developing which uses goal programming as the framework. We use the latest ideas of implying the future stochastic structure of the world from the market prices of standard European options together with a multi-objective formulation. This approach has the potential to allow realistic modelling of the risks in financial markets and allowing realistic constraints on the risk management process to be incorporated.

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Figure 1: Delta Hedging under Proportional Transaction Costs Control Limits

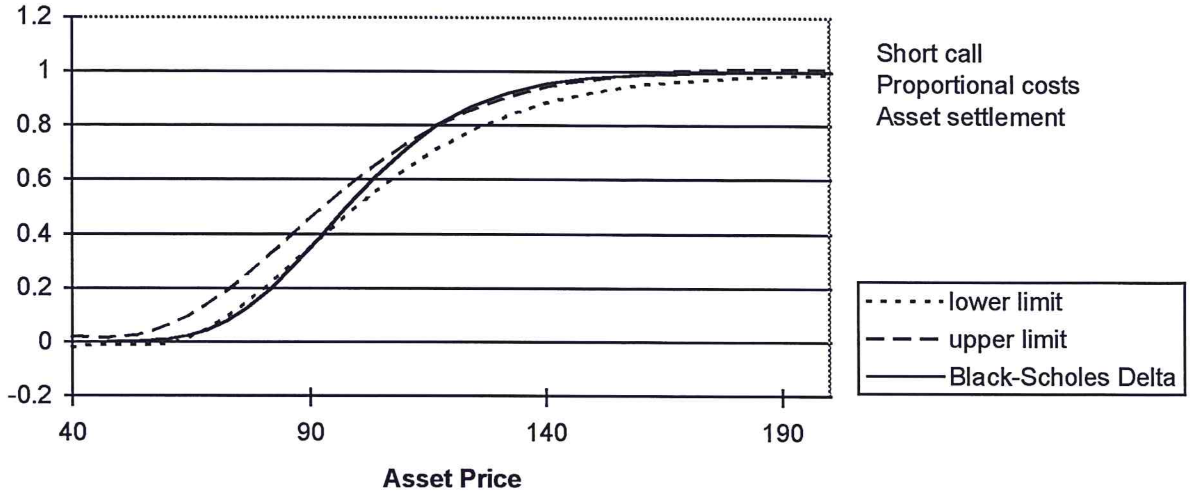

Figure 2: Static Hedge for Up-and-out Call Option (Clewlow and Pascoa (1996))