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Computational Aspects of Term Structure Models and Pricing Interest Rate Derivatives

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1 Introduction

In this chapter we give an overview of some of the computational aspects involved in implementing interest rate models for pricing and hedging derivatives. One way of constructing a no-arbitrage model for interest rates is in terms of the process followed by the short rate, r. The process for the short rate in a risk-neutral world determines the current term structure and how it can evolve. Arbitrage pricing theory tells us that discount bond prices are given by

$$P(t,s) = \hat{E}_t\left[\exp\left(-\int_t^s r(\tau)d\tau\right)\right] \quad (1)$$

where $\hat{E}_t$ denotes expectations (with the information set at time t) in a risk-neutral world, with $r(\tau), \tau \in [t,s]$ denoting the path of the short rate from t to s. Time t interest rate derivative prices, $\varphi(t)$, are determined in an analogous way:

$$\varphi(t) = \hat{E}_t\left[\exp\left(-\int_t^T r(\tau)d\tau\right)\varphi(T)\right] \quad (2)$$

where $\varphi(T)$ is the payoff to the derivative at time T.

In the first part of this chapter we concentrate on models which start with a stochastic process for the short rate, which are represented by, for example, the papers by Black, Derman, and Toy (BDT) [1990], and Hull and White (HW) [1992, 1994, 1994b]. The BDT model is used as an example to describe the general procedure for building trees to which are consistent with the observed yield curve or term structure of interest rates and yield volatilities or term structure of interest rate volatilities. We then describe how the range of standard interest rate derivatives; coupon bond options, caps, floor, collars, and swaptions can be priced using the short rate tree. We then go on to describe how path dependent interest rate exotics can be priced using the example of an Index Amortising Swap (IAS). Finally, we look at an approach which models the whole yield curve directly, which can be represented by the papers Heath, Jarrow, and Morton [1992] and Carverhill [1995]. Equation (2) is still applicable, but it is an implication of the pricing methodology and not its starting point. All the previous models are nested within this general framework. We describe how this very general model can be implemented efficiently using Monte Carlo simulation.

2 The Traditional Approach

During the late 1970's and the early 1980's most models for pricing interest rate derivatives were based on models originally developed to explain the term structure of interest rates. Two of the best known are the Vasicek [1977] and Cox, Ingersoll and Ross (CIR) [1985] models which can be characterised by their assumptions about the short term interest rate which is assumed to be the single source of uncertainty:

$$dr = \alpha(\gamma - r)dt + \sigma(r)dz \qquad (3)$$

$r = r(t)$ is the level of the short rate, at time t, with dz an increment in a Wiener process. The instantaneous drift represents the process as mean reverting back towards some long term level γ, at a speed α. In the Vasicek paper the short rate is assumed to follow the Ornstein-Uhlenbeck diffusion process with the volatility of the process set equal to a constant σ. In the CIR paper the volatility of the short rate increases with the square root of the rate itself. In both of these models the term structure of interest rates and the volatilities associated with those rates are uniquely determined once the risk-adjusted parameters of (3) have been fixed. We are also able to obtain closed form solutions for bond option prices.

In recognition of the fact that bond prices are not instantaneously perfectly correlated, a number of authors have put forward models which involve more than a single source of uncertainty. Two of the more recent of these come from Longstaff and Schwartz [1992] and Fong and Vasicek [1992] who propose stochastic volatility models of the term structure. Both of these models can be represented by processes of the form:

$$\begin{aligned} dr &= \alpha_r(t)dt + \sqrt{v}dz_1 \\ dv &= \alpha_v(t)dt + \sigma_v(t)dz_2 \end{aligned} \tag{4}$$

where $\alpha_r(t)$ and $\alpha_v(t)$ are the risk-adjusted drift terms of the short rate and the variance of the short rate respectively. $\sigma_v(t)$ is the instantaneous variance of the short rate variance v. Again, both the terms structures of rates and rate volatilities are determined once the parameters of the risk-adjusted process are determined.

This approach to pricing interest rate derivatives has the important advantage that all interest rate derivatives are valued on a common basis, i.e. with respect to the term structure derived by the model. However, it has the severe disadvantage that the resulting term structures can only come from a limited family which do not necessarily price all traded bonds correctly. By valuing interest rate derivatives with reference to a theoretical yield curve rather than the actually observed curve, the traditional models produce contingent claims prices that disregard key market information affecting the valuation of any interest rate derivative security. Many models currently appearing in the literature seek to overcome this shortcoming. They are formulated to be consistent with the observed yield curve and with some the observed yield volatilities. We call models of this type "term structure consistent" models.

3 The Term Structure Consistent Approach

Models of this approach set out to model the dynamics of the entire term structure in a way that is automatically consistent with the initially observed market data. We can further sub-divide models in this approach into those that fit the term structure of interest rates only, and those that fit both the term structure of rates and the term structure of rate volatilities. For models that don't fit the market volatility structure mathematical relationships within the model determine this curve. There are two basic ways of achieving consistency with the yield curve. One is to specify a process for the short rate, as in the traditional approach, and then effectively increase the parameterisation of the model by using time dependent parameters until all initial market data can be matched[1]. The second starts by taking the initial yield curve and the yield volatilities as given and then determining the drift structure that makes the model arbitrage free.

A number of models which start from an initial short rate specification are popular with practitioners due to their level of analytical tractability. For example, the models of Ho and Lee [1986] and Hull and White [1992] both achieve closed form solutions for pricing European options on discount bond. The stochastic differential equations for these models are

Ho-Lee $$dr = \theta(t)dt + \sigma dz \qquad (5)$$

Hull-White $$dr = [\theta(t) - \alpha r]dt + \sigma dz \qquad (6)$$

The single time-dependent function in the drift of both models is chosen to ensure consistency with the initial yield curve. The constant parameters explicitly determine the volatility structure of spot rates. For more general specifications of the short rate process, for lognormal models, and for pricing non-European options, we loose the analytical tractability of these models and so need to implement numerical techniques. We now go on to describe a methodology for building trees for the short rate which are constructed to be consistent with the observed yield and yield volatility curves.

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1 In much the same way that the Black-Scholes model, for pricing options on stocks, can be inverted to obtain the implied volatility of stock prices consistent with the option price, it can be reasoned that the same principle could be applied to the pricing of bonds.

4 Building trees consistent with the observed yield and yield volatility curves

The idea behind constructing short rate trees is the same as tree construction for the underlying asset price in, say, the binomial framework of Cox, Ross, and Rubenstein [1979]. We will use as an example the model of Black, Derman, and Toy [1990] (BDT). BDT develop a single factor Markov model to match observed term structure data, and which conversations with practitioners suggest is currently popular. BDT developed the model algorithmically, describing the evolution of the entire term structure in a discrete-time binomial lattice framework. A binomial tree is constructed for the short rate in such a way that the tree automatically returns the observed yield function and the volatilities of different yields. In a short rate tree the variable at each node is the Δt period interest rate and the movements of the rate in the tree are chosen to match some process which in the limit becomes a continuous time process. For pricing derivatives, interest rate trees work similarly to stock price trees except that the discount rate used varies from node to node.

Although the algorithmic construction means that the model is rather opaque with regard to its assumptions about the evolution of the short rate, several authors have shown that the implied continuous time limit of the BDT model is given by the following stochastic differential equation

$$d\ln r(t) = \left[\theta(t) - \frac{\sigma'(t)}{\sigma(t)}\ln r(t)\right] dt + \sigma(t)dz \qquad (7)$$

This representation of the model allows us to better understand the assumptions implicit in the model. The model incorporates two independent functions of time; $\theta(t)$, chosen so that the model fits the term structure of spot interest rates, and $\sigma(t)$, so that it fits the term structure of spot rate volatilities. Once $\theta(t)$ and $\sigma(t)$ are chosen, the future short rate volatility is , by definition, entirely determined and an unfortunate consequence of the model is that for certain specifications of the volatility function, i.e. if the future short rate volatility declines over time, the short rate can be mean-fleeing rather than mean-reverting. Changes in the short rate are lognormally distributed, with the resulting advantage that interest rates cannot become negative but with the disadvantage that analytic solutions for the prices of bonds or the prices of bond options are no longer available. The model also has the advantage that the short rate volatility is expressed as a percentage, conforming with the market convention.

If the model is fitted to the rate structure only, with future short rate volatility held constant, then the convergent limit reduces to the following which is a lognormal version of Ho-Lee;

$$d\ln r(t) = \theta(t)dt + \sigma dz \tag{8}$$

We now go on to describe how a binomial short rate tree for the BDT model can be constructed using forward induction[2]. Jamshidian [1991] shows that the level of the short rate at time t in the BDT model is given by,

$$r(t) = U(t)\exp(\sigma(t)z(t)) \tag{9}$$

where $U(t)$ is the median of the (lognormal) distribution for r at time t, $\sigma(t)$ is the level of short rate volatility and $z(t)$ is the level of the Brownian motion. In order to fit the model simultaneously to market yield and volatility curves we have to determine both $U(t)$ and $\sigma(t)$ at each time step. If the model is implemented to fit just the yield curve, we only require to determine the median.

We divide the life of the instrument underlying the interest rate derivative into, say, $i = 1,\ldots,N$ equal segments each of length Δt and define the following functions that describe the initial yield and volatility curves

$P(i)$: price at time 0 of a pure discount bond maturing at time $i\Delta t$

$R(i)$: yield at time 0 on a bond maturing at time $i\Delta t$

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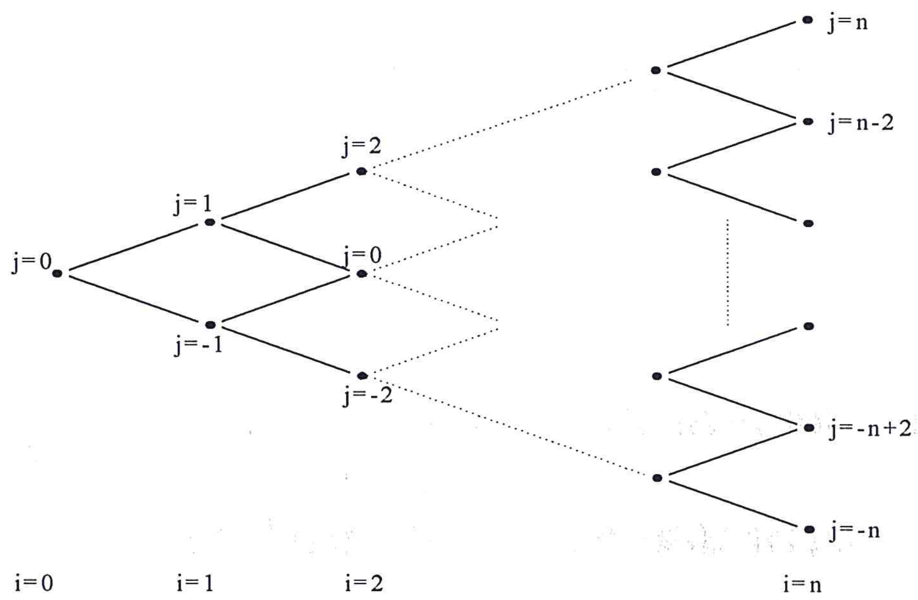
2 The original paper to apply forward induction for constructing yield curve models is due to Jamshidian [1991]

$V(i)$: proportional volatility at time 0 of yield $R(i)$

The risk-neutral transitional probabilities in the tree are assumed to be equal to one half. We label the states as in Figure 1.

Figure 1: Black-Derman-Toy Short Rate Tree



The value of r on the tree at time 0 is the initial short rate r_0, and is the yield on a discount bond maturing at Δt. At time $i = 0$ there is a single state $j = 0$. At time $i = 1$ there are two states $j = -1$ and $j = 1$. At time $i = n$ there are $(n + 1)$ states $j = -n, -n + 2, ..., n - 2, n$. j therefore has a centralised binomial distribution with mean 0 and variance n. $j\sqrt{dt}$ is therefore distributed with mean 0 and variance t which is the same as the random walk, implying that as dt approaches 0 then the binomial process $j\sqrt{dt}$ converges to the Weiner process $z(t)$. Therefore we can represent the level of the short rate in the tree as[3]:

$$r(i,j) = U(i)\exp(\sigma(i)j\sqrt{\Delta t}) \qquad (10)$$

3 C.f. equation (9).

i.e. replace t by i and $z(t)$ by $j\sqrt{dt}$. We refer to $r(i,j)$ as the short rate at the (i,j) node. To build a tree for the short rate (i.e. determining $r(i,j)$ for all i and j) therefore requires us to determine $U(i)$ and $\sigma(i)$. In order to determine these time dependent functions we can use Arrow-Debreu (or state or pure security) prices. Define $Q(i,j)$ as the value, at time 0, of a security that pays the following;

$1 if node (i,j) is reached and

$0 otherwise

the $Q(i,j)$'s are therefore prices of Arrow-Debrue securities which we can think of as discounted probabilities. Arrow-Debrue securities are the building blocks of all securities; in particular pure discount bond prices can be expressed as the following

$$P(i+1) = \sum_j Q(i,j)p(i,j) \tag{11}$$

where $p(i,j)$ denotes the price at time $i\Delta t$ and state j of the zero-coupon bond maturing at time (i.e. the one-period discount factor) and where the summation takes place across all of the nodes j at time i. For simple compounding we have:

$$p(i,j) = \frac{1}{1+r(i,j)\Delta t} \tag{12}$$

Forward induction involves accumulating the state prices as we progress thru the tree. Specifically, the pure security prices at time step $i+1$ are updated from the known values at time step i according to:

$$\begin{aligned} Q(i+1,j) &= \frac{1}{2}Q(i,j-1)p(i,j-1) + \frac{1}{2}Q(i,j+1)p(i,j+1) \\ &= \frac{1}{2}Q(i,j-1)\frac{1}{1+r(i,j-1)\Delta t} + \frac{1}{2}Q(i,j+1)\frac{1}{1+r(i,j+1)\Delta t} \end{aligned} \tag{13}$$

i.e. for the two nodes that lead into node $(i+1, j)$ we sum the product of their state prices, one period discount factors, and transitional probabilities[4].

Firstly we look at fitting the lognormal model to one yield curve. Let $\sigma(t)$ be equal to a constant σ, therefore equation (9) becomes:

$$r(i,j) = U(i)\exp(\sigma j\sqrt{\Delta t}) \quad (14)$$

Using (11) and (14) the time step $i+1$ pure discount bond prices can be constructed as:

$$\begin{aligned} P(i+1) &= \sum_j Q(i,j)\frac{1}{1+r(i,j)\Delta t} \\ &= \sum_j Q(i,j)\frac{1}{1+U(i)\exp(\sigma j\sqrt{\Delta t})\Delta t} \end{aligned} \quad (15)$$

As the initial discount function is known, the only unknown in equation (15) is $U(i)$. Unfortunately, we cannot rearrange equation (15) to obtain $U(i)$ explicitly, and need to use a numerical search technique such as Newton-Raphson. Once $U(i)$ has been determined we can use (14) to determine all the rates in the tree for that time step.

The extended methodology of fitting the model to both the rate and volatility structures has the same general form. In order to fit the term structure of volatilities $\sigma(t)$ must now be made time dependent and so the level of the short rate at node (i,j) is given by equation (10). Let the upnode from the first time step (node (1,1)) be denoted by U and the downnode (node (1,-1)) be denoted by D. Also, let $P_U(i)$ and $P_D(i)$ (for $i \geq 1$) be the discount function as seen from nodes U and D respectively with $R_U(i)$ and $R_D(i)$ the corresponding bond yields. Specifying both the spot rate yield and volatility curves at the initial time $i = 0$, is equivalent to specifying, at period $i = 1$, the two discount functions $P_U(i+1)$ and $P_D(i+1)$.

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4 For the extreme nodes $(i+1, j+1)$ and $(i+1, -j-1)$ there is a unique transitional path.

Firstly, we need to determine $R_U(i)$ and $R_D(i)$ for all $i \geq 1$. These must be consistent with the known values of $R(i)$ and $V(i)$. Therefore we need to solve simultaneously:

$$e^{-r_0\Delta t}\left[0.5e^{-(i-1)R_U(i)\Delta t} + 0.5e^{-(i-1)R_D(i)\Delta t}\right] = e^{-iR(i)\Delta t} \tag{16}$$

$$V(i)\sqrt{\Delta t} = 0.5\log\frac{R_U(i)}{R_D(i)} \tag{17}$$

These equations are solved as;

$$R_D(i) = R_U(i)e^{-2V(i)\sqrt{\Delta t}} \tag{18}$$

and where $R_U(i)$ is found as the solution to[5]:

$$e^{-(i-1)R_U(i)\Delta t} + e^{-(i-1)R_U(i)\exp(-2V(i)\sqrt{\Delta t})\Delta t} = 2e^{(r_0-i)R(i)\Delta t} \tag{19}$$

For $i = 2$ equations (18) and (19) determine the short rate for the two nodes at time Δt;

$$r_U = r(1,1) = R_U(2), \quad r_D = r(1,-1) = R_D(2)$$

The tree is constructed from time Δt onward using a procedure similar to the fitting the one yield curve case described earlier. Now we have two equations similar to (11)

$$P_U(i+1) = \sum_j Q_U(i,j)p(i,j) \tag{20}$$

$$P_D(i+1) = \sum_j Q_D(i,j)p(i,j) \tag{21}$$

where $p(i,j) = \dfrac{1}{1+r(i,j)\Delta t} = \dfrac{1}{1+U(i)\exp(\sigma(i)j\sqrt{\Delta t})\Delta t}$

5 This must be solved using a numerical search technique.

$Q_U(i,j)$ and $Q_D(i,j)$ are defined as the values, as seen from nodes U and D respectively, of a security that pays off $1 if node (i,j) is reached and zero otherwise. Therefore (20) and (21) are 2 equations with two unknowns, $U(i)$ and $\sigma(i)$. These can be solved by a two dimensional Newton-Raphson technique.

The same general procedure is used by Hull and White to build trinomial short rate trees consistent with observed yield and yield volatilities. Trinomial trees generally have better stability and convergence properties than binomial trees. The extra branch from each node gives them more flexibility so that the changes in the short rate between the nodes can be kept fixed and only the probabilities changed to achieve the fitting. A further improvement can be obtained by arranging for the central path of the tree to follow the expected behaviour of the short rate. This ensures that the tree most efficiently approximates the future distribution of the short rate.

A major problem with using deterministic time dependent parameters of the short rate process to fit the model to the observed yield volatilities is that the yield volatilities are then constrained to evolve deterministically into the future. In particular, if the initial yield volatility curve does not have a negative exponential form, then it will not keep the same shape. Starting from typical market yield volatilities, the volatility structure will be forced to evolve to be flat. The approach Heath, Jarrow and Morton/Carverhill approach we discuss in section 7 deals with this problem.

5 Pricing interest rate derivatives using trees

Once the tree has been constructed we know the short rate at every time and every state of the world consistent with our original assumptions about the process, and we can use it to price a wide range of interest rate derivatives in the usual manner via backwards induction. Let $C(i,j)$

represent the value of the contingent claim at node (i,j). Its value is related to the two connecting nodes at time step $i+1$ according to the usual discounted expectation:

$$C(i,j) = \tfrac{1}{2} p(i,j)\left[C(i+1, j+1) + C(i+1, j-1)\right] \qquad (22)$$

As an example, in the following we use the tree to price discount bond options. Assume that we are pricing a T maturity option on a s-maturity discount bond ($T \leq s$) with a strike price of K. Let *mats* and *matT* represent the number of time steps for the maturity of the bond and option respectively (so $s = mats \times \Delta t$ and $T = matT \times \Delta t$). We assume that the short rate tree has been constructed out as far as *mats*. Let $P(i,j)$ represent the value of the s-maturity bond at node (i,j).

Firstly, set the maturity condition for the bond underlying the option, $P(mats, j) = 1$ for all j, and then perform backwards induction for the bond price calculating the value of the s-maturity bond for every node in the tree[6].

$$P(i,j) = \tfrac{1}{2} p(i,j)\left[P(i+1, j+1) + P(i+1, j-1)\right] \qquad (23)$$

Next, evaluate the maturity condition for the option.

$$C(matT, j) = \max\{0, P(matT, j) - K\} \quad \text{for all } j \qquad (24)$$

For European options the value is obtained by applying equation (22) back through to the origin of the tree. For American options we need to allow for the possibility of early exercise in the normal way by taking the maximum of the discounted expectation and the intrinsic value of the option at each node.

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6 Obviously we only have to do back as far as *matT* for European options.

$$C(i,j) = \max\left\{P(i,j) - K, \tfrac{1}{2}p(i,j)\left[C(i+1,j+1) + C(i+1,j-1)\right]\right\} \tag{25}$$

A more efficient procedure for valuing European option prices utilises the fact that the pure security prices are equivalent to discounted probabilities. The value of any European option can be calculated directly from the tree as the sum of the product of the maturity condition of the option and the state price, for each node at the maturity time, i.e. for a call option;

$$call\ value = \sum_j Q(matT, j)\max\{0, P(matT, s) - K\} \tag{26}$$

6 Pricing interest rate exotics using short rate trees

So far we have concentrated on pricing interest rate derivatives where the payoff to the derivative at maturity only depends on the level of the short rate at each node and not on the path that the short rate took to achieve that level. We now show how one particular "path dependent" security can be valued with the short rate tree framework. In the past 12 months, or so, anecdotal evidence suggests that these path dependent or "exotic" interest rate derivatives are becoming popular amongst the practitioner community.

We focus on the case of index amortising rate (IAR) swaps. IAR swaps are agreements to exchange fixed for floating rate payments on pre-specified dates on a principal amount that may decline through time; the reduction in principal depending on the level of interest rates. A typical principal reduction schedule is illustrated in table 1. The schedule determines how the principal underlying the swap will be reduced according to some index. Throughout this example we take as the index 3 month LIBOR, but other popular choices include LIBOR of different tenor and Treasury constant maturity rates (CMT).

Table 1: Principal Reduction Schedule for an Index Amortising Swap

Index Level Relative to Base Rate X	**Principal Reduction**
X - 300 bps or lower	100
X - 200	60
X - 100	40
X	20
X + 100	10
X + 200	5
X + 300 or higher	0

Consider a 3 year IAR swap with a starting principal of $100m where the principal repayments are determined according to the above schedule. The base rate, X, is determined at the outset of the swap at, say 10%. Payments are only made on the quarterly interest payment dates and the principal reduction may not exceed the outstanding principal. Suppose that 3 month LIBOR is currently 10%. The annual amortisation rate is therefore 20%. A LIBOR level of 9.5% would lead to an amortisation rate of 30%[7].

Like a plain vanilla swap, the value of an index IAR swap is the difference between the value of the two bonds, which we denote by B_{fix} for the fixed side, and B_{fl} for the floating side, underlying the swap. The floating side equals the swap principal immediately after a reset date, whilst the fixed side must be calculated from the short rate tree. The value of the fixed rate bond depends on the level of interest rates and the outstanding principal PR. PR satisfies the condition that its level at time $t + \Delta t$ can be calculated from the level at time t and the interest rate at time $t + \Delta t$, and so we can apply the techniques in Hull and White [1993].

Depending on the path of the short rate through the tree, the number of alternative principal amounts that can be realised at any node grows quickly with the number of time steps. Instead of keeping track of all the possible alternatives we compute the value of the fixed rate bond at any node only for certain values of the principal. Specifically, we evaluate the maximum and

7 We are assuming that the terms of the deal specify that linear interpolation applies to the reduction schedule.

minimum values of PR at each node and then approximate the set of all possible PR's with M (say 4) equally spaced values at each node. Define $PR_{i,j,k}$ $(k = 0,1,\ldots,M)$ for the outstanding principal at node (i,j) and $B_{i,j,k}$ for the value of B_{fix} at (i,j) when PR has this value. To calculate the outstanding principal we apply forward induction[8].

The principal amounts at node (i,j) depend on the relevant levels at the nodes which lead into (i,j) and the LIBOR rate at the node. We can determine the 3 month LIBOR at node (i,j), which we denote by $L(i,j)$, according to the following;

$$L(i,j) = \frac{1}{0.25}\left[\frac{1}{P(i,j,0.25)} - 1\right] \qquad ()$$

where $P(i,j,0.25)$ is the price of a pure discount bond with 3 months to maturity, which can be determined from the short rate tree. $L(i,j)$ is then used to determine the percentage principal reduction , pr, according to the scheme in table 1. If we are moving from node $(i-1,j+1)$ and if the amortisation rate applies to the outstanding principal, then the principal reduction is given by $pr \text{ x } PR_{i-1,j+1,k}$ which is used to determine the maximum and minimum principals at the (i,j)'th node. If the rate applies to the original principal, PR_0, then, in order that the principal cannot become negative, the reduction is given by $\min(pr \text{ x } PR_0, PR_{i-1,j+1,k})$. A similar procedure is performed for the other parent node $(i-1,j-1)$. This process continues until the end of the life of the swap.

Once the forward induction step is completed, the value of the bond at maturity, $B_{n,j,k}$, can be calculated for all j and all k

$$B_{n,j,k} = PR_{n,j,k}$$

8 As we perform forward induction we only update $PR_{i,j,0}$ and $PR_{i,j,M}$, the maximum and minimum values of PR at node (i,j) respectively.

To calculate the value of B_{fix} at node (i,j) for $i < n$ we use backwards induction. Suppose $PR_{i,j,k}$ leads to $PR_{i+1,j+1,k_u}$ when there is an up movement in the short rate and $PR_{i+1,j-1,k_d}$ when there is a down movement. For a European derivative this implies that

$$B_{i,j,k} = e^{-r(i,j)\Delta t}\left[\frac{1}{2}B_{i+1,j+1,k_u} + \frac{1}{2}B_{i+1,j-1,k_d} + c_{i,j}\right]$$

where $c_{i,j}$ is the cashflow (interest plus principal reduction) during the period $(i, i+\Delta t)$.

Due to the nature of the forward induction (we only hold M values of the principal at each node) $B_{i+1,j+1,k_u}$ and $B_{i+1,j-1,k_d}$ might not be known at time step $i+1$, and so we interpolate from the set of known values. For example, we interpolate $B_{i+1,j+1,k_u}$ from $B_{i+1,j+1,k_1}$ and $B_{i+1,j+1,k_2}$ where k_1 and k_2 are the closest values of PR to $PR_{i+1,j+1,k_u}$ such that $PR_{i+1,j+1,k_1} \le PR_{i+1,j+1,k_u} \le PR_{i+1,j+1,k_2}$. This procedure is repeated for all k at node (i,j). We determine $B_{i+1,j-1,k_d}$ similarly.

The value of the swap for the receiver of fixed payments is then the difference between $B_{fix} - 100$.

7 Non-Markovian Short Rate Models and Monte Carlo Simulation

An alternative approach to directly modelling the short rate was introduced by Heath, Jarrow and Morton (1992). Their idea was to take the whole forward rate curve together with a set of forward rate volatility curves as given by the market and specify the no-arbitrage evolution of the forward rate curve via a system of stochastic differential equations

$$df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{n} \sigma_i(t,T,)dB_i(t) \tag{27}$$

where $f(t,T)$ and $\sigma_i(t,T,)$ are the instantaneous forward rate at date t for date T in the future and its volatilities, and the $dB_i(t)$ are independent Brownian motions. The volatility functions in general can be functions of the entire history of the entire forward rate curve, Carverhill (1995) provides an excellent exposition of this model and the technical conditions required. We will concentrate on the Gaussian case, that is the volatilities are functions of time and maturity only. The drifts of the forward rates under the risk neutral measure are determined by no-arbitrage to be

$$\alpha(t,T) = \sum_{i=1}^{n}\left\{\sigma_i(t,T,)\left[\int_t^T \sigma_i(t,u,)du\right]\right\} \tag{28}$$

so that the expected return on a pure discount bonds is the riskless rate. The model can equivalently be formulated in terms of pure discount bond prices

$$\frac{dP(t,T)}{P(t,T)} = (r(t))dt + \sum_{i=1}^{n} v_i(t,T)dB_i(t) \tag{29}$$

where

$$v_i(t,T) = -\int_t^T \sigma_i(t,s)ds \tag{30}$$

The implied process for the short rate is

$$dr = \left[\frac{\partial f(0,t)}{\partial t} + \sum_{i=1}^{n}\left\{\int_0^t v_i(u,t)\frac{\partial^2 v_j}{\partial t^2}(u,t) + \frac{\partial v_j}{\partial t}(u,t)^2 du + \int_0^t \frac{\partial^2 v_i}{\partial t^2}(u,t)dB_i(u)\right\}\right]dt + \sum_{i=1}^{n}\frac{\partial v_j}{\partial t}(u,t)\bigg|_{u=t} dB_i(t) \tag{31}$$

The drift of the short rate depends on integrals of the second derivatives of the pure discount bond volatilities over the Brownian paths. It is therefore non-Markovian in general and if we costructed a tree for the short rate it would be non-recombining or exploding. Monte Carlo simulation provides a way of dealing with this problem, although the use of Monte Carlo simulation makes pricing American style options more computationally difficult. We will describe how Monte Carlo simulation for the model can be implemented efficiently by judicious choice of the formulation.

The pure discount bond price formulation of the model described by equation (29) has a number of advantages for pricing and hedging derivatives. Since the model is formulated in terms of prices then there is a natural measure (the forward measure) under which prices are Martingales. This change of measures removes the necessity of computing the short rate directly. All interest rate derivatives depend, either directly or indirectly, on the prices of pure discount bonds and most, for example coupon bond options or swaptions, depend directly on these prices. Therefore in pricing a derivative we only need to model a relatively small number of PBD's. In contrast, the forward rate formulation (27) requires integration over the forward rate curve to obtain PDB prices. In implementation this requires a very large number of points on the forward rate curve to be modelled in order to obtain accurate prices.

In order to price derivatives we firstly transform to the forward measure so that equation (29) becomes

$$\frac{dP(t,T)}{P(t,T)} = +\sum_{i=1}^{n} v_i(t,T)d\widetilde{B}_i(t) \tag{32}$$

where $d\widetilde{B}_i(t) = dB_i(t) - \dfrac{r(t)}{v_i(t,T)}dt$.

Now consider pricing an option on a coupon bond with certain cash flows C_k at dates s_k, $k = 1,\ldots,m$[9]. The price of a call option on this coupon bond is given by (see Carverhill (1995))

$$Call_{CB}(t,K,T,\{s_k\})=\tilde{\mathrm{E}}_t\left[\max(0,\sum_{k=1}^{m}C_k P(T,s_k)-KP(T,T))\right] \qquad (33)$$

This can be written as

$$Call_{CB}(t,K,T,\{s_k\})=\tilde{\mathrm{E}}_t\left[\max(0,\sum_{k=1}^{m}C_k P(t,s_k)Y(t,T,s_k)-KP(t,T)Y(t,T,T))\right] \qquad (34)$$

where

$$Y(t,T,s) = \exp\left[\sum_{i=1}^{n}\left\{\int_t^T v_i(u,s)dB_i(u)-\frac{1}{2}\int_t^T v_i(u,s)^2\,du\right\}\right]$$

and are exponential Martingales. We can obtain an estimate of the expectation in equation (34) by Monte Carlo simulation. Let M be the number of simulations, N be the number of time steps of size Δt, and ε be a standard normal random number, then the Monte Carlo estimate of the option value is given by

$$Call_{CB}(t,K,T,\{s_k\})=\frac{1}{M}\sum_{j=1}^{M}\left[\max(0,\sum_{k=1}^{m}C_k P(t,s_k)Y_j(t,T,s_k)-KP(t,T)Y_j(t,T,T))\right] \qquad (35)$$

where

$$Y_j(t,T,s) = \exp\left[\sum_{l=1}^{n}\sum_{i=1}^{N} v_l(u_i,s)\varepsilon_l(u_i)\sqrt{\Delta t}-\frac{1}{2}v_l(u_i,s)^2\,\Delta t\right]$$

So we only need to simulate the pure discount bond prices underlying the coupon bond rather than the entire forward rate curve. Essentially the pure discount bond prices provide us with

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[9] Caps, floors, collars, and swaptions can all be considered as options on coupon bonds (see Strickland (1996).

exactly the summary statistics on the non-Markovian evolution which we need to price the option. However, we must use small time steps to accurately represent the stochastic behaviour of the term structure so the volatility functions must be stored or computed to the resolution of the simulation time step. Carverhill and Pang (1995) show how the accuracy of this Monte Carlo estimate can be improved by using the exponential Martingales $Y_j(t,T,s)$ as control variates. Consider taking the Monte Carlo estimate as

$$Call_{CB}(t,K,T,\{s_k\})=\frac{1}{M}\sum_{j=1}^{M}\left[\begin{array}{l}\max(0,\sum_{k=1}^{m}C_kP(t,s_k)Y_j(t,T,s_k)-KP(t,T)Y_j(t,T,T))\\ -\beta_1(Y_j(t,T,T)-1)-\sum_{k=1}^{m}\beta_{k+1}(Y_j(t,T,s_k)-1)\end{array}\right] \quad (36)$$

Now we have $\widetilde{E}_t\left[Y_j(t,T,s_k)-1\right]=0$ and so the expectation of the expression in the square brackets will be unaffected by the addition of the extra terms. But the $Y_j(t,T,s)$ are correlated with the payoff and so by taking short positions in them they act like a hedge, reducing the variance of the payoff and thus increasing the accuracy of the Monte Carlo estimate. The β's can be estimated by least squares regression as follows; rewrite the Monte Carlo estimate with the payoff (y_j) and control variates ($x_{j,k}$) for each path j. The estimate of the option value C is

$$C=\tfrac{1}{M}\sum_{j=1}^{M}(y_j-\sum_{k=1}^{m}\beta_k x_{j,k}) \quad (37)$$

A least squares regression estimate of C and the β's is

$$\beta=(X'X)^{-1}X'Y \quad (38)$$

where $\beta=(C,\beta_1,\ldots,\beta_m)$, X is a matrix whose rows are $(1,x_{j,1},\ldots,x_{j,m})$ and Y is the vector of payoffs. The matrices $X'X$ and $X'Y$ can be accumulated as the simulation proceeds using

$$(X'X)_{k,l}^{j+1}=(X'X)_{k,l}^{j}+x_{j+1,k}x_{j+1,l} \quad (39)$$

$$(X'Y)_k^{j+1} = (X'Y)_k^j + x_{j+1,k} y_{j+1} \qquad (40)$$

where $x_{j,0} = 1$, and $x_{j,k}$ and y_j are the values returned by the jth simulation. The final estimate of the option value should be obtained via (36) after estimating the β's by the least squares regression, otherwise the estimate will be biased. This is because with a simultaneous estimation the β's will be correlated with the $Y_j(t,T,s)$'s and so the control variates will no longer have mean zero. Note that the β's remain constant along the path and so we can interpret the role of the control variates in equation (36) as a static hedge. Clewlow and Strickland (1996) show how the technique can be extended to use a dynamic hedge and achieve much greater variance reduction.

8 Conclusions

In this chapter we have attempted to give an overview of the current state of interest rate term structure modelling and derivative pricing. In particlular we have tried to show how theory meets the practice .

References

Black F, E Derman and W Toy, 1990, "A One-Factor Model of Interest Rates and It's Application to Treasury Bond Options", *Financial Analysts Journal*, Jan-Feb, pp 33-39

Carverhill A, 1995, "A Simplified Exposition of the Heath, Jarrow and Morton Model", Stochastics and Stochastics Reports Vol. 53, pp 227 - 240.

Carverhill A and K Pang, 1995, "Efficient and Flexible Bond Option Valuation in the Heath, Jarrow, and Morton Framework", Journal of Fixed Income Vol. 5, No. 2, pp 70 - 77.

Clewlow L and C Strickland, 1996, "Monte Carlo Pricing of American Options in the Gaussian Heath, Jarrow, and Morton Model", Financial Options Research Centre Working Paper, Warwick University.

Cox, Ingersoll, and Ross. 1985, "A Theory of the Term Structure of Interest Rates", *Econometrica*, Vol 53, March.

Cox, Ross, and Rubenstein, 1979, "Option Pricing: A Simplified Approach", *Journal of Financial Economics*, Vol 7, September, pp 229-263

Fong, H. G., and O. A. Vasicek, 1992, "Interest Rate Volatility as a Stochastic Factor", Gifford Fong Associates Working Paper.

Heath, D., R Jarrow, and A Morton. 1992, "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claim Valuation", *Econometrica*, Vol 60, (1), pp 77-105.

Ho T S Y, and S-B Lee, 1986, "Term Structure Movements and Pricing Interest Rate Contingent Claims", *Journal of Finance*, Vol 41 (5), December pp 1011-1029.

Hull J and White A, 1990, "Pricing Interest Rate Derivative Securities", *The Review of Financial Studies*, Vol 3 (4).

Hull J, and A White, 1993, "Efficient Procedures for Valuing European and American Path-Dependent Options", *The Journal of Derivatives*, Fall, pp 21-31.

Jamshidian F, 1991, "Forward Induction and Construction of Yield Curve Diffusion Models", *Journal of Fixed Income*, Vol 1 (1)

Longstaff F A, and E S Schwartz, 1992, "Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model", *The Journal of Finance*, Vol XLVII (4), September, pp 1259-1282.

Strickland C, 1996, "Models for Pricing Interest Rate Derivative Securities", European Journal of Finance ...

Vasicek O, 1977, "An Equilibrium Characterisation of the Term Structure" *Journal of Financial Economics*, No. 5.

Hull J and White A, 1994, "Numerical Procedures for Implementing Term Structure Models I: Single Factor Models", *The Journal of Derivatives*, Fall 1994, pp 7-16.

Hull J, and A White, 1994, "Numerical Procedures for Implementing Term Structure Models I: Single Factor Models", *The Journal of Derivatives*, Winter1994, pp 37-48.