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Empirical Properties of Asset Price Processes

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EMPIRICAL PROPERTIES OF ASSET PRICE PROCESSES

Abstract

The motivation for this work was to produce models that would satisfy the empirical regularities most often found in financial data, but starting from the data, rather than from any particular model. Our approach is to find the key empirical regularities present in the data, and then find and test the models that have properties that are compatible with those regularities. Most practitioners still use the Normal distribution to describe financial asset returns, or consider a conditional fat tailed distribution like Student's-*t* together with a GARCH-type volatility model enough to capture most of the empirical regularities of these returns. Our study shows that to develop a model that truly captures the empirical characteristics of financial asset returns, a symmetric fat tailed distribution is not enough, and we need jumps and a stochastic volatility type-model.

1. INTRODUCTION

The first studies about the distributional properties of financial asset returns were based on the Normal distribution (Osborne, 1959). This distribution is still being used in practice (e.g. Black-Scholes, Risk Metrics), despite the fact that ever since Fama and Mandelbrot (1963, 1965) published their, now famous, papers, more and more evidence has been gathered against it.

There is clear evidence that many short term financial return series are leptokurtic (Fama 1965, Blattberg and Gonedes 1974, Bookstaber and McDonald 1987, Eberlein and Keller 1995, etc.), meaning that their empirical distributions are more peaked around the mean and have fatter tails than a Normal distribution. This excess kurtosis remains 'surprisingly' high for longer horizons, failing to converge to the levels of the Normal distribution even after 20 log-returns have been aggregated (approximately one month returns). The presence of non-linear dependencies in the data of the type generated by (finite but) time-varying variances, as proposed by GARCH-type models, could explain this phenomenon.

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The unconditional distribution may not be Normal, but the conditional distribution may still be. It has been shown (see Bollerslev 1986) that a GARCH-type volatility combined with a conditional Normal distribution for daily returns would produce excess kurtosis in the unconditional distribution, but, in general, this excess kurtosis is not high enough, and conditional fat-tailed distributions still cannot account for the extreme outliers present in financial time series.

A second possible explanation for the excess kurtosis could be the presence of infinite higher moments. Our results suggest that the variance is finite but time-varying, inducing non-linear dependencies in the returns. Higher moments like the kurtosis, however, might not be finite or might also be time varying (a concept introduced by Hansen (1992), as 'conditional heterokurtosis' and 'conditional heteroskewness'), but the tests available at the moment are not conclusive enough for us to differentiate between an infinite or conditionally time varying higher moment

The paper has been organised as follows. Section 2 describes the data and sample sizes employed in the estimations. Section 3 analyses the characteristics exhibited by the unconditional distributions, as well as investigating the possible presence of 'day-of-the-week' effects in mean and variance. Conditional distributions and GARCH models are described and analysed in section 4. As mentioned before, the behaviour of the higher moments seems to suggest that they are either infinite or also conditionally time-varying. Section 5 explores this possibility and briefly describes the behaviour of the tails for longer horizons. Our main conclusions are summarised in section 6.

2. THE DATA

The analysis was done on four time series, the S & P 500 and FT-SE 100 daily indexes (divided in full and post crash samples) , and the Deutsche Mark-US Dollar and Yen-Dollar daily exchange rates. The source for the data was Datastream.

Returns were calculated in the usual way, as continuously compounded returns:

$$x_t = ln(S_t) - ln(S_{t-1})$$

where x_t is the continuously compounded return for day t and S_t is the corresponding price for day t.

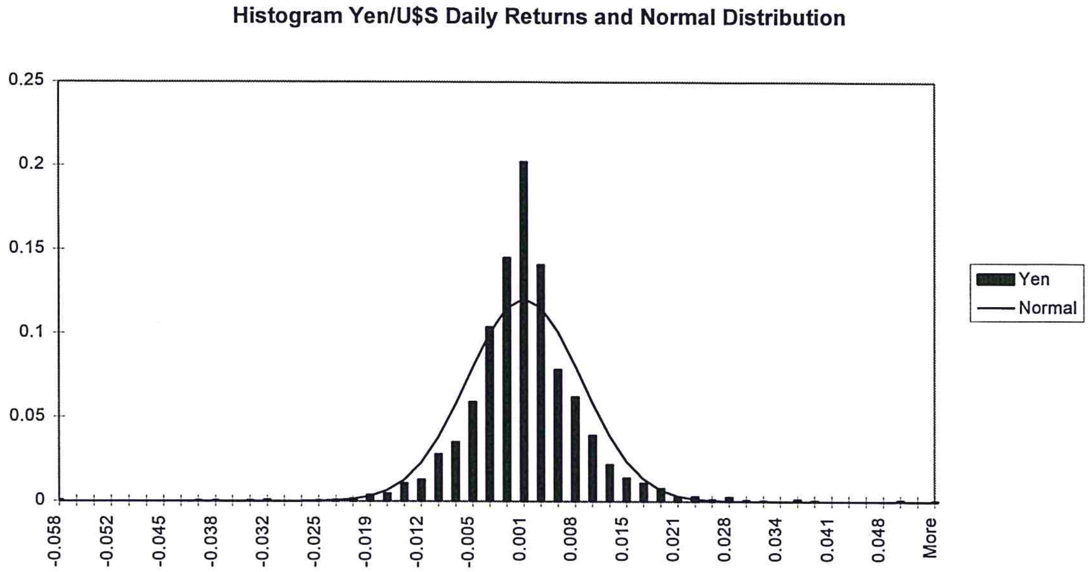
The data ranges from 01/01/85 to 13/02/96 for the indexes, and from 02/01/86 to 13/02/96 for the exchange rates. A sub-sample was also created for the daily indexes, starting in January 1988, to avoid the effects of the 1987 crash.

3. TESTING THE CLASSICAL ASSUMPTIONS ABOUT THE UNCONDITIONAL DISTRIBUTIONS

Bachelier (1900) and later Osborne (1959) developed a distributional theory for asset returns that was to be the basis of most of the work carried out in the following years. They start with the assumption that price changes from transaction to transaction in an individual security are independently and identically distributed (i.i.d.) random variables, transactions are uniformly spread across time, and the variance of approximate returns is finite. Then, the central limit theorem leads to the conclusion that daily, weekly and monthly returns are distributed normal, and their variances are proportional to the sampling interval.

Kendall (1953) questioned the applicability of this model, as it was noted that the data exhibited characteristics that were not compatible with the Gaussian model, essentially, there were too many values near the mean and out in the extreme tails of the distribution (leptokurtosis). To assess how far the unconditional distributions of our data deviated from the Normal, we plotted the frequency histograms and compared them with the relative frequency that would correspond to each bin under a

Normal distribution with mean and standard deviations equal to the estimated unconditional mean and standard deviation.



As can be seen from the figure, the data exhibit the 'non-Normal' characteristics usually found in financial time series, namely 'fat' tails (represented by non-negligible observations more than three standard deviations away from the mean) and peakedness around the mean. These features are still present in the index returns even after we exclude all data before and during the crash of 1987. It is interesting to note that, in general, the departures from 'normality' exhibited by the exchange rates are less marked than those exhibited by the indexes.

Table 3.1: Summary Statistics, Full Sample

STAT	S & P 500	FT-SE 100	YEN/U$S	DM/U$S
Mean	0.00047365	0.00038356	0.000235	0.0001888
Stddev	0.0097139	0.0092382	0.007269	0.007291
Kurtosis	115.78701	28.63376	9.8648	5.1482
Skewness	-4.83411	-1.66815	0.3433	-0.09393

Table 3.2: Summary Statistics, Post-Crash

STAT	S & P 500	FT-SE 100
Mean	0.00046448	0.00036988
Stddev	0.0077535	0.0080097
Kurtosis	10.34388	5.27822
Skewness	-0.64015	0.12190

Table 3.3: 5% Confidence Intervals

Series	Kurtosis	Skewness
S & P 500 Pre Crash	$2.8217 < k < 3.1783$	± 0.08915
FT-SE 100 Pre Crash	$2.8217 < k < 3.1783$	± 0.08915
YEN/U$S	$2.8131 < k < 3.1869$	± 0.09347
DM/U$S	$2.8131 < k < 3.1869$	± 0.09347
S & P 500 Post Crash	$2.7913 < k < 3.2087$	± 0.1044
FT-SE 100 Post Crash	$2.7913 < k < 3.2087$	± 0.1044

Another very useful graphic tool to assess how well a proposed theoretical distribution is supported by a set of data is assessing the linearity of the Quantile-Quantile plots (for a collection of theoretical Q-Q plots see Fowlkes, 1987).

The points in a Q-Q plot are derived as follows:

If z_i ($i = 1, 2, \ldots, n$), are n sample values of the z's arranged in ascending order, then a particular z_i is an estimate of the p_i fractile of the distribution of z, where the value of p_i is the empirical cumulative probability corresponding to z_i given by

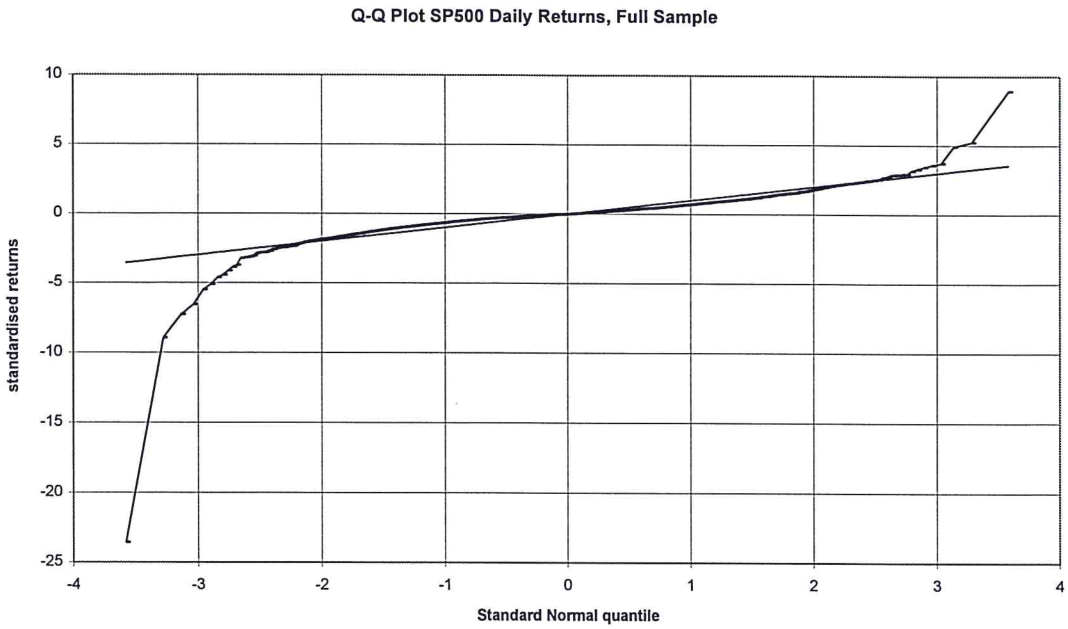
$$p_i = (i - 0.5)/n$$ [1]

Then we solve

$$p_i = \int_{-\infty,\ qi} f(t)dt$$

for q_i, where $f(t)$ is the density function of the hypothesised distribution. We plot every z_i against its corresponding q_i to compare every one of the standardised values of the returns from the sample (z_i) with the corresponding values under the hypothesised distribution (Standard Normal) that would accumulate the same probability p_i. Therefore, if the hypothesised distribution were a good distribution for returns, the plot should be linear (and, since we are working with standardised values, close to the 45 degree line).

[1] This particular convention for estimating p_i is only one possibility of many. Some authors have used (3i-1)/(3n-1) or simply i/n to estimate the same probability value. Nevertheless, given the sample sizes with which we are working, the actual convention adopted for estimating the p's wouldn't have any effect on our results.



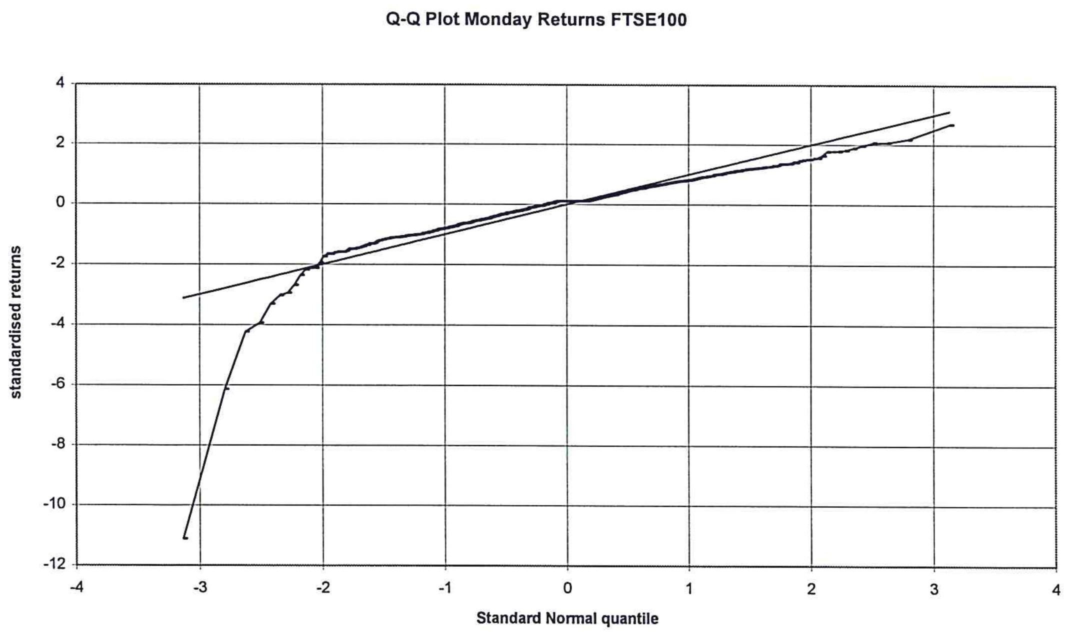
It should be noted that a Q-Q plot tends to emphasise the appearance of differences in the tails.

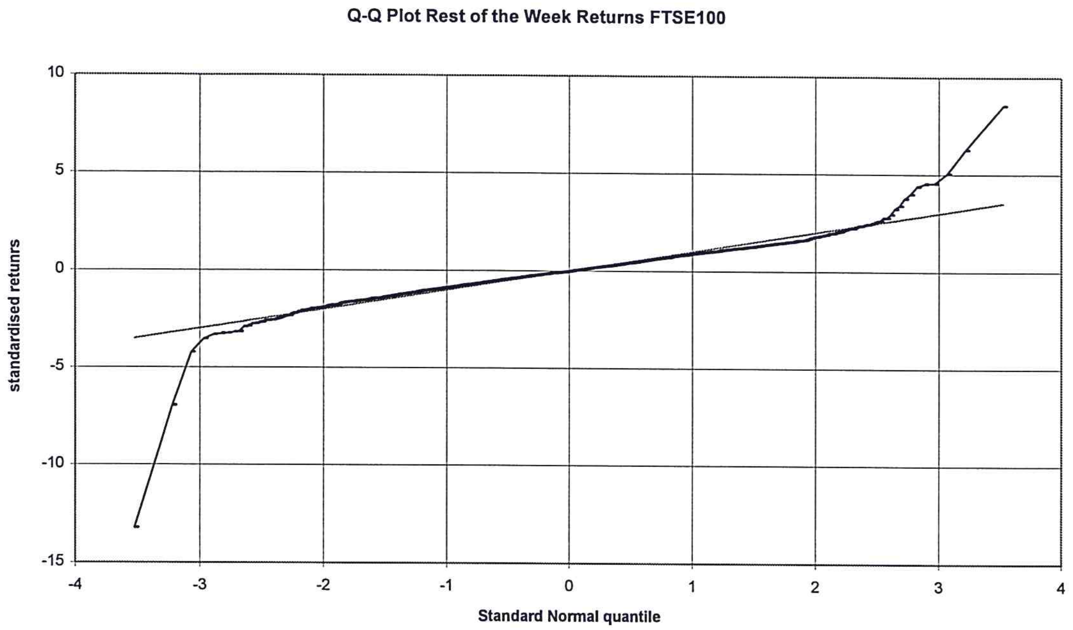
Again, we found the same deviations present in all data sets, the most obvious departure from linearity ('normality') is in the behaviour of the tails (with, in the case of the S & P 500 full sample returns, values even more than 23 standard deviations away from the mean). We should note that, even if we do not consider the period containing the crash (our post crash sample starts on 01/01/1988), we still find observations more than five standard deviations from the mean with relatively (to the Normal distribution) high frequencies. Therefore, these 'fat-tails' are not an effect of the crash of 1987, but a more permanent characteristic of the unconditional distributions.

These results are not affected by the way returns were standardised (by their general means and standard deviations). We did Q-Q plots for non-standardised data (comparing the returns with the numbers that would accumulate the same probability but under the assumption that the distribution were Normal with a mean and variance equal to the population mean and variance), and to further investigate the effect of standardisation on the plots and taking into account some evidence found in the literature of the presence of 'day of the week' effects in the mean and variance (see, for example, Mills 1995) we re-plotted the series but standardising them with the mean and variance corresponding to each day of the week. No major difference in the plots could be found.

The Kolmogorov-Smirnov (Lilliefors) non-parametric test also rejected the hypothesis that the Normal distribution is an appropriate model to describe the unconditional distribution of daily returns, (see Kolmogorov 1933, Lilliefors 1967 and Smirnov 1939).

An idea that would be interesting to consider is that if daily returns were distributed Normal, but with different means and variances for different days of the week, then daily returns considered as a whole would exhibit skewness and excess kurtosis like the ones found in our series. But (assuming there were no time non-stationarity), then Mondays and rest-of-the-week returns should plot linearly when considered separately. Nevertheless, Mondays as well as rest-of-the-week returns present non-linear plots, suggesting that they are not distributed Normal.





It is surprising that for both indexes the rest of the week returns plot similar to the full sample but Monday returns exhibit a thin right hand tail, shown by standardised returns lying bellow the 45 degree line. This result suggests that some further work would be desirable, we should perhaps look at other equity indexes and individual stocks.

Given that the presence of significant 'day-of-the-week' effects in either mean or variance would affect our analysis, we performed several non-parametric tests on the data to try and detect them.

Wilcoxon's (Wilcoxon 1945) test for differences in the mean of two samples (Mondays and rest-of-the-week) only rejects the null of equal means for the FT-SE 100 daily returns, but this rejection is marginal.

However, we are more concerned with day seasonality in the variance than in the mean. As a preliminary way to test whether different days have a significantly different variance, we computed the percentage proportion of one week's variance corresponding to each day of the week, as proposed by Taylor (1986).

Table 3.4: Percentage Proportions of One week's Variance

Day	Proportion			
	Yen/U$S	DM/U$S	S & P 500	FT-SE 100
Monday	0.1922	0.2143	0.2037	0.2085
Tuesday	0.2128	0.1996	0.2162	0.1912
Wednesday	0.1931	0.1781	0.1819	0.2190
Thursday	0.1987	0.1920	0.1924	0.1852
Friday	0.2033	0.2160	0.2058	0.1961

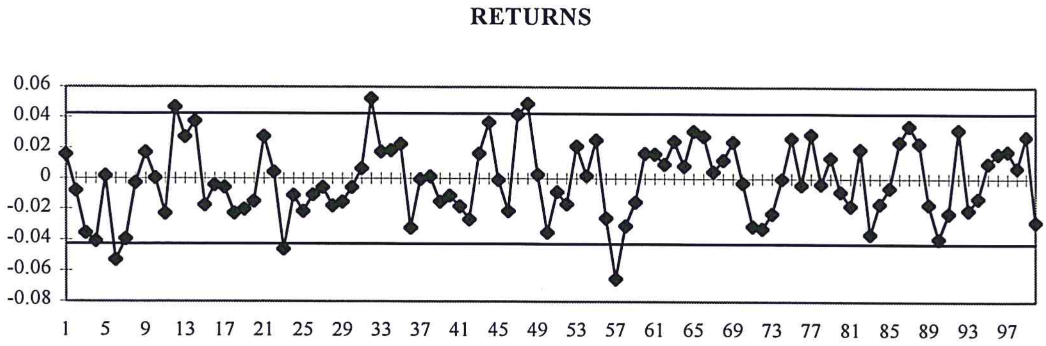
Even though, to our knowledge, there are no confidence intervals to estimate the significance of differences in these statistics, all numbers are close enough to suggest that Mondays do not make a greater contribution to each week's variance than that made by any other day of the week, suggesting that Mondays will not have a significantly different variance to the rest of the week.

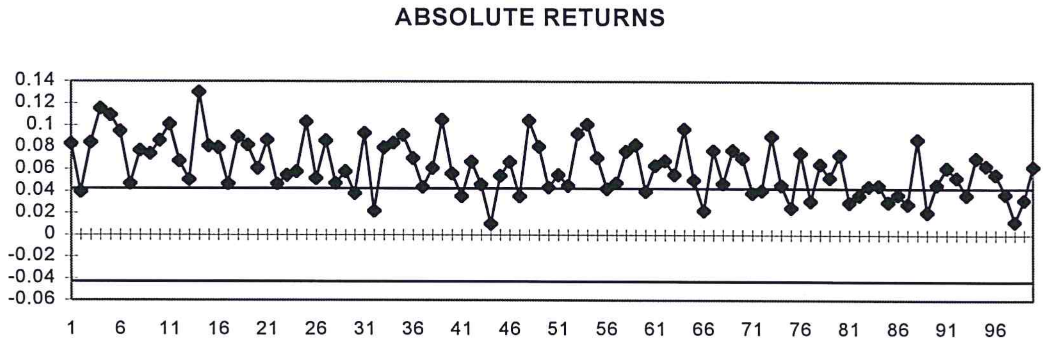
We performed the Siegel-Tukey test for differences in variance on Monday and rest-of-the-week returns considered as separate series (Siegel and Tukey 1960). In this case only the Yen/U$S exchange rate returns rejected the null of equal variance at a 95% confidence level. However, this test is particularly sensitive to other differences being present in the data (like differences in mean) so the results for the FT-SE 100 index returns should, somehow, be taken carefully.

We also performed the Wald-Wolfowitz test for general differences in distribution (Wald and Wolfowitz 1940) on the same data. All series rejected the hypothesis of equal distributions for Monday and rest-of-the-week returns considered as two separate samples.

It is generally accepted that even though not independent, returns are uncorrelated. We wanted to test the randomness of our sample data, and begun by plotting the standard autocorrelogram with the 5% confidence intervals calculated as $\pm 1.96/\sqrt{n}$, with n = sample size. We should note that strictly speaking, this is only a test for i.i.d. returns and not for autocorrelation, as a series can be dependent but still uncorrelated.

Autocorrelograms S & P 500 Day-Centred Post Crash Returns





The results presented by the graphs are clear. Autocorrelation in returns, if present, is weak, but returns are not independent.

The Runs test (Neave and Worthington 1988) confirmed this idea, all series except the Yen/U$S exchange rate returns have statistics that are smaller than the 5% critical value of 1.96, determining the acceptance of the uncorrelatedness hypothesis. In the case of the Yen/U$S returns, however, the statistic is slightly larger than 1.96 (1.99).

Table 3.5: Runs Test Statistics

Series	Statistic
S & P 500	0.5846
FT-SE 100	1.1493
Yen/U$S	1.9969*
DM/U$S	1.9190

* reject 5% level

Our general conclusion after considering all the evidence in favour and against autocorrelatedness in the series is that, there is some correlatedness present in our returns, but it is very small, and none of the series considered here possess the characteristics of an i.i.d. processes.

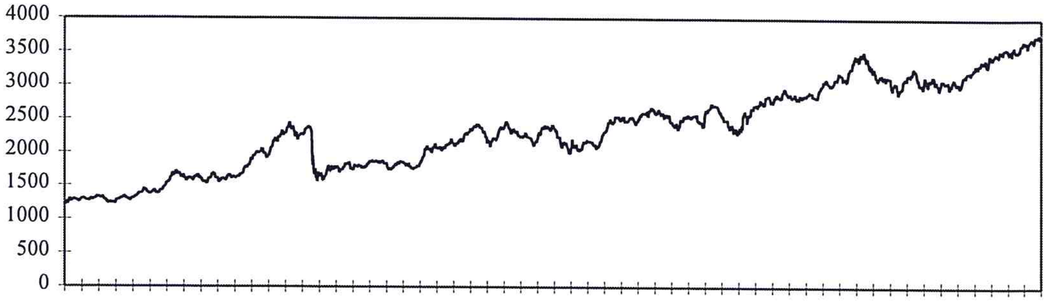
The underlying distributions are far from being Normal, specially with respect to the behaviour of the tails.

4. ANALYSIS OF CONDITIONAL DISTRIBUTIONS

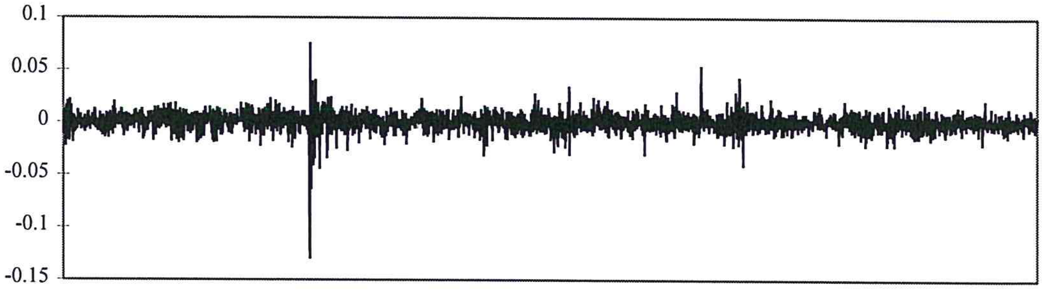
4.1 The Generalised Autorregressive Conditional Heteroscedasticity Hypothesis

In empirical studies of various time series it has often been noticed that large returns tend to be followed by large returns (of either sign), small returns seem to be followed by small returns (of either sign), and that the autocorrelations of square or absolute returns tend to remain significantly positive for very long lags, even if returns themselves seem to be uncorrelated, revealing a high degree of non-linear dependency (see for example, Granger et al 1993). As mentioned before, most financial time series exhibit significant skewness and excess kurtosis.

FT-SE 100 Index Full Sample

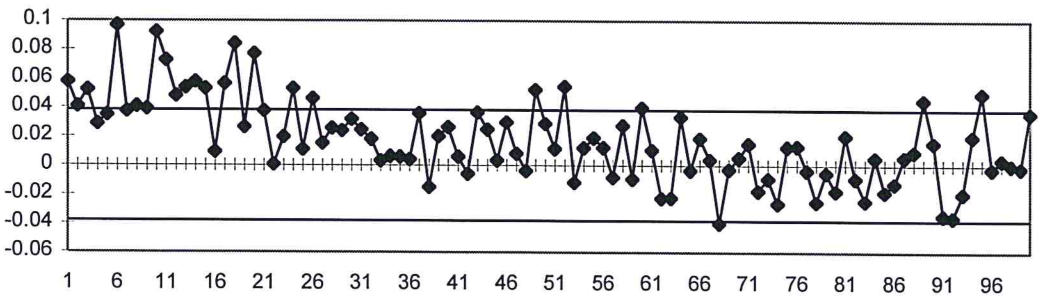


FT-SE 100 Index Daily Returns Full Sample

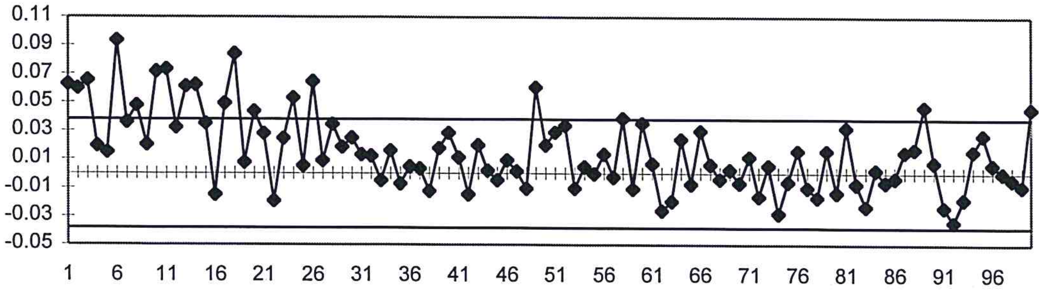


Autocorrelogram DM/U$S Exchange Rate Returns

ABSOLUTE RETURNS



SQUARED RETURNS



It has also been recognised that the uncertainty of speculative prices as measured by the variances and covariances changes over time. Recently we have seen great progress and interest in modelling ARCH effects. We will review key innovations in this literature before describing our own empirical work.

The ARCH (Autorregressive Conditional Heteroscedastic) process introduced by Engle (1982) allows the conditional variance to change over time as a function of past errors leaving the unconditional variance constant, and provides one of the first serious models which seems to capture these characteristics of the data.

Bollerslev (1986) generalised Engle's basic ARCH(q) model including past values of the conditional variance in the conditional variance equation.
GARCH(p,q) model:

$$y_t = x_t'\xi + \varepsilon_t$$

$$\varepsilon_t|\psi_{t-1} \sim F(0,h_t)$$

$$h_t = \alpha_0 + \Sigma_{i=1,q}\alpha_i\varepsilon^2_{t-i} + \Sigma_{i=1,p}\beta_i h_{t-i}$$

$$h_t = \alpha_0 + A(L)\varepsilon^2 + B(L)h_t$$

where $p \geq 0,\ q > 0,\ \alpha_0 > 0,\ \alpha_i \geq 0\ i = 1,\ldots,q,\ \beta_i \geq 0\ I = 1,\ldots,p.$

For $p = 0$ the process reduces to the ARCH(q) model, and for $p = q = 0$ it becomes a white noise. If all the roots of $[1\text{-}B(z)] = 0$ lie outside the unit circle, then the conditional variance equation can be re-written as a distributed lag of past ε's.

$$h_t = \alpha_0(1 - B(L))^{-1} + A(L)(1 - B(L))^{-1}\varepsilon_t^2$$

This is equivalent to an infinite order ARCH with parameters φ_i equal to the coefficient corresponding to L^i in the expansion of

$$\varphi_i = A(L)(1 - B(L))^{-1},\ i = 1,\ 2,\ 3\ldots.$$

$$h_t = \varphi_0 + \varphi(L)\varepsilon_t^2$$

Bollerslev (1986) also showed that the GARCH model has an ARMA representation for the square residuals. This ARMA representation gives a tool to determine the order of the GARCH model required. Note that for the autocorrelations of the square residuals to exist, ε_t must have a finite fourth moment and therefore a bound kurtosis.

In practice, it has been noticed that a GARCH(1,1) model is usually enough to capture the desired characteristics of the data.

The coefficient of excess kurtosis of a GARCH(1,1) process is

$$k = (E(\varepsilon^4) - 3E(\varepsilon^2)^2)E(\varepsilon^2)^{-2} = 6\alpha_1^{\,2}(1 - \beta_1^{\,2} - 2\alpha_1\beta_1 - 3\alpha_1^{\,2})$$

which is greater than zero by assumption, therefore, the GARCH(1,1) process is leptokurtic (given the condition on the fourth moment of ε_t). One of the interesting features of GARCH models is the fact that, even though the conditional variance changes over time, the unconditional variance remains constant, so, unconditionally, the GARCH process is homoscedastic.

Nelson and Cao (1992) show that the non-negativity restrictions necessary to ensure a positive conditional variance do not need to be applied to all coefficients, for example, a GARCH(1,2) model can have a negative α_2 provided all other coefficients are greater or equal to zero, and that the inequality constrains on the conditional variance equation do not need to be imposed *a priori*, as a violation would not necessarily mean that the conditional variance is misspecified.

The estimation of a GARCH regression model is usually done by maximum-likelihood. Since this is a recursive estimation, we need to specify starting values for h_t and ε_t. The algorithm most widely used for the estimation is the Berndt, Hall, Hall and Hausman (BHHH) algorithm:

$$\theta^{(i+1)} = \theta^i + \lambda_i[\Sigma_{t=1,T}\partial f_t/\partial\theta \times \partial f_t/\partial\theta']^{-1}\Sigma_{t=1,T}\partial f_t/\partial\theta$$

with θ^i = parameter estimates after *i-th* iteration.

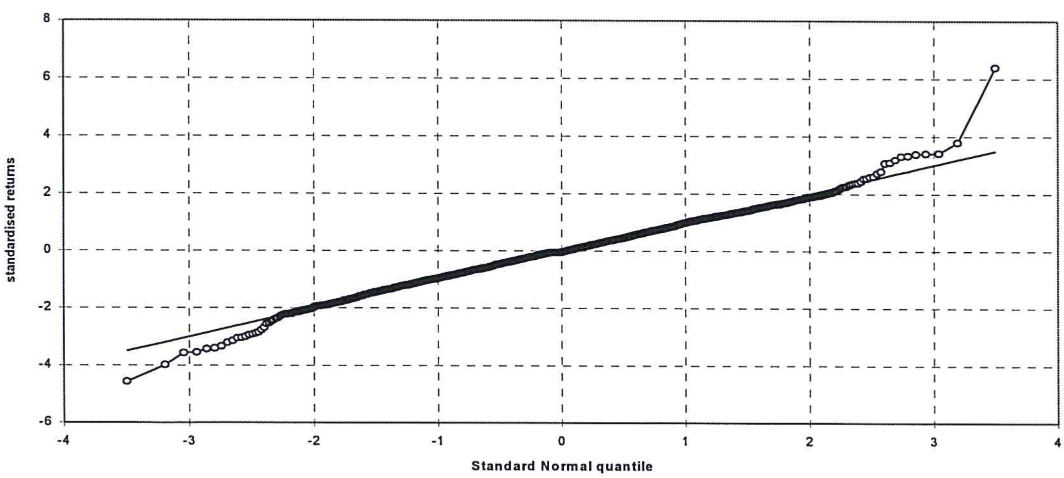
Despite being able to capture several of the main characteristics exhibited by economic and financial data, it has been recognised that basic GARCH models with conditional Normal distributions fail to account for some important characteristics also presented by these data, like, for instance, 'leverage effects'. Besides, it has also been noted that, though a GARCH model with a conditional Normal distribution is leptokurtic, this degree of excess kurtosis is not high enough, as standardised returns still exhibit 'fat' tails.

Estimations

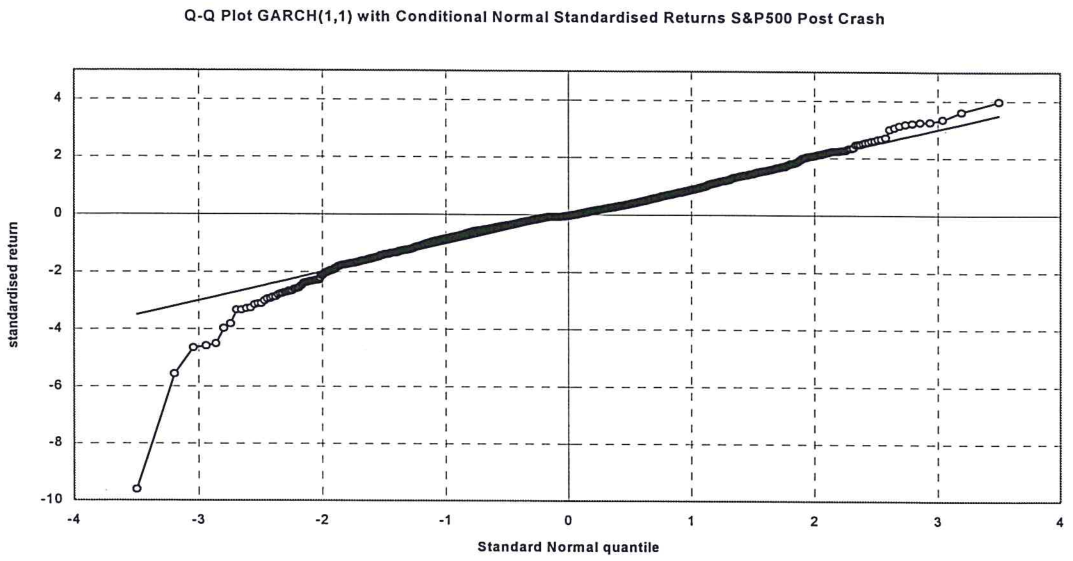
We fitted GARCH(1,1) to the four samples and two sub-samples, using a conditional Normal distribution. The empirical results confirmed the general characteristics often presented by these models, the sum of the parameters was very close to one , and both the AR and MA parameters of the conditional variance equation were significantly different from zero. Other orders were fitted, but the results did not change significantly[2].

Nevertheless, our results showed that the conditional Normal distribution is not appropriate to model standardised returns (see Q-Q plots below). Even if a GARCH(1,1) with a conditional Normal distribution does account for most of the non-linear dependencies exhibited by the data, it fails to provide enough flexibility in terms of the kurtosis. Returns standardised using a GARCH(1,1) with a conditional Normal distribution still exhibit significant fat tails, as shown by their deviations from the straight line. Even when not considering the sample including the crash, GARCH residuals for the indexes range from -5 to 6. These values suggest that a fat-tailed conditional distribution would be more appropriate to model standardised returns.

Q-Q Plot GARCH(1,1) with Conditional Normal Standardised Returns FTSE100 Post Crash Sample



[2] When necessary, the data was corrected for any dependencies in the conditional mean. Our estimations show that there seems to be a significant day seasonality in the unconditional mean equation when returns are taken separately (as oposed to belonging to two samples, Mondays and rest-of-the-week). This seasonality was taken account of in the GARCH estimations. Nevertheless, our results still reject the presence of day seasonality in the conditional variance equation.



A natural alternative candidate for the conditional distribution, proposed by Bollerslev (1986), is the Student *t* distribution, as, by changing its degrees of freedom parameter, one can obtain a wider range of possible kurtosis (as opposed to the constant 3 of the Normal), including the possibility of infinite kurtosis when the degrees of freedom are less or equal to four.

The estimated AR and MA parameters of the conditional variance equation exhibit the same characteristics as the Normal case, the sum is very close to one, and both are highly significant. The estimated degrees of freedom seem to suggest that, in most cases, the distributions are close to the limit between infinite and finite kurtosis. The correlation analysis of the GARCH residuals shows that once the non-linear dependencies created by the time-varying volatility are taken into account, the series tend to be much closer to i.i.d.

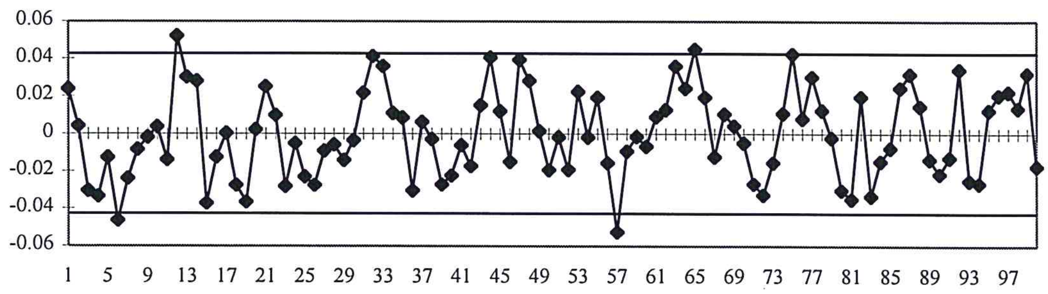
Table 4.1: Results GARCH(1,1) with Conditional Student's-*t* Distribution

Series	α_0	α_1	β_1	ν
S & P 500 full	7.824E-7 (2.11E-7)	0.03712 (0.0066)	0.95257 (0.0073)	4.26 (0.3473)
S & P 500 post	2.196E-7 (1.08E-7)	0.01972 (0.0052)	0.97642 (0.0055)	4.49 (0.4760)
FT-SE 100 full	3.278E-6 (8.53E-7)	0.06831 (0.0109)	0.88278 (0.0172)	10.77 (0.8782)
FT-SE 100 post	2.137E-6 (8.84E-7)	0.04780 (0.0119)	0.91729 (0.2318)	11.85 (2.1505)
DM/U$S	1.336E-6 (5.28E-7)	0.04170 (0.0094)	0.93657 (0.0154)	4.51 (0.5182)
YEN/U$S	2.401E-6 (7.62E-7)	0.06145 (0.0146)	0.90857 (0.0198)	3.23 (0.2577)

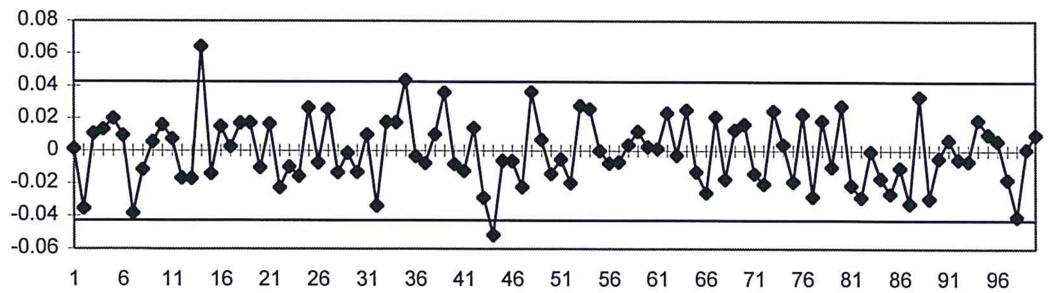
Standard Errors in brackets
ν = Degrees of Freedom

Autocorrelograms GARCH(1,1) with Conditional Student-*t* Standardised post Crash Returns

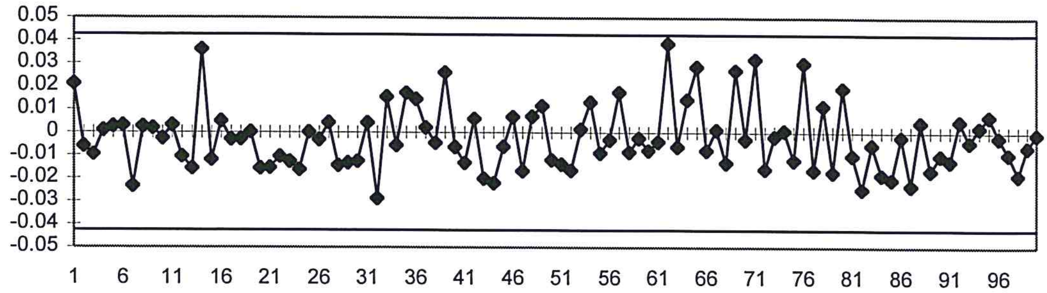
RETURNS



ABSOLUTE RETURNS

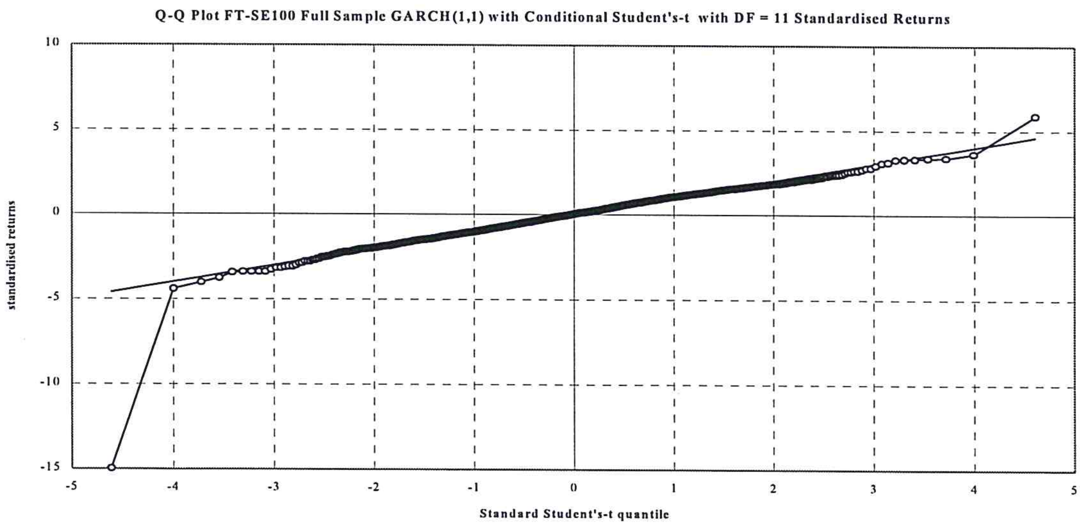


SQUARED RETURNS

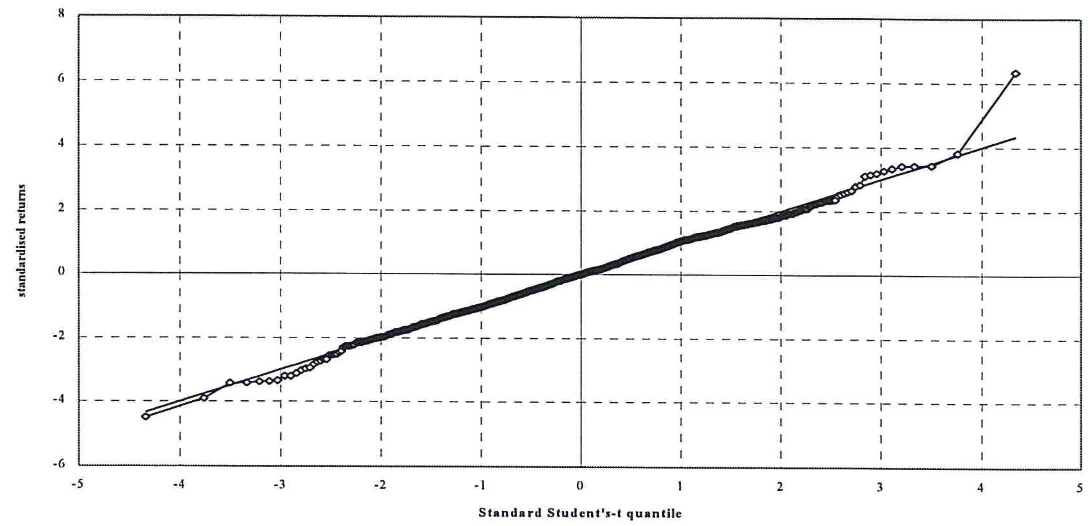


Note: Confidence Intervals calculated as before

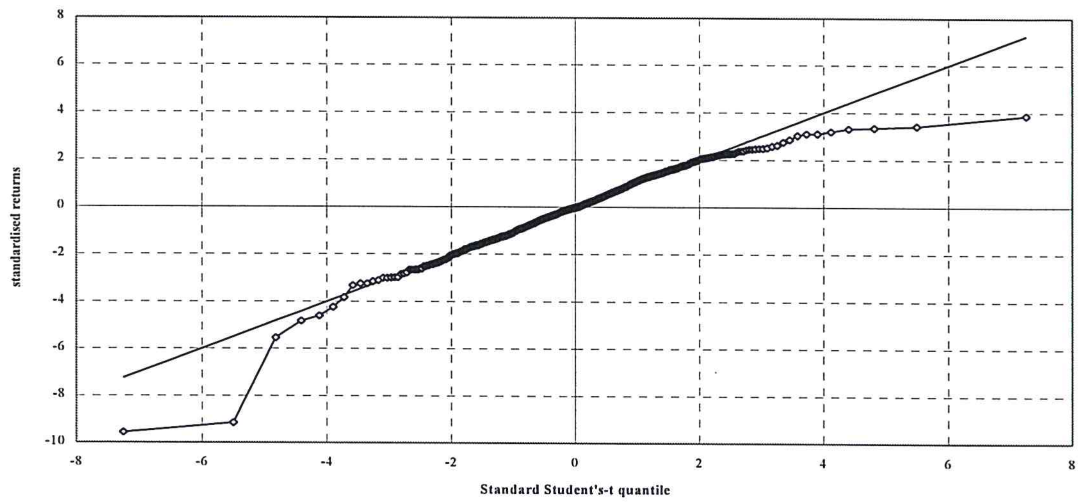
Q-Q Plots of the GARCH residuals with respect to the values of the corresponding Student-t's show that, even if the fit of this distribution is better than that of the Normal, specially in the tails, a distribution with maybe one more parameter and therefore more flexible might be necessary. What seems to be clear is that, except for these outliers, the Student t seems to capture the conditional excess kurtosis quite well. Nevertheless it still cannot account for the extreme outliers present in the data sets. The necessity of a more flexible conditional distribution applies specially to the S & P 500 index returns, as they still exhibits quite a skewed distribution after the standardisation.



Q-Q Plot FT-SE100 Post Crash GARCH(1,1) with Conditional Student's-t with DF = 12 Standardised Returns



Q-Q Plot SP500 Post Crash Garch(1,1) with Conditional Student's-t with DF = 4 Standardised Returns



It is possible that this result is due to the 'leverage effect' mentioned before. Basically, the presence of a 'leverage effect' means that the volatility is negatively correlated with past returns, and therefore the conditional variance equation should be asymmetric with respect to ε_t. Several alternative asymmetric relationships have been proposed to take account of this effect (e.g. QTARCH, PNP, GJR, EGARCH). Maybe the best known and more commonly used is the EGARCH formulation of Nelson (1991):

$$y_t = x_t'\xi + \varepsilon_t$$

$$\varepsilon_t|\psi_{t-1} \sim F(0,h_t)$$

$$\eta_t \sim F(0,1)$$

$$\varepsilon_t = \eta_t \times \sqrt{h_t}$$

$$log(h_t) = \alpha_0 + \sum_{i=1,p}\beta_i log(h_{t-i}) + \sum_{i=1,q}\alpha_i g(\eta_{t-i})$$

$$g(\eta_t) = \gamma[|\eta_t| - E(|\eta_t|)] - \theta\eta_t$$

where $F(0,h_t)$ is such that $\varepsilon_t/\sqrt{h_t} \sim F(0,1)$

$g(\eta_t)$ is the innovation to the conditional variance. The term $[|\eta_t| - E(|\eta_t|)]$ gives the ARCH effect, as (assuming $\theta = 0$) a large shock ($|\eta_t| > E(|\eta_t|)$) will increase the conditional variance. The parameter θ measures the presence of the 'leverage effect', assuming $|\eta_t|$ to be equal to $E(|\eta_t|)$ or $\gamma = 0$, then a positive θ means that when η_t is negative the innovation to the conditional variance will be positive.

Nelson (1991) and Taylor(1994) estimated this model not only for a conditional Normal distribution, but also for Student's *t* distribution and the Generalised Error Distribution (GED) of Box and Tiao (1962, 1973). It should be noted that all conditional distributions considered in the literature are symmetric, all have time varying second and maybe first conditional moments but time invariant and finite higher conditional moments (like skewness and kurtosis) except for the special case of a Student *t* distribution with degrees of freedom equal or less than 4, for which the kurtosis is not defined.

Other specifications account for more specific characteristics of different data sets employed, and usually are computationally more difficult. But, as Bera and Higgins (1993) put it "whether any inadequacies in the EGARCH functional form for volatility justifies the additional computational effort of estimating a more flexible model may depend on the peculiarities of the individual data set and the ultimate purpose of the empirical analysis".

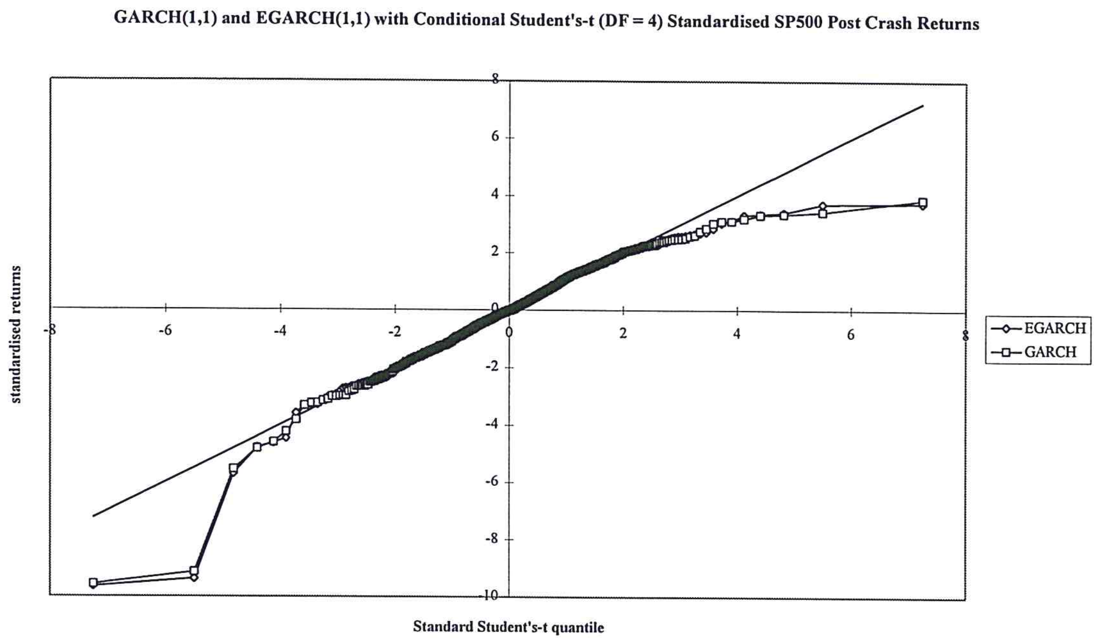
Table 4.2: Results, EGARCH(1,1) with Conditional Student-*t* Distribution

Series	α_0	α_1	β_1	θ	ν
S & P 500 full	-0.09258 (2.761E-2)	0.08825 (1.433E-2)	0.98976 (2.882E-3)	0.34240 (1.385E-1)	4.3030 (0.3526)
S & P 500 post	-0.05952 (2.799E-2)	0.06569 (1.492E-2)	0.99342 (2.885E-3)	0.45177 2.031E-1)	4.4791 (0.4508)
FT-SE 100 full	-0.30289 (7.957E-2)	0.15456 (1.669E-2)	0.96093 (8.469E-3)	0.19193 (8.069E-2)	10.0541 (0.8495)
FT-SE 100 post	-0.16081 (7.888E-2)	0.08165 (2.137E-2)	0.97910 (8.687E-3)	0.35795 (1.372E-1)	10.7340 (1.7985)
DM/U$S	-0.19905 (8.090E-2)	0.09603 (1.957E-2)	0.97882 (8.311E-3)	-0.20705* (1.124E-1)	4.50395 (0.5173)
YEN/U$S	-0.39049 (1.290E-1)	0.14389 (2.829E-2)	0.96012 (1.322E-2)	-0.12554* (1.085E-1)	3.10042 (0.2466)

Parameterization:
$h_t = exp(\alpha_0 + \beta_1 log(h_{t-1}) + \alpha_1 g_{t-1})$
$g_t = |\eta_t| - E(|\eta_t|) - \theta\eta_t$
$\eta_t = \varepsilon_t / \sqrt{h_t}$
$\varepsilon_t \sim t_\nu(0,h_t)$Standard Errors in brackets
ν = Degrees of Freedom
* = Non-Significant at 5% Level

These results suggest the presence of a significant 'leverage effect' in the conditional variance of both indexes for the two samples considered. All θ's have the 'expected' sign, with negative (positive) shocks inducing positive (negative) innovations in the conditional variance. On the other hand, both currencies exhibit non-significant θ's, and therefore their conditional variances do not seem to exhibit a significant 'leverage effect'. All other parameters in the model are highly significant and have values that are consistent with other empirical work done in the area. In particular, β_1 is very close to one, implying that shocks to the variance are persistent and die out slowly.

To see whether accounting for the 'leverage effect' would reduce (or eliminate) the skewness exhibited by some of the conditional distributions (in particular the S & P 500 post crash index) we standardised the returns by the square root of the EGARCH(1,1) estimated conditional variance and re-plotted the Q-Q plot with a Student-*t* distribution with degrees of freedom equal to the estimated ν's as the reference.



As can be seen from the plot, the presence of a significant 'leverage effect' cannot account for the skewness present in the conditional distribution of the S & P 500 index.

Again, and except for the FT-SE 100 index returns, all conditional *t* distributions seem to be somehow in the limit between having a finite or infinite fourth moment.

5. Convergence Tests for Higher Moments

As a way of testing if conditional higher moments are finite, we could test whether the estimated ν's (degrees of freedom) are significantly equal or below particular levels (three for the skewness, four for the kurtosis). However, since we are not confident that the Student-*t* distribution is the right specification (for instance, the DM/U$S exchange returns seem to have a symmetric conditional distribution with tails that are somewhere in between those of the Normal distribution and those of the Student-*t* distribution with the estimated degrees of freedom), we prefer tests that are based on much weaker assumptions, like Embrecht's (1997) test for finite or infinite moments, that is based on an i.i.d. assumption only. We performed this test on the moments of the conditional distributions of EGARCH-*t* residuals.

5.1 Embrecht's Test for Finite Moments

If

$$E|x|^p < \infty$$

then

$$max_{i=1,\dots,n} |x_i|^p / \Sigma_{i=1,n} |x_i|^p$$

should tend to zero almost surely as n (the estimating sample size) increases.

Applying this concept to the kurtosis means that if:

$$E(x-x_{mean})^4 < \infty,$$

then:

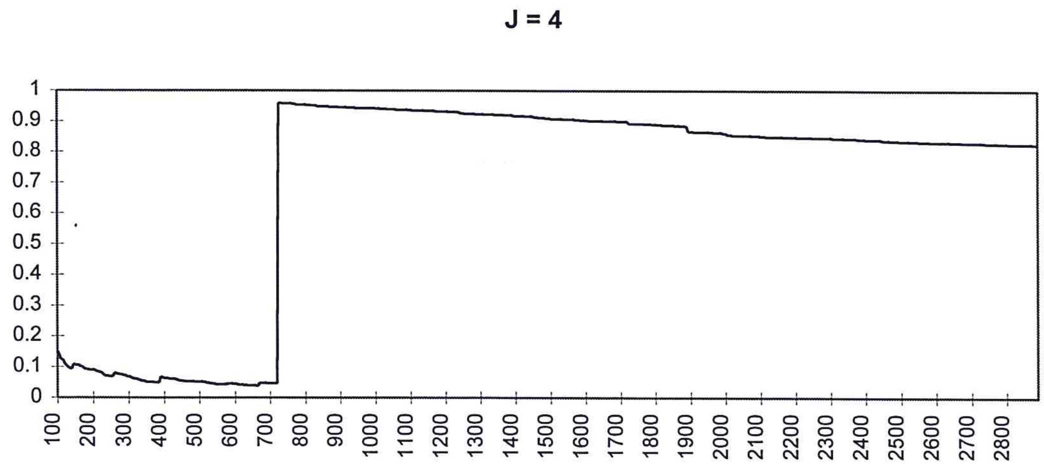
$$max_{i=1,\dots,n} |x_i-x_{mean}|^4 / \Sigma_{i=1,n} |x_i-x_{mean}|^4$$

should tend to zero almost surely for increasing n's, as

$|x|^4 = (x)^4$.

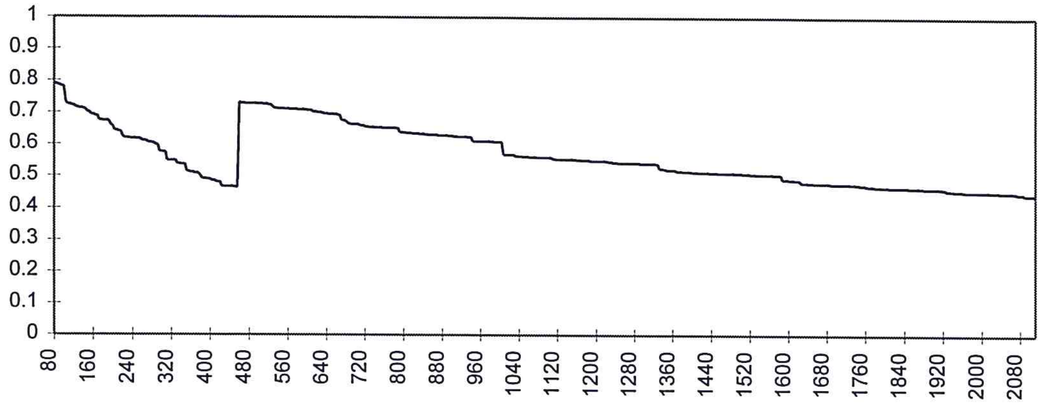
Simulations of this test done for several theoretical distributions (Stable with $\alpha = 2.5$, Student t with degrees of freedom ranging from 3 to 6, Standard Normal, etc.) show that, when the underlying distribution is in the limit of finite or infinite kurtosis, the convergence or non convergence is not that clear for data sets of the size we are considering. Convergence is very clear in the Standard Normal case as is non-convergence for the Stable with infinite variance, but the Student t's with degrees of freedom from 3 to 5 presented cases which didn't seem to converge even though the theoretical kurtosis was bounded and vice versa.

Embreght's Test on FT-SE 100 Full Sample Returns



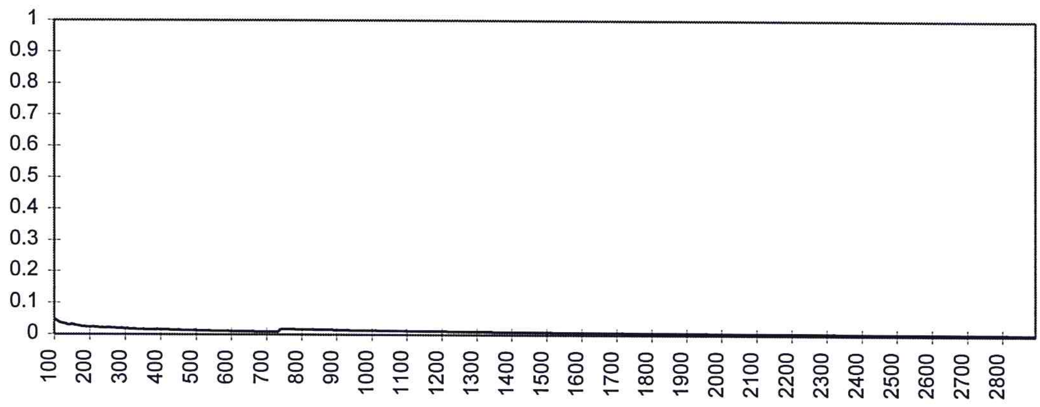
Embrecht's Test on S&P 500 Post Crash Returns

J = 4

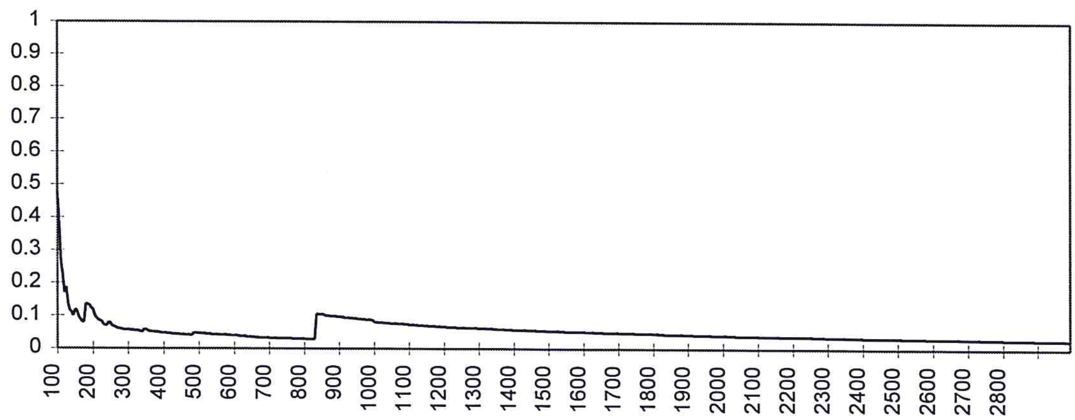


Simulation of Embrecht's Test for Student's-t Distribution with 4 Degrees of Freedom

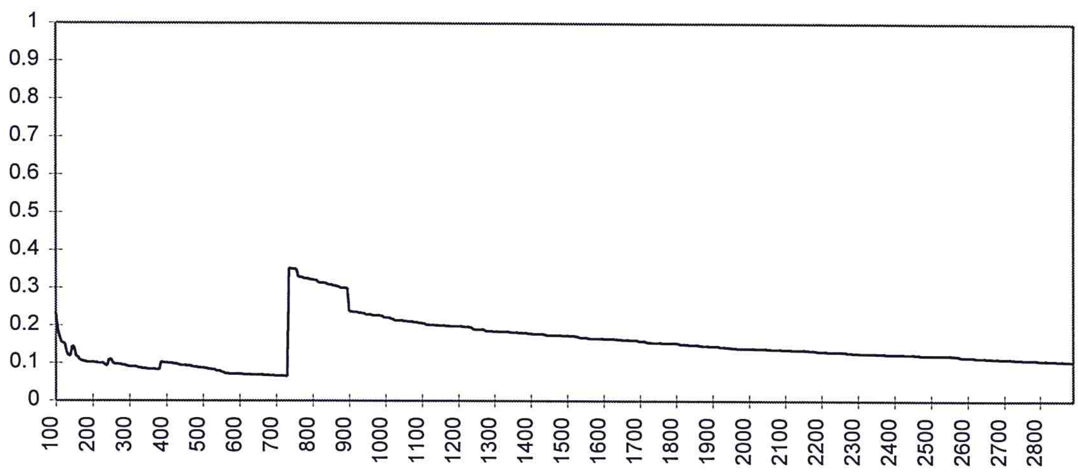
J = 1



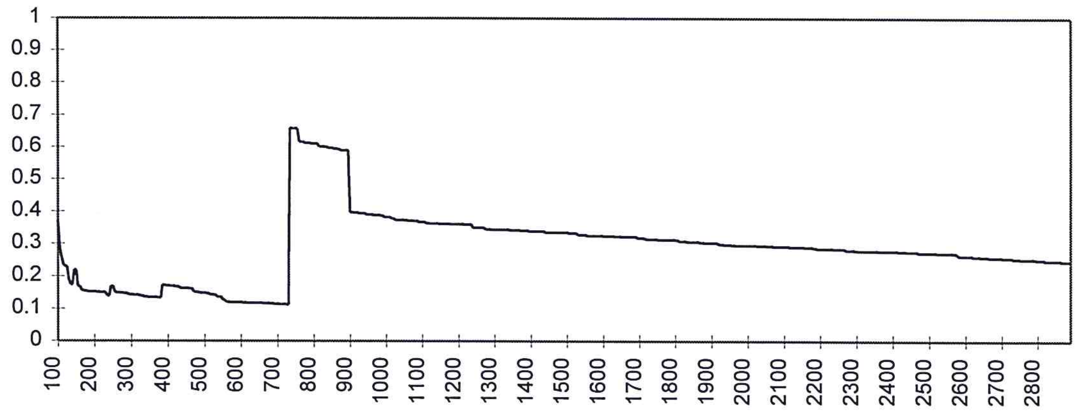
J = 2



J = 3



J = 4



Note: $statistic = max_{i=1,\dots,n} |x_i|^j / \Sigma_{i=1,\,n} |x_i|^j$

The same conclusion is derived from the Convergence Tests performed on the GARCH residuals. These tests are based on Granger and Orr's 1972 paper. The tests were originally applied to assess the validity of the Stable Paretian hypothesis by testing whether the variance of the return distribution was finite or infinite. The method is based on the estimation of the sample variance for increasing sample sizes as:

$$s^2_n = \Sigma_{t=1,\,n} (x - x_{mean})^2/(n-1)$$

Assuming all x's (returns) come from the same distribution, the estimated sample variance (s^2_n) should converge to a finite value if the population's unconditional distribution has a finite variance. Nevertheless, they admit that non-convergence does

not necessarily imply the existence of an infinite variance. As pointed out by Granger and Orr (1972), non-convergence may be caused by a non-stationary series, with a population variance changing over time, and convergence would be slowed, even for a stationary series, by the presence of autocorrelation between the observations (these objections are dealt with by applying the test to the conditional moments).
Perry (1983) uses a test for infinite variances based on the same principle, that, as the sample size increases, the estimated sample variance should converge to a finite number if the population variance is itself finite. Taking into account the two possibilities considered above, that the variance might be finite but conditionally changing over time or that the presence of autocorrelation might slow the convergence (this possibility must be taken into account, given the sample sizes we are considering might not give a long enough series to overcome the slow convergence) Perry proposes three orders of the data, one chronological, one backwards through time, and a randomised sample.
The first two might give an insight into how the 'finite' variance is changing over time, as non-convergence when the sample variance is increased could happen because of a population variance that is finite but steadily increasing over time. The randomised sample is used to overcome the problems of autocorrelation and non-stationarity.

When applied to the non-standardised data, this tests suggested that the variance was finite for all series, but time-varying. Therefore, we concentrated our analysis on the conditional higher moments.

Simulations of the test present the same result as with Embrecht's test, convergence is clear for a Normal distribution or Student's *t* with five degrees of freedom, as is non-convergence in the case of a Symmetric Stable distribution with alpha = 1.5. But not so when we analyse the case of a distribution like the Student-*t* with, for example, four degrees of freedom. In this case, the sample size we are able to work with seems to be insuficient to determine non convergence or convergence.

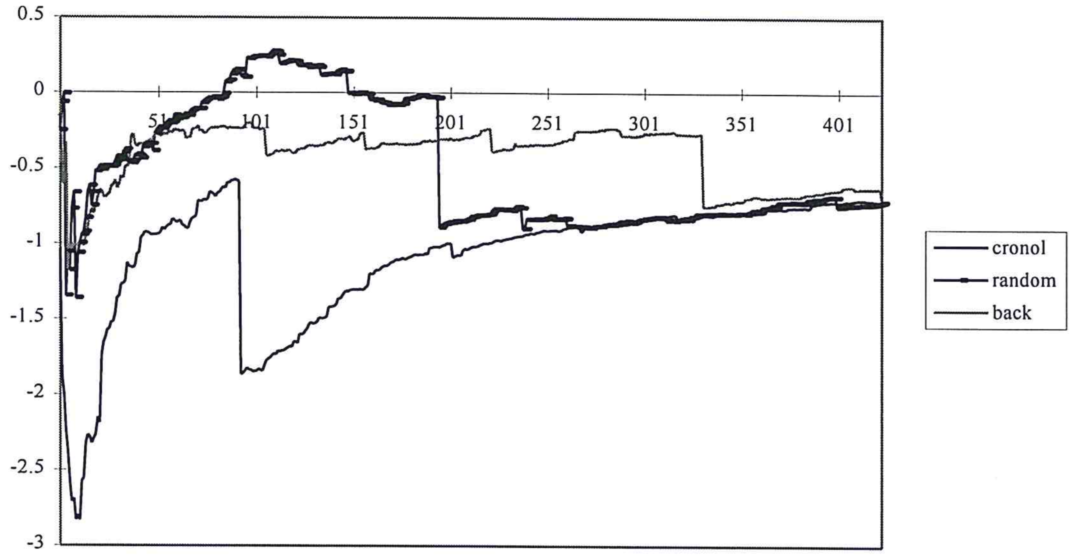
As to the behaviour of the tails for longer horizons, results for the standardised data suggest the DM achieves levels of skewness and kurtosis compatible with a Normal distribution quite rapidly, but convergence is a lot slower for the equity indexes and the YEN, however, the statistics seemed to be converging, even if slowly. Unfortunately, the sample sizes we are working with do not provide enough independent returns for longer horizons to assess this convergence properly.

Perry's test indicates that convergence of the skewness and kurtosis is clear for the DM/U$S exchange rate standardised returns. In the case of the index returns, the crash of 1987 affects the conditional distribution to the point of making its higher moments look infinite. Results for post crash samples are not so clear, maybe due to the limited number of observations available, but do not provide sufficient proof against the possibility of infinite kurtosis. The tests on the S & P 500 residuals confirms the presence of significant conditional skewness mentioned before, even after the period containing the crash has been dropped out of the estimating sample.

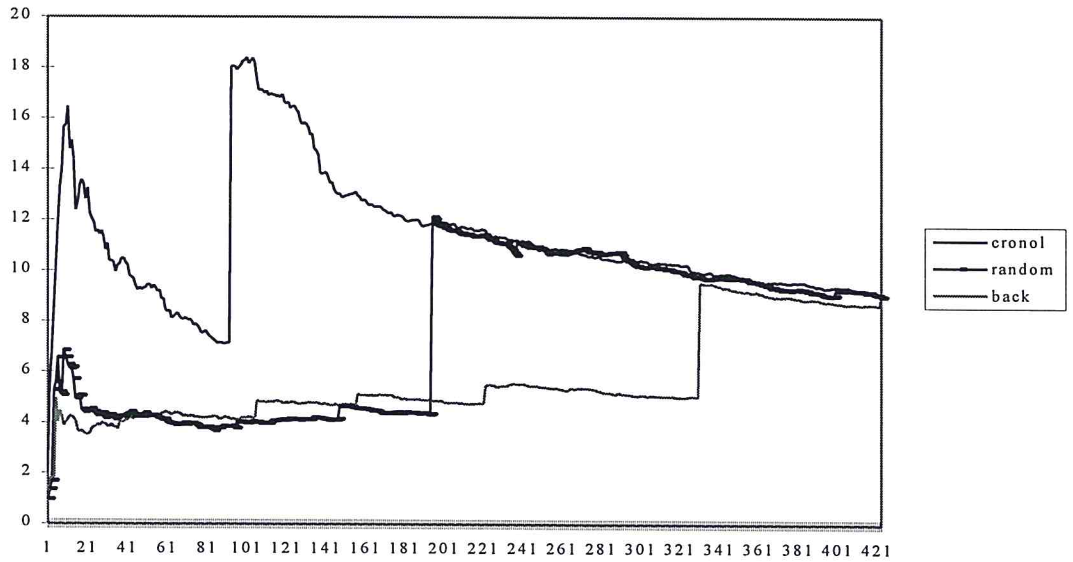
The behaviour of the chronological and backward sample skewness and kurtosis seems to suggest that the conditional third and fourth moments might be changing over time. Neither a significantly negative left hand tail, nor a conditionally time varying third moment can be modelled with the distributions most commonly used in practice.

Skewness and Kurtosis for Increasing Sample Sizes, S & P 500 Post Crash Standardised (GARCH(1,1) with Conditional *t*) Index Returns

SKEWNESS



KURTOSIS



6. MAIN CONCLUSIONS

All tests carried on the unconditional distributions support the idea that these exhibit 'fat tailedness'. This characteristic seems to be present even after aggregating 20 log-day returns (non-overlapping), suggesting Normality is not yet achieved for monthly returns, except in the case of the DM/U$S exchange rate returns, for which convergence to 'Normal' levels seems to be attained faster than for all other series.

Another regularity that seems to be present in all data sets is that, contrary to the Stable hypothesis, variances are finite, but conditionally time-varying.
This would induce non-linear dependencies in the returns series. Even though they do not seem to be strongly correlated, they have characteristics that are not compatible with the i.i.d hypothesis. A somewhat 'surprising' result is that, contrary to what we expected, there doesn't seem to be strong evidence of day-seasonality in the variance.

Summary of Tests

Hypothesis Tested	Test	Conclusion
Normality	Kolmogorov-Smirnov & Q-Q Plots	Returns are non-Normal and exhibit leptokurtosis
Day-seasonality in Mean	Wilcoxon	Mondays and rest-of-the-week returns have, in general, the same mean
Day-seasonality in Variance	Siegel-Tukey, Taylor's Percentage of one week's variance	Only Yen/U$S returns rejected the null: no seasonality. No strong evidence of day-of-the-week effects in variance
Mondays and rest-of-the-week returns have same distribution	Wald-Wolfowitz	Monday returns seem to come from a different distribution than the rest of the week
Randomness	Autocorrelograms, Runs Test	Returns are not strongly correlated but are not independent, presence of non-linear dependencies
Conditional Heteroscesasticity	GARCH with Normal	Normal distribution is not a good candidate, need conditional 'fat'-tailedness. Significant day-seasonality in unconditional mean.
Conditional Heteroscesasticity with 'fat' tailed distribution	GARCH with *t*	Better fit in the tails though not able to account for 'outliers'. Presence of conditional asymmetry
Leverage effects	EGARCH with *t*	Significant leverage effects in indexes but not in currencies
Infinite or time-varying higher moments	Embreght, Granger and Orr, and Perry	Higher moments and especially kurtosis may be infinite, also evidence of possible time-varying skewness

Although returns may be uncorrelated, they are not independent, and variances seem to be time-varying. This could be explained by a GARCH-type specification. Conditional variances do change over time, with shocks that seem to persist over long horizons.

The analysis of returns standardised with conditional standard deviations obtained with a basic GARCH model with conditional Normal distribution suggests that the Normal is not a good candidate, and that conditional distributions exhibit fat tails. In particular, the S & P daily index returns might even exhibit significant skewness. Therefore, a more flexible family of distributions or more general specification seems necessary. Skewness and kurtosis might also be time-varying and the kurtosis might even be infinite. An interesting possibility suggested by our results is that the seemingly infinite conditional kurtosis might actually be finite but time-varying as suggested by Hansen(1992). The estimation of an Autorregressive Conditional Density would allow for higher moments to be also modelled as a function of the variables included in the information set. A general conclusion about the data confirmed throughout the analysis is that equity index returns exhibit much stronger irregularities than those of the exchange rates.

Even using a model with a conditional distribution that provides 'fatter' tails than allowed by the Normal distribution (and in some cases the Student-*t* seems to be too 'fat', as for example with the DM/U$S returns while in others it is not 'fat' enough), we wouldn't be able to account for the extreme outliers exhibited by some data series, specially the indexes. This leads us to the key conclusion of this study, that we can only represent price processes adequately with a combination of stochastic volatility and conditional fat tails (or jumps). Thus, neither the conventional type of stochastic volatility model (e.g. Hull and White 1987) nor an i.i.d. jump diffusion process (e.g. Merton 1976) are sufficient. What we would need is a model that combines the properties of both.

Given that different types of series seem to exhibit different types of behaviour, we should perhaps expand the analysis to futures data, other equity indices and exchange rates, etc. A more general GARCH specification with a candidate conditional distribution that gives more flexibility seems to be necessary, especially, it would be interesting to investigate distributions with a shape parameter controlling the skewness and or kurtosis a more flexible candidate distribution would allow us to model a time-varying skewness/kurtosis, and the effect this has on the behaviour of the tails.

REFERENCES

Bachelier, L. J. B. A., *Théorie de la Speculation*, Paris: Gauthier-Villars, 1900. Reprinted in: *The Random Character of Stock Markets*, Paul H. Cootner (ed.). Cambridge (Mass): M.I.T. Press, 1964, pp 17-78.

Bera, A. K., and H. L. Higgins, 'ARCH Models: Properties, Estimation and Testing', *Journal of Economic Surveys*, 1993, vol 7(4), pp 305-366.

Blattberg, R. C., and N. J.Gonedes, 'A comparison of the Stable and Student Distributions as Statistical Models for Stock Prices', *Journal of Business,* 1974, vol 47, pp 244-280.

Bollerslev, T., 'Generalised Autoregressive Conditional Heteroskedasticity', *Journal of Econometrics,* 1986, vol 31, pp 307-327.

Bollerslev, T., R. F. Engle, and D. B. Nelson, 'ARCH Models', *Handbook of Econometrics*, Vol. 4, pp 2961-2984, 1994, Engle and McFadden Eds., Elsevier, Amsterdam.

Bookstaber, R. M., and J. B. McDonald, 'A general Distribution for Describing Security Price Returns', *Journal of Business,* 1987, vol 60(3), pp 401-424.

Box, G. E. P., and G. C. Tiao, 'A Further Look at Robustness via Bayes theorem', *Econometrika,* 1962, 49(3&4), pp 419-432.

Bayesian Inference in Statistical Analysis, Reading: Addison-Wesley, 1973.

Campbell, J. Y., A. W. Lo, and A. C. MacKinlay, *The Econometrics of Financial Time Series*, Chichester: Princeton University Press, 1997.

Ding, Z., C. W. J. Granger and R. F. Engle, 'A Long Memory Property of Stock Market Returns and a New Model' *Journal of Empirical Finance,* 1993, vol 1, pp 83-106.

Eberlein, E., and U. Keller, 'Hyperbolic Distributions in Finance', *Universitat Freiburg Discussion Paper,* 1995.

Embrechts, P. C. Kluppelberg and T. Mikosch, *Modelling Extreme Events,* Springer-Verlag, London, 1997.

Engle, R., 'Autoregressive Conditional Heteroskesasticity with Estimates of Variance of the United Kingdom', *Econometrica*, 1982, Vol. 50, pp 987-1007.

Fama, E. F., 'The Behaviour of Stock Market Prices', *Journal of Business,* 1965, vol 38, pp 34-105.

Fama, E. F., and R. Roll, 'Some Properties of Symmetric Stable Distributions', *Journal of the American Statistical Association*, 1968, vol 63, pp 817-836.

Fowlkes, E. B., *A Folio of Distributions*, New York: Marcel Dekker Inc., 1987.

Granger, C., and D. Orr, 'Infinite Variance and Research Strategy in Time Series Analysis', *Journal of the American Statistical Association',* 1972, vol 67, pp 275-285.

Hagerman, R. L., 'More Evidence on the Distributions of Security Returns', *Journal of Finance*, 1978, vol 33(4), pp 1213-1221.

Hansen, B. E., 'Autoregressive Conditional Density Estimation', *Rochester Center for Economic Research Working Paper No 332*, 1992.

Harvey, A., 'Long Memory in Stochastic Volatility', 1993, *LSE Discussion Paper,* Dept. of Statistics.

Hull, J., and A. White, 'The Pricing of Options on Assets with Stochastic Volatility', *Journal of Finance*, 1987, vol 42(2), pp 281-300.

Johnson, N. L., and S. Kotz, *Continuous Univariate Distributions, Vol 1 and 2.* New York: John Wiley & Sons, 1970.

Kendall, M. G., 'The Analysis of Economic Time Series', *Journal of the Royal Statistical Society* (Ser. A), 1953, vol 96, pp 11-25.

Kolmogorov, A. N., 'Sulla Determinazione Empirica di una Legge di Distribuzione', *Giornale Ist. Ital. Attuari*, 1933, vol 4, pp 83-91.

Kon, S. J., 'Models of Stock Returns - A Comparison', *Journal of Finance*, 1984, vol 39(1), pp 147-165.

Lilliefors, W. H., 'On the Kolmogorov-Smirnov Test for Normality with Mean and Variance Unknown', *Journal of the American Statistical Association*, 1967, vol 62, pp 399-402.

Mandelbrot, B., 'The variation of Certain Speculative Prices', *Journal of Business,* 1963, vol 36, pp 394-419.

Merton, R. C., 'Option Pricing when Underlying Stock Returns are Discontinuous', *Journal of Financial Economics*, 1976, vol 3, pp 125-144.

Mills, T. C., 'Modelling Skewness and Kurtosis in the London Stock Exchange FT-SE Index Return Distributions', *The Statistician*, 1995, vol 44(3), pp 323-332.

Mills, T. C., and J. A. Coutts, 'Calendar Effects in the London Stock Exchange FT-SE Indices', *European Journal of Finance*, 1995, vol 1, pp 79-93.

Mitchnik, S., and S. Rachev, 'Modelling Asset Returns with Alternative Stable Distributions', *Econometric Reviews,* 1993, vol 12(3), pp 261-330.

Neave, H. R., and P. L. Worthington, *Distribution-Free Tests*, London: Unwin Hyman, 1988.

Nelson, D. B., 'Stationarity and Persistence in the GARCH(1,1) Model', *Econometric Theory*, 1990, vol 6, pp 318-334.

'Conditional Heteroskedasticity in Asset Returns: A New Approach', *Econometrica*, 1991, Vol. 59(2), pp 347-370.

Nelson, D. B., and C. Q. Cao, 'Inequality Constraints in the Univariate GARCH Model', *Journal of Business and Economic Statistics*, 1992, vol 10, pp 229-235.

Officer, R., 'The Distribution of Stock Returns', *Journal of the American Statistical Association*, 1972, vol 67, pp 807-812.

Osborne, M. F. M., 'Brownian Motions in the Stock Market', *Operations Research*, 1959, vol 7, pp 145-173.

Perry, 'More evidence on the Nature of the Distribution Security Returns', *Journal of Financial and Quantitative Analysis,* 1983, Vol. 18, pp 211-221.

Praetz, P. D., 'The Distribution of Share Price Changes', *Journal of Business*, 1972, vol 45, pp 49-55.

Siegel, S., and J. W. Tukey, 'A Non-Parametric Sum of Ranks Procedure for Relative Spread in Unpaired Samples', *Journal of the American Statistical Association*, 1960, vol 55, pp 429-445. Errata, *Ibid*, 1961, vol 56, pp 1005.

Smirnov, N. V., 'Estimate of Deviation Between Empirical Distribution Functions in Two Independent Samples', *Bulletin Moscow University*, vol 2, pp 3-16.

Taylor, S. J., *Modelling Financial Time Series*. Chichester: John Wiley & Sons, 1986. Reprinted, 1992.

'Modeling Stochastic Volatility: a Review and Comparative Study', *Mathematical Finance*, 1994, vol 4(2), pp 183-204.

Wald, A., and J. Wolfowitz, 'On a Test Whether Two Samples are from the Same Population', *Annals of Mathematical Statistics*, 1940, vol 11, pp 147-162.

Wilcoxon, F., 'Individual Comparisons by Ranking Methods', *Biometrics*, 1945, vol 1, pp 80-83.