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Abstract

In this paper we describe a formulation of the multi-factor Gaussian interest rate model of Carverhill (1995) closely related to the Heath, Jarrow and Morton model (1992). Our formulation allows us to efficiently price complex interest rate derivatives. We illustrate this by describing analytical solutions for caps and floors and efficient numerical integration and approximation formulae for European swaptions. Finally we present some numerical results for the pricing of European coupon bond options in a three factor implementation.

1 Introduction

In 1992 Heath, Jarrow, and Morton [1992] (HJM) described a very general multi-factor arbitrage-free model for pricing interest rate derivatives. Their approach was to take the initial instantaneous forward rate curve and its volatility structure as given and derive the arbitrage free stochastic evolution of the forward rate curve. Carverhill [1995] showed how the model could be reformulated more easily in terms of pure discount bond prices. Our starting point is the model formulation of Carverhill [1995].

A great deal of research has been directed at the HJM model mainly in the search for restricted forms of the model which retain the advantageous properties whilst simplifying its implementation. We show in this paper that restricting the model to the Gaussian family is all that is required to obtain tractibility. An important property required for tractibility is to be able to price caps/floors and European swaptions efficiently so that the volatility functions can be implied from the market prices of these liquidly traded interest rate derivatives. The model can then be used to consistently price and hedge more complex interest rate derivatives such as average rate caps or American swaptions.

In this paper we develop efficient valuation techniques for pricing caps and floors (a similar result has previously been obtained by Brace and Musiela [1994]) and European swaptions under a multi-factor Gaussian interest rate model. The models of Hull and White [1990, 1993] and Ritchken and Sankarasubrmanian [1995a, 1995b] are special cases of our model in that they require restrictions on the form of the volatility functions, we do not require any restrictions on the volatility functions (apart from the usual technical conditions; see Carverhill [1995]).

2 The Model

Heath, Jarrow, and Morton [1992] extend the early term structure consistent framework of Ho and Lee [1986] proposing the following stochastic differential equation (SDE) for the evolution of the instantaneous forward rate curve:

$$df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{n} \sigma_i(t,T,f(t,T))dz_i(t) \tag{1}$$

where $f(0,t)$ is the initially observed forward curve and dz_i are independent Wiener processes. Equation (1) is the most general formulation of the HJM model with n sources of randomness and with the volatilities of forward rates allowed to be dependent on the level of the individual rates. As shown by HJM the drift rate $\alpha(t,T)$ is determined by no arbitrage;

$$\alpha(t,T) = \sum_{i=1}^{n} \left\{ \sigma_i(t,T,f(t,T)) \left[\int_t^T \sigma_i(t,u,f(t,u))du \right] \right\} \tag{2}$$

and hence the forward rate process is completely specified by the volatility functions. In the following analysis we concentrate on Gaussian versions of the approach dropping the dependence of the volatility function on the forward rate levels. Although the original formulation of the HJM approach is in terms of forward rates the model can be equivalently restated in terms of pure discount bond prices. Under this formulation (see Carverhill [1995]) bond price returns satisfy the SDE;

$$\frac{dP(t,T)}{P(t,T)} = r(t)dt + \sum_{i=1}^{n} v_i(t,T)dz_i(t) \tag{3}$$

where $P(0,T)$ is the initially observed discount function and represents the price of a pure discount bond which pays one unit of cash, with certainty, at time T. The T-maturity bond price return volatility is related to the forward rate volatility function via;

$$v_i(t,T) = -\int_t^T \sigma_i(t,u)du \tag{4}$$

The process for bond price returns is more intuitive than the forward rate process; because the bonds are traded assets the drift in a risk-neutral world is simply the instantaneous short rate. For the rest of this paper we will work with this formulation as numerical implementation is both more efficient and intuitive. It can be shown (Carverhill [1995]) that the instantaneous spot interest rate, $r(t)$, which is given by $f(t,t)$, satisfies the following SDE;

$$\begin{aligned} dr = &\left[\frac{\partial f(0,t)}{\partial t} + \sum_{i=1}^{n} \left\{ \int_0^t v_i(u,t)\frac{\partial^2 v_i(u,t)}{\partial t^2} + \frac{\partial v_i(u,t)^2}{\partial t} du + \int_0^t \frac{\partial^2 v_i(u,t)}{\partial t^2} dz_i(u) \right\} \right] dt \\ &+ \sum_{i=1}^{n} \left. \frac{\partial v_i(u,t)}{\partial t} \right|_{u=t} dz_i(t) \end{aligned} \tag{5}$$

The last component of the drift term, involving integration over the Brownian motions, indicates that the short rate is non-Markovian in general formulations of the HJM model since the evolution of the short rate could depend on the entire path taken by the driving Wiener processes.

3 Pricing Interest Rate Caps

In this section we first discuss the pricing of pure discount bond options. Any implementation amounts to evaluating the expected discounted payoff value under an appropriate measure. For an European call with strike K, maturity date T on a pure discount bond maturing at time s the price at time t ($t \leq T \leq s$) under the usual risk-neutral measure is given by:

$$Call(t,K,T,s) = \mathrm{E}_t\left[\exp(-\int_t^T r(u)du)\max(0, P(T,s)-K)\right] \tag{6}$$

In the first stage of our formulation we change the numeraire to the savings account

$$\beta(T) = \beta(t)\exp(\int_t^T r(u)du) \tag{7}$$

where usually $\beta(t) = 1$, and we obtain a stochastic differential equation representing bond price returns under this new measure;

$$\frac{dP(t,T)}{P(t,T)} = +\sum_{i=1}^{n} v_i(t,T)d\tilde{z}_i(t) \tag{8}$$

The effect of the change in measure, which converts all prices into today's dollars, is to eliminate the drift of the return process, making it a Martingale. The discounted expectation in equation (6) becomes;

$$Call(t,K,T,s)=\tilde{\mathrm{E}}_t\left[\max(0,P(T,s)-KP(T,T))\right] \tag{9}$$

Now, we can rewrite equation (8) in its integral form;

$$P(T,s)=P(t,s)\exp\left[\sum_{i=1}^{n}\left\{-\frac{1}{2}\int_t^T v_i(u,s)^2\,du+\int_t^T v_i(u,s)\,dz_i(u)\right\}\right] \tag{10}$$

If we think in terms of implementing (8) via Monte Carlo simulation, elimination of the drift saves us first inflating the pure discount bond prices at the short rate and then discounting the terminal payoff of the derivative at the short rate. This saving can be considerable given the complex non-Markovian short rate process given by equation (5).

Next we change numeraire again, this time to $P(t,T)$ (the forward measure to the maturity date of the option) and so equation (9) becomes;

$$Call(t,K,T,s)=P(t,T)\,\mathrm{E}_t^T\left[\max\left(0,\frac{P(T,s)}{P(T,T)}-K\right)\right] \tag{11}$$

where $E_t^T[]$ indicates expectation with respect to the $P(t,T)$-numeraire and where

$$\frac{P(T,s)}{P(T,T)} = \frac{P(t,s)}{P(t,T)} \exp\left[\sum_{i=1}^{n}\left\{-\frac{1}{2}\int_t^T \left(v_i(u,s) - v_i(u,T)\right)^2 du + \int_t^T (v_i(u,s) - v_i(u,T))dz_i(u)\right\}\right] \tag{12}$$

The integration within the exponential is now performed over the difference in the two bond price volatilities. From equation (12) the relative bond prices $P(T,s)/P(T,T)$ are seen to be lognormally distributed and so the natural logarithm of the relative bond prices are normally distributed

$$\ln\left(\frac{P(T,s)}{P(T,T)}\right) \sim N\left[\ln\left(\frac{P(t,s)}{P(t,T)}\right) - \frac{1}{2}\sum_{i=1}^{n}\left\{\int_t^T \left(v_i(u,s) - v_i(u,T)\right)^2 du\right\}, \sum_{i=1}^{n}\left\{\int_t^T \left(v_i(u,s) - v_i(u,T)\right)^2 du\right\}\right]$$

the solution of equation (11) is therefore straightforward and analogous to the Black and Scholes (1973) equation;

$$Call(t,K,T,s) = P(t,s)N(h) - KP(t,T)N(h-w) \tag{13}$$

where

$$h = \frac{\ln\left(\frac{P(t,s)}{P(t,T)K}\right) + \frac{1}{2}w}{\sqrt{w}} \quad , \quad w = \sum_{i=1}^{n}\left\{\int_t^T \left(v_i(u,s) - v_i(u,T)\right)^2 du\right\},$$

and w is the variance of the log relative discount bond prices.

The implication of equation (13) is that for multi-factor Gaussian versions of the HJM model, evaluating European options on pure discount bonds is as straightforward as using the Black-Scholes equation. The calculation requires only univariate integrations involving the volatilities of the discount bonds maturing at the time of the option and the bond underlying the option. This integration will usually be analytical, for example if the volatility functions are polynomials in time and maturity, and if not standard numerical routines can be used to efficiently perform the numerical integration (see for example Press *et al.* [1992]).

It is well known that interest rate caps (floors) can be priced as portfolios of European put (call) options on pure discount bonds (see Strickland [1996]) and therefore we can use the results above for pricing these instruments. Pricing of interest rate caps can therefore be performed extremely efficiently, enabling us to calibrate our model to this set of market data[1].

4 Pricing European Interest Rate Swaptions

We now turn our attention to pricing European options on interest rate swaps, or European swaptions. A well known result concerning swaption pricing is that European payer (receiver) swaptions can be priced as put (call) options on coupon bonds where the strike price of the option is set equal to the principal underlying the swap (see for example Strickland [1996]). In a single factor world Jamshidian [1989] has shown that an option on a coupon bond can be

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1 See Clewlow *et al* [1997] for empirical estimates of HJM volatility functions calibrated to money market caps and swaptions data.

valued as a portfolio of options on pure discount bonds. However, in a multi-factor world bond returns are not perfectly correlated and this decomposition is no longer possible. In this section we show how coupon bond option values can be efficiently calculated. The nature of our methodology implies that adding extra factors comes at the expense of only fractional increases in computation times. We concentrate on pricing a call option, with maturity T, on a coupon bond paying cashflows C_k at dates s_k, $k=1,\ldots,m$ ($s_k \geq T$). Under the forward measure of section (3) the discounted expectation of the option payoff is given by;

$$Call_{CB}(t,K,T,\{s_k\}) = P(t,T)\mathrm{E}_t^T\left[\max\left(0,\sum_{k=1}^{m} C_k \frac{P(T,s_k)}{P(T,T)} - K\right)\right] \tag{14}$$

The expectation is taken over the m-dimensional normal distribution of the correlated log-relative discount bond prices, $\ln\left(\frac{P(T,s_k)}{P(T,T)}\right)$, where m is the number of cashflows accruing to the coupon bond after the maturity of the option. In order to perform Monte-Carlo simulation to evaluate (14) we must therefore compute the $m*m$ covariance matrix Σ;

$$\begin{aligned}\Sigma_{kj} = Cov\left[\ln\left(\frac{P(T,s_k)}{P(T,T)}\right),\ln\left(\frac{P(T,s_j)}{P(T,T)}\right)\right] = \\ \sum_{i=1}^{n}\left\{\int_t^T \left(v_i(u,s_k)-v_i(u,T)\right)\left(v_i(u,s_j)-v_i(u,T)\right)du\right\}\end{aligned} \tag{15}$$

In order to efficiently sample from this covariance matrix we compute the orthogonal representation of the covariance matrix which gives us the m eigenvectors $\underline{w}_i$ and associated m eigenvalues λ_i such that

$$\Sigma = \Gamma\Lambda\Gamma' \tag{16}$$

where $\Gamma = \begin{vmatrix} w_{11} & w_{21} & \dots & w_{m1} \\ w_{12} & w_{22} & \dots & w_{m2} \\ \dots & \dots & \dots & \dots \\ w_{1m} & w_{2m} & \dots & w_{mm} \end{vmatrix}$ and $\Lambda = \begin{vmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda_m \end{vmatrix}$

The columns of Γ are the eigenvectors. Let M be the number of samples or simulations and ε_i, $i = 1,\dots,m$ be independent standard normal random numbers. Therefore, we have;

$$Call_{CB}(t,K,T,\{s_k\}) = P(t,T)\frac{1}{M}\sum_{j=1}^{M}\left[\max\left(0, \sum_{k=1}^{m} C_k \frac{P(t,s_k)}{P(t,T)} Y_j(t,T,s_k) - K\right)\right] \tag{17}$$

where $Y_j(t,T,s_k) = \exp\left[-\frac{1}{2}\sum_{i=1}^{m}\left\{w_{ik}{}^2\lambda_i\right\} + \sum_{i=1}^{m}\left\{w_{ik}\sqrt{\lambda_i}\varepsilon_i\right\}\right]$

and $\tilde{w}(t,T,s_k) = \sum_{i=1}^{m}\left\{w_{ik}{}^2\lambda_i\right\} = \sum_{i=1}^{n}\left\{\int_t^T (v_i(u,s_k) - v_i(u,T))^2 du\right\}$

Notice that with this representation we are only required to simulate the pure discount bonds which correspond to the cashflows of the underlying coupon bond. Also, because of the Gaussian nature of our framework we effectively jump straight to the end of the life of the option under each simulation, rather than the more usual practice of simulating at a large number of small discrete time steps until maturity. Finally this formulation gives a natural way to trade off accuracy with speed by truncating the integration space dependent on the sizes of the eigenvalues.

Although we have an efficient procedure for simulation of coupon bond option prices (we only have to simulate the bond prices corresponding to coupon cashflows and can jump straight to the end of each simulation) convergence is still slow. As in standard simulation we can also apply antithetic techniques by computing the terminal coupon bond value based upon the negative of the standard normal increments. We now describe the application of an analytical approximation for coupon bonds described by Pang [1996] as a control variate.

The coupon bond underlying the option is a weighted sum of the lognormally distributed pure discount bonds corresponding to the cashflows, and as such will not itself be lognormally distributed. However, since the final cashflow is much larger than the others then the coupon bond price will be highly correlated with the price of this pure discount bond. This implies that we can hedge it with an appropriate position in an option on a discount bond with a maturity corresponding to the final cashflow (s_m).

In our formulation we can compute the mean and variance of the coupon bond at the maturity date of the option analytically, as well as the value of the hedge discount bond option (via equation (13)). Our variance reduction technique consists of valuing a hedged portfolio - long the coupon bond option and short the discount bond option which matches the mean and variance[2]. This hedged portfolio has much smaller variance than the coupon bond option and so we obtain a much more accurate estimate of the option value.

The value of the coupon bond at the maturity of the option, $CB(T)$, is given under the forward measure as:

$$CB(T) = \sum_{k=1}^{m} C_k \frac{P(T,s_k)}{P(T,T)} \tag{18}$$

Therefore the expected value of the payoff to the coupon bond is given by:

$$E_t^T[CB(T)] = \sum_{k=1}^{m} C_k \frac{P(t,s_k)}{P(t,T)} \tag{19}$$

In order to compute the variance of the coupon bond we need the expectation of the squared payoff to the bond:

$$E_t^T\left[CB(T)^2\right] = \sum_{j=1}^{m}\sum_{k=1}^{m} C_j C_k \frac{P(t,s_j)}{P(t,T)}\frac{P(t,s_k)}{P(t,T)} \exp\left(Cov\left[\ln\left(\frac{P(T,s_k)}{P(T,T)}\right), \ln\left(\frac{P(T,s_j)}{P(T,T)}\right)\right]\right) \tag{20}$$

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2 The principle behind this variance reduction technique is similar to that used by Clewlow and Strickland [1996]. In that paper the authors simulated a 'delta-hedged' portfolio for the two-factor stochastic interest rate model of Fong and Vasicek [1991].

Now we can equate the variance of the coupon bond with the variance of the discount bond with maturity s_m and with principal L.

$$Var[CB(T)] = L^2 \frac{P(t,s_m)^2}{P(t,T)^2}\left\{\operatorname{var}\left(\ln\left(\frac{P(T,s_m)}{P(T,T)}\right)\right) - 1\right\} \tag{21}$$

Equation (21) can then be inverted to determine L

$$L = \sqrt{\frac{Var[CB(T)]}{\frac{P(t,T')^2}{P(t,T)^2}}\left\{\operatorname{var}\left(\ln\left(\frac{P(T,T')}{P(T,T)}\right)\right) - 1\right\}} \tag{22}$$

Finally, we want the mean of our hedging instrument to match the mean of the target instrument and so we adjust the strike of the coupon bond option by the difference between the means to obtain the strike for the discount bond option, K';

$$K' = K - \left(E_t^T[CB(T)] - L\frac{P(t,T')}{P(t,T)}\right) \tag{23}$$

5 Numerical Results

In order to illustrate the efficiency of the techniques described in this paper we look at up to a 3 factor version of the Gaussian HJM model. For our example and without loss of generality we consider that each of the pure discount bond return volatility functions can be described by an equation of the following form;

$$v_i(u,s) = \frac{\sigma_i}{\alpha_i}\left(\exp(-\alpha_i(s-u)) - 1\right) \qquad i = 1,2,3 \tag{23}$$

i.e. they have Vasicek [1977] form. Therefore for the variance of the log relative discount bond prices (w in equation (13)) we obtain ;

$$\sum_{i=1}^{3}\left\{\int_t^T \left(v_i(u,s) - v_i(u,T)\right)^2 du\right\} =$$
$$\frac{\sigma_1^2}{2\alpha_1^3}\left(1-e^{-\alpha_1(s-T)}\right)^2\left(1-e^{-2\alpha_1(T-t)}\right)+\frac{\sigma_2^2}{2\alpha_2^3}\left(1-e^{-\alpha_2(s-T)}\right)^2\left(1-e^{-2\alpha_2(T-t)}\right) \tag{24}$$
$$+\frac{\sigma_3^2}{2\alpha_3^3}\left(1-e^{-\alpha_3(s-T)}\right)^2\left(1-e^{-2\alpha_3(T-t)}\right)$$

From equations (13) and (24) it can be seen that pure discount bond options can be computed almost instantaneously and that adding extra factors does not add significantly to the computation time. We assume the initial yield curve is flat at 5% (continuously compounded), the parameter values for the three volatility functions are $\alpha_1 = 0.10$, $\sigma_1 = 0.0095$, $\alpha_2 = 1.00$, $\sigma_2 = 0.0025$, $\alpha_3 = 5.00$, and $\sigma_3 = 0.0019$ and we price a European coupon bond call option with one year to maturity and a strike price equal to the current bond price on a 5% semi-annual coupon bond with a range of coupons remaining from six (a three year bond) to the twenty (a ten year bond). Table 1 shows the results.

Table 1 : Computation Times for European Coupon Bond Option

Number of Coupons	Strike Price	Option Value	Standard Error	Time Taken (seconds)
12	0.996574	0.013555	0.000003	0.11
14	0.996105	0.016138	0.000005	0.17
16	0.995661	0.018343	0.000006	0.26
18	0.995240	0.020234	0.000010	0.38
20	0.994841	0.021884	0.000009	0.51

6 Summary and Conclusions

In this paper we have described a formulation of a multi-factor Gaussian interest rate model based on Carverhill (1995) and closely related to the Heath, Jarrow and Morton (1992) model. Our formulation allows us to efficiently price complex interest rate derivatives, as examples we have described analytical solutions for caps and floors and efficient numerical integration and approximation formulae for European coupon bond options and swaptions. In related papers we use these results to calibrate the model to market prices of caps and swaptions (Clewlow *et al* (1997)) and show how to price a range of American and path dependent interest rate derivatives in this framework (Clewlow and Strickland (1997)).

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