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Abstract

We study a generalization of the Arc-Sine Law. In particular we provide new results about the distribution of the time spent by a BM with drift inside a band, giving the Laplace transform of the characteristic function. If one of the extremes of the band goes to infinity, our formula agrees with the results given in Akahori (1995) and Takàcs (1996). We apply these results to the pricing of exotic option contracts known as corridor derivatives. We then discuss the inversion problem comparing different numerical methods.

1 Introduction

In this paper we obtain new results on a generalization of the Lévy arc-sine law. Lévy studied the density of the time spent by a Standard Brownian Motion (SBM) below a given level. We will provide results about the case of the Brownian Motion with drift below a given level and inside a given band. This problem has been solved for the case of Brownian Motion with drift below a given level by Akahori [3], Dassios [8], Takacs [23], Embrechts et al.[10]. In this paper we derive the same expression as in [23] and simpler than that given in [3] and [8]. So the main results are related to the case of the time spent by the Brownian motion (without drift and with drift) inside a band.

The results have also financial applications to the pricing of corridor and hurdle options as we illustrate in section 2. Other applications are, as suggested in Taleb [25] (page. 66-67), to the management of a portfolio for the computation of the expected amount of time a trader is expected to spend in the red. A similar problem for the Standard Brownian Excursion can be found in chemistry as well and in particular in the theory of ring polymers as studied in Jansons [15].

In next section we will illustrate the problem of the corridor derivative pricing. In order to price such a contract we need the knowledge of the distribution function of the occupation time. In section 3 we will give the expression for the Laplace transform of the characteristic function of the occupation time of the Brownian Motion of the interval $(-\infty; l]$ and $[l; u]$. In section 4, we discuss different inversion techniques (univariate and multidimensional) and we provide numerical examples with a comparison with the MonteCarlo simulation method.

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2 The Corridor Derivative

Corridor derivatives are exotic options paying at expiry an amount that depends on the time spent by a reference index, usually an exchange rate or an interest rate, below a given level or inside a band. The structure of the payoff is common to FX range floaters and boost structures as described in Hull [14], Pechtl [20], Tucker et al.[26] and Turnbull[27]. This kind of product is suitable for investors believing in stable markets, because with the actual low interest rate level they allow a higher performance than investing in bonds or directly in stocks.

In order to price the contract, we apply the well known result that in a arbitrage-free market, according to the well known Harrison-Kreps theorem of asset pricing [13], the price of any contract is just the expected value, under the risk-neutral measure, of the discounted payoff of the contract. So our aim will be to find the distribution function of the occupation time.

Let us suppose that the price of the underlying asset is described by a stochastic differential equation:

$$\begin{aligned} dP_t &= rP_t dt + \sigma P_t dW_t, \\ P_0 &= p \end{aligned}$$

where r is the instantaneous risk free interest rate, so the dynamics of the asset price under the martingale measure is described by a Geometric Brownian Process and $\ln P(t)$ has normal distribution with mean $\ln p + \left(r - \sigma^2/2\right) t$ and variance $\sigma^2 t$.

If we define the random variable:

$$\tau(t, p; L, U) = \int_0^t \mathbf{1}_{(L<P(s)<U)} ds$$

then, a corridor option (hurdle option if $L = 0$) at maturity has payoff given by $\max[\tau - K; 0] := (\tau - K)^+$ and the price at time 0 of the contract having a residual life equal to t and a strike $K < t$, is given by:

$$\begin{aligned} & e^{-rt}\mathbf{E}_{0,p}\left(\max\left[\tau(t, p; L, U) - K; 0\right]\right) = \\ & e^{-rt}\left(\int_0^t (s - K)^+ f_\tau(s, t, p; L, U)\, ds + (t - K)^+ \Pr_{0,p}\left[\tau(t, p; L, U) = t\right]\right) \end{aligned} \tag{1}$$

where $f_\tau(s, t, p; L, U)$ is the density function of the r.v. $\tau(t, p; L, U)$ calculated at s, when $0< s < t$. Note that in order to calculate the price we need to take into account the fact that the index can always stay inside the band or below the level U (when $L = 0$). This explains the presence of the term $(t - K)^+ \Pr\left[\tau(t, p; L, U) = t\right]$. In the following we write $f_\tau(s, t, p)$ to mean $f_\tau(s, t, p; L, U)$. We observe that:

$$\begin{aligned} \mathbf{1}_{(L<P(s)<U)} &= \mathbf{1}_{\left(L<p\exp\left(\left(r-\frac{\sigma^2}{2}\right)s+\sigma W(s)\right)<U\right)} \\ &= \mathbf{1}_{\left(\frac{1}{\sigma}\ln\left(\frac{L}{p}\right)<\frac{1}{\sigma}\left(r-\frac{\sigma^2}{2}\right)s+W(s)<\frac{1}{\sigma}\ln\left(\frac{U}{p}\right)\right)} \end{aligned}$$

So we can calculate the density function of the r.v. $\tau(t, p; L, U)$ using the density of the occupation time of the SBM $W(t)$, if $\left(r - \frac{\sigma^2}{2}\right)/\sigma = 0$, and of the BM with drift $W^{(m)}(t) := mt + W(t)$, where $m = \left(r - \sigma^2/2\right)/\sigma \neq 0$. In both cases the barriers are fixed at the levels

$l = (\ln L)/\sigma$ and $u = (\ln U)/\sigma$ and with starting value $x = (\ln p)/\sigma$. We study this problem in the following sections.

It is natural to see that we have the same problem to solve if the barriers increase exponentially at rate δ : $a(t) = Le^{\delta t}$ and $b(t) = Ue^{\delta t}$. Indeed:

$$\mathbf{1}_{\left(Le^{\delta t} < P(t) < Ue^{\delta t}\right)} = \mathbf{1}_{\left(l < \left(r - \delta - \frac{\sigma^2}{2}\right)\frac{1}{\sigma}t + W(t) < u\right)}$$

so the problem is equivalent to consider the occupation time for a Brownian Motion with an adjusted drift equal to $\left(r - \delta - \frac{\sigma^2}{2}\right)/\sigma$ and fixed barriers as above.

We observe that if the strike price is set to zero, we have a corridor bond (hurdle bond if $L = 0$) and the price can be obtained discounting the following expression:

$$\mathbf{E}_{0,x}\left(\tau\left(t, p; L, U\right)\right) = \mathbf{E}_{0,x}\left[\int_0^t \mathbf{1}_{(l < ms + W(s) < u)} ds\right] \tag{2}$$

and applying the Fubini's theorem, we obtain:

$$\begin{aligned} &= \int_0^t \mathbf{E}_{0,x}\left(\mathbf{1}_{(l<ms+W(s)<u)}\right) ds \\ &= \int_0^t \mathrm{Pr}_{0,x}\left(l < ms + W(s) < u\right) ds \\ &= \int_0^t \left(\Phi\left(h\left(x, u, s\right)\right) - \Phi\left(h\left(x, l, s\right)\right)\right) ds \end{aligned}$$

where $\Phi(x) = \int_{-\infty}^x \left(\exp\left(-\frac{w^2}{2}\right)/\sqrt{2\pi}\right) dw$ is the cumulative normal distribution and $h(x, l, t) = \frac{1}{\sqrt{t}}\left(l - x - mt\right)$.

3 The Characteristic Function of the Occupation Time of the interval $[l; u]$

In order to price corridor derivatives, we are interested in the evaluation of the distribution of the r.v.:

$$\begin{aligned} \tau(t, x) \quad &: \quad = \tau(t, x; u, l, m) = \int_0^t \mathbf{1}_{(l < W(s) + ms < u)} ds \\ W(0) \quad &= \quad x \end{aligned}$$

representing the amount of time spent inside the interval $[l; u]$ up to time t by a Brownian Motion with drift m and starting at x. Special cases are obtained when the drift is zero or when the lower barrier goes to -∞.

If we define the characteristic function of the r.v. $\tau(t, x)$:

$$\begin{aligned} &v(t, x) := v(t, x; u, l, m) = \mathbf{E}_{0,x}\left[e^{i\mu\tau(t,x)}\right] \\ &= \int_0^t e^{i\mu\tau} f_\tau\left(\tau, t, x; u, l, m\right) d\tau + 1 \times \mathrm{Pr}_{0,x}\left[\tau\left(t, x; u, l\right) = 0\right] + e^{i\mu t} \times \mathrm{Pr}_{0,x}\left[\tau\left(t, x; u, l\right) = t\right] \end{aligned} \tag{3}$$

using the Feynman-Kac formula, [16] pag. 366, it can be shown that $v(t, x)$ satisfies the following partial differential equation (PDE):

$$-\frac{\partial v(t, x)}{\partial t} + \frac{1}{2}\frac{\partial^2 v(t, x)}{\partial^2 x} + m\frac{\partial v(t, x)}{\partial x} + i\mu\mathbf{1}_{(l<x<u)} v(t, x) = 0 \tag{4}$$

with initial condition:

$$v(0,x) = 1,\ \forall x \in (-\infty;+\infty) \tag{5}$$

and boundary conditions:

$$v(t,\pm\infty) = 1,\ \forall t > 0 \tag{6}$$

Given a function of time t, we denote with $\mathcal{L}[\cdot;t \to \gamma]$ its Laplace transform with respect to the variable time t, and with $\mathcal{L}^{-1}[\cdot;\gamma \to t]$ the inverse Laplace transform. We have the following result:

Theorem 1 *: The characteristic function of the r.v. $\tau(t,x;u,l,m)$ admits the following representation:*

$$v(t,x;l,u,m) = \Omega(t,\mu,x;l,u,m) + \\ + \begin{cases} 1 \times \Pr_{0,x\in[u,+\infty)}\left(\inf\limits_{0\le s\le t} ms + W(s) > u\right) \\ e^{i\mu t} \times \Pr_{0,x\in(u,l)}\left(\sup\limits_{0\le s\le t} ms + W(s) < u;\ \inf\limits_{0\le s\le t} ms + W(s) > l\right) \\ 1 \times \Pr_{0,x\in(-\infty,l]}\left(\sup\limits_{0\le s\le t} ms + W(s) < l\right) \end{cases} \tag{7}$$

where:

$$\Omega(t,\mu,x;l,u,m) := \int_0^t e^{i\mu\tau} f_\tau(\tau,t,x;u,l,m)\,d\tau = e^{-mx-\frac{m^2}{2}t}\mathcal{L}^{-1}[\omega(\gamma,\mu,x;l,u,m);\gamma \to t]$$

and:

$$\omega(\gamma,\mu,x;l,u,m) = \\ = \begin{cases} \mathbf{1}_{(x\ge u)} e^{-\sqrt{2}(x-u)\sqrt{\gamma}}\mathcal{L}[y(t,1);\gamma] \\ \mathbf{1}_{(l<x<u)} \frac{\mathcal{L}[y(t,0);\gamma]\sinh\left(a\pi\left(\frac{u-x}{u-l}\right)\right)+\mathcal{L}[y(t,1);\gamma]\sinh\left(a\pi\left(\frac{x-l}{u-l}\right)\right)}{\sinh(a\pi)} \\ \mathbf{1}_{(x\le l)} e^{-\sqrt{2}(l-x)\sqrt{\gamma}}\mathcal{L}[y(t,0);\gamma] \end{cases}$$

and:

$$\mathcal{L}[y(t,1);t\to\gamma] = \frac{e^{mu}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{m}{\sqrt{2}}\right)} - \frac{c}{2\sqrt{\gamma}}(s(\gamma)+d(\gamma))$$

$$\mathcal{L}[y(t,0);t\to\gamma] = \frac{e^{ml}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{m}{\sqrt{2}}\right)} + \frac{c}{2\sqrt{\gamma}}(s(\gamma)-d(\gamma))$$

with:

$$
\begin{aligned}
\frac{c}{\sqrt{\gamma}}d(\gamma) = {} & \frac{\sqrt{\gamma-i\mu}\sinh(a\pi)}{\left(\sqrt{\gamma-i\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)+1)\right)}\left(\frac{e^{mu}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{m}{\sqrt{2}}\right)}+\frac{e^{ml}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{m}{\sqrt{2}}\right)}+\right. \\
& \left.+\frac{\left(-\frac{m}{\sqrt{2}}\left(e^{ml}-e^{mu}\right)(\cosh(a\pi)+1)-\sqrt{\gamma-i\mu}\left(e^{mu}+e^{ml}\right)\sinh(a\pi)\right)}{\sqrt{\gamma-i\mu}\sinh(a\pi)\left(\gamma-i\mu-\frac{m^2}{2}\right)}\right) \\
\frac{c}{\sqrt{\gamma}}s(\gamma) = {} & \frac{\sqrt{\gamma-i\mu}\sinh(a\pi)}{\sqrt{\gamma-i\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)-1)}\left(\frac{e^{mu}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{m}{\sqrt{2}}\right)}-\frac{e^{ml}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{m}{\sqrt{2}}\right)}+\right. \\
& \left.-\frac{\left(-\frac{m}{\sqrt{2}}\left(e^{mu}+e^{ml}\right)(\cosh(a\pi)-1)+\sqrt{\gamma-i\mu}\left(e^{mu}-e^{ml}\right)\sinh(a\pi)\right)}{\sqrt{\gamma-i\mu}\sinh(a\pi)\left(\gamma-i\mu-\frac{m^2}{2}\right)}\right)
\end{aligned}
\tag{8}
$$

$$
a\pi=\sqrt{\frac{\gamma-i\mu}{c^2}};\quad \alpha=-m;\quad \beta=-\frac{m^2}{2}\quad c^2=\frac{1}{2(u-l)^2} \tag{9}
$$

Moreover we can express the density function $f(\tau,t,x;u,l,m)$ of the occupation time for a generic starting point x and for $0<\tau<t$ in terms of the density function of the occupation time when $x=u$ and $x=l$ in the following way:

$$
\begin{aligned}
& f_\tau(\tau,t,x;u,l,m)= \\
& =\begin{cases}
1_{(x\geq u)}e^{-m(x-u)}\displaystyle\int_\tau^t \frac{x-u}{\sqrt{2\pi(t-\eta)^3}}e^{-\frac{(x-u)^2}{2(t-\eta)}-\frac{m^2}{2}(t-\eta)}f_\tau(\tau,\eta,u)\,d\eta \\
1_{(l<x<u)}2\pi c^2\displaystyle\sum_{n=1}^{\infty}n\sin\left(n\pi\left(\frac{x-l}{u-l}\right)\right)\int_0^\tau e^{-\left(\frac{m^2}{2}+\lambda_n\right)\xi}\times \\
\times\left(e^{-m(x-l)}f_\tau(\tau-\xi,t-\xi,l)-(-1)^n e^{m(u-x)}f_\tau(\tau-\xi,t-\xi,u)\right)d\xi \\
1_{(x\leq l)}e^{m(l-x)}\displaystyle\int_\tau^t \frac{(l-x)}{\sqrt{2\pi(t-\eta)^3}}e^{-\frac{(l-x)^2}{2(t-\eta)}-\frac{m^2}{2}(t-\eta)}f_\tau(\tau,\eta,l)\,d\eta
\end{cases}
\end{aligned}
\tag{10}
$$

3.1 Remarks

In the appendix, we solve the PDE and prove the theorem. We can make now some remarks.

1) A natural way of solving the PDE (4) could be to take the Laplace transform respect to t and then obtain three second order differential equations. The continuity and differentiability of the solution at the barriers and its boundedness at $\pm\infty$, require then the determination of four constants, generalizing the example in Karatzas and Shreve [16] pag. 273. However, using this approach we can incur in two problems.

a) The final expression of the solution will be the Laplace transform of the characteristic function $v(t,x)$ and then it will include the Laplace transform of the mass of probabilities concentrated at $\tau=0$ and $\tau=t$. This fact as explained in Abate and Whitt [1] can create problems in the numerical inversion and it is advisable to remove the atoms of probability before the inversion. Attacking directly the PDE and using the Laplace transform only in a successive step, we avoid this problem. Indeed we are able to identify in the expression of the c.f. these probabilities and so we can give the Laplace transform of the function $\Omega(t,\mu,x;l,u,m)$ and not directly of the c.f..

b) If we want to use two univariate numerical inversions for limiting the programming effort and to use well-tested numerical inversion routines, as discussed in the next section, we should provide the Laplace transform of the real part and of the complex part of the function $\Omega(t,\mu,x;l,u,m)$. Using our approach, this consists in solving (24) separating the real and the complex part of the functions $D(t)$ and $S(t)$ and then, taking the Laplace transform, we obtain two linear systems of just two equations[1]. Instead if we take directly the Laplace transform of (4), we should solve three systems of two differential equations each and then the continuity and differentiability and boundedness of the solution will require the determination of eight constants in a linear system with eight equation.

2) The characteristic function is continuous and differentiable at $x=l$ and $x=u$, because as we will show later the PDE (4) has been solved requiring continuity and differentiability of the solution at these points. This property will be transmitted to the price of the corridor option.

3) Comparing expression (3) and (7), we obtain the natural results:

$$\Pr_{0,x}\left[\tau(t,x;u,l)=0\right]=\begin{cases}\mathbf{1}_{(x>u)}\Pr_{0,x}\left(\inf\limits_{0\le s\le t} ms+W(s)>u\right)\\ 0;\ l\le x\le u\\ \mathbf{1}_{(x<l)}\Pr_{0,x}\left(\sup\limits_{0\le s\le t} ms+W(s)<l\right)\end{cases}$$

and:

$$\Pr_{0,x}\left[\tau(t,x;u,l)=t\right]=\begin{cases}\mathbf{1}_{(l<x<u)}\Pr_{0,x}\left(\sup\limits_{0\le s\le t} ms+W(s)<u;\inf\limits_{0\le s\le t} ms+W(s)>l\right)\\ 0;\ x\le l \text{ or } u\le x\end{cases}$$

The expressions for these quantities can be found in equations (33), (34) and (39) in the Appendix and can be compared with the same expressions in Borodin and Salminen (BS) [4].

4) We can show that the above expressions allow us to recover known results. In particular, now we discuss the following cases: a) $m=0$, i.e. the case of the occupation time of the SBM of the interval $[l,u]$, b) $m=0$ and $l=-\infty$, i.e. the time spent below the level u by the SBM and we obtain the Lévy arc-sine law, c) $l=-\infty$, i.e. the time spent by a BM with drift below the upper barrier u, case studied by Akahori [3], Dassios [8] and Takacs [23].

a) If $m=0$, we are considering the occupation time of the SBM of the interval $[l;u]$. The expression for the functions $d(\gamma)$ and $s(\gamma)$, $\mathcal{L}[y(t,0);t\to\gamma]$ and $\mathcal{L}[y(t,1);t\to\gamma]$ simplify to:

$$\begin{aligned}&d(\gamma)=\tfrac{1}{c}\frac{2\sqrt{\gamma}\sqrt{\gamma-i\mu}\sinh(a\pi)}{\left(\sqrt{\gamma-i\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)+1)\right)}\left(\frac{1}{\gamma}-\frac{1}{\gamma-i\mu}\right)\\ &s(\gamma)=0\\ &\mathcal{L}[y(t,0);t\to\gamma]=\mathcal{L}[y(t,1);t\to\gamma]=\frac{\sqrt{\gamma}\sinh(a\pi)+\sqrt{\gamma-i\mu}(\cosh(a\pi)+1)}{\left(\sqrt{\gamma-i\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)+1)\right)}\frac{1}{\sqrt{\gamma}\sqrt{\gamma-i\mu}}\end{aligned}$$

[1]The Laplace transforms of the real and immaginary part are available on request.

and:

$$\omega(\gamma,\mu,x;l,u,m) = $$

$$= \begin{cases} \mathbf{1}_{(x\geq u)} \frac{e^{-\sqrt{2}(x-u)\sqrt{\gamma}}}{\sqrt{\gamma}} \frac{\sqrt{\gamma}\sinh(a\pi)+\sqrt{\gamma-i\mu}(\cosh(a\pi)+1)}{\sqrt{\gamma-i\mu}\left(\sqrt{\gamma-i\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)+1)\right)} \\ \mathbf{1}_{(l<x<u)} \frac{\sinh\left(a\pi\left(\frac{u-x}{u-l}\right)\right)+\sinh\left(a\pi\left(\frac{x-l}{u-l}\right)\right)}{\sinh(a\pi)} \mathcal{L}\left[y(t,1);\gamma\right] \\ \mathbf{1}_{(x\leq l)} \frac{e^{-\sqrt{2}(l-x)\sqrt{\gamma}}}{\sqrt{\gamma}} \frac{\sqrt{\gamma}\sinh(a\pi)+\sqrt{\gamma-i\mu}(\cosh(a\pi)+1)}{\sqrt{\gamma-i\mu}\left(\sqrt{\gamma-i\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)+1)\right)} \end{cases}$$

This expression does not seem to admit a simple analytical inverse, although in BS [4], formula 1.7.4 pages 140-141, is given a very complicated expression. We remark that $\mathcal{L}\left[v(t,0);t\to\gamma\right] = \mathcal{L}\left[v(t,1);t\to\gamma\right]$, a consequence of the reflection principle.

b) If we let $l\to-\infty$ in the above expression, we are considering the time spent below the level u by the SBM, so we should recover the Lévy arc-sine law. We get:

$$\lim_{l\to-\infty}\mathcal{L}\left[y(t,1);t\to\gamma\right] = \frac{1}{\sqrt{\gamma}\sqrt{\gamma-i\mu}}$$

and then for $x>u$:

$$\omega(\gamma,\mu,x;l,u,m) = \mathbf{1}_{(x\geq u)}\frac{e^{-\sqrt{2}(x-u)\sqrt{\gamma}}}{\sqrt{\gamma}\sqrt{\gamma-i\mu}}$$

and inverting we obtain:

$$\lim_{l\to-\infty} f_T(\tau,t,x;u,l,0) = \begin{cases} 1_{(x>u)}\frac{1}{\pi}\frac{e^{-\frac{1}{2}\frac{(x-u)^2}{t-\tau}}}{\sqrt{\tau(t-\tau)}} \\ 1_{(x=u)}\frac{1}{\pi}\frac{1}{\sqrt{\tau(t-\tau)}} \\ 1_{(x<u)}\frac{e^{-\frac{1}{2}\frac{(u-x)^2}{\tau}}}{\pi\sqrt{\tau(t-\tau)}} \end{cases} \tag{11}$$

where the expression for the case $x<u$ has been found exploiting the symmetry property $\tau(t,x;u,-\infty,m) = t-\tau(t,-x;-u,-\infty,-m)$, compare Takacs [23]. Expression (11) is the well known arc-sine law, Lévy [18] and BS[4], formula 1.4.4 page 129 where is given the time spent above the level u.

c) If we let $l\to-\infty$, we are considering the time spent by BM with positive drift below the level u. This case has been studied by Akahori [3], Dassios [8], although Takacs [23] provides a simpler expression.

From equation (8), we obtain[2]:

$$\begin{aligned}\lim_{l\to-\infty}\frac{c}{\sqrt{\gamma}}d(\gamma)&=e^{-\alpha u}\frac{\sqrt{\gamma-i\mu}}{\sqrt{\gamma-i\mu}+\sqrt{\gamma}}\left(\frac{1}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)}-\frac{\sqrt{\gamma-i\mu}+\frac{\alpha}{\sqrt{2}}}{\sqrt{\gamma-i\mu}\left(\gamma-i\mu-\frac{\alpha^2}{2}\right)}\right)\\ \lim_{l\to-\infty}\frac{c}{\sqrt{\gamma}}s(\gamma)&=e^{-\alpha u}\frac{\sqrt{\gamma-i\mu}}{\sqrt{\gamma-i\mu}+\sqrt{\gamma}}\left(\frac{1}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)}-\frac{\sqrt{\gamma-i\mu}+\frac{\alpha}{\sqrt{2}}}{\sqrt{\gamma-i\mu}\left(\gamma-i\mu-\frac{\alpha^2}{2}\right)}\right)\end{aligned}\tag{12}$$

and then:

$$\begin{aligned}\lim_{l\to-\infty}\mathcal{L}\left[y(t,1);t\to\gamma\right]&=\frac{e^{-\alpha u}}{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma-i\mu}-\frac{\alpha}{\sqrt{2}}\right)}\\ &=\frac{e^{-\alpha u}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}-\frac{\alpha}{\sqrt{2}}\right)}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma-i\mu}-\frac{\alpha}{\sqrt{2}}\right)}\\ &=\frac{e^{-\alpha u}}{\left(\sqrt{\gamma-i\mu}-\frac{\alpha}{\sqrt{2}}\right)}\left(\frac{1}{\sqrt{\gamma}}-\frac{\alpha/\sqrt{2}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)}\right)\end{aligned}$$

so when $x>u$:

$$\begin{aligned}&\Omega(t,\mu,x;l,u,m)=\\ &=e^{-m(x-u)-\frac{m^2}{2}t}\mathcal{L}^{-1}\left[\frac{e^{mu}}{\left(\sqrt{\gamma-i\mu}+\frac{m}{\sqrt{2}}\right)}\left(\frac{e^{-\sqrt{2}(x-u)\sqrt{\gamma}}}{\sqrt{\gamma}}+\frac{m}{\sqrt{2}}\frac{e^{-\sqrt{2}(x-u)\sqrt{\gamma}}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{m}{\sqrt{2}}\right)}\right);\gamma\to t\right]\\ &=1_{(x>u)}e^{-m(x-u)-\frac{m^2}{2}t}\left(\int_0^t e^{i\mu\theta}\left(\frac{1}{\sqrt{\pi\theta}}-\frac{m}{\sqrt{2}}e^{\frac{m^2}{2}\theta}\,\mathrm{Erfc}\left(\frac{m\sqrt{\theta}}{\sqrt{2}}\right)\right)\right)\times\\ &\times\left(\frac{e^{-\frac{(x-u)^2}{2(t-\theta)}}}{\sqrt{\pi(t-\theta)}}+\frac{m}{\sqrt{2}}e^{\frac{m^2}{2}(t-\theta)-m(x-u)}\,\mathrm{Erfc}\left(\frac{-m\sqrt{t-\theta}}{\sqrt{2}}+\frac{(x-u)}{\sqrt{2(t-\theta)}}\right)\right)d\theta\end{aligned}$$

where we have used the inversion formulas in Abramowitz and Stegun [2] and the convolution property of the Laplace transform.

For $x<u$ and $l\to-\infty$, we can again use the symmetry argument in Takacs [23], so we then obtain that the density function of the occupation time with only one barrier is given

[2]The limits can appear to depend on the value of m, but we can suppose to have $m>0$ without loss of generality. Indeed the key in the determination of the density is the symmetry property in [23] that allows us to find an expression for the density when $x<u$ in terms of the density when $x>u$. So if the drift is negative, we can suppose that $x<l$ and we let $u\to\infty$ so we consider the time spent above the level l. Then using the symmetry property, we find the expression for $x>l$ as well. So the result does not depend on the sign of m.

by:

$$\lim_{l\to-\infty} f_\tau(\tau,t,x;u,l,m) = \left\{ \begin{array}{l} 1_{(x\geq u)}\left(\frac{e^{-\frac{m^2}{2}\tau}}{\sqrt{\pi\tau}} - \frac{m}{\sqrt{2}}\,\mathrm{Erfc}\left(\frac{m\sqrt{\tau}}{\sqrt{2}}\right)\right)\times \\ \times\left(\frac{e^{-\frac{1}{2}\frac{((x-u+m(t-\tau)))^2}{t-\tau}}}{\sqrt{\pi(t-\tau)}} + \frac{m}{\sqrt{2}}e^{-2m(x-u)}\,\mathrm{Erfc}\left(\frac{(x-u)-m(t-\tau)}{\sqrt{2(t-\tau)}}\right)\right) \\ \\ 1_{(x<u)}\left(\frac{e^{-\frac{m^2}{2}(t-\tau)}}{\sqrt{\pi(t-\tau)}} + \frac{m}{\sqrt{2}}\,\mathrm{Erfc}\left(-\frac{m\sqrt{t-\tau}}{\sqrt{2}}\right)\right)\times \\ \times\left(\frac{e^{-\frac{1}{2}\frac{((u-x-m\tau))^2}{\tau}}}{\sqrt{\pi\tau}} - \frac{m}{\sqrt{2}}e^{+2m(u-x)}\,\mathrm{Erfc}\left(\frac{(u-x)+m\tau}{\sqrt{2\tau}}\right)\right) \end{array}\right. \tag{13}$$

In order to make comparable the expression above with equation (12) in Takacs [23], where is considered the average time spent below the level $x < u$, it is necessary to set in Takacs [23] $\alpha = (u-x)/\sqrt{t}$ and substitute m with -$m\sqrt{t}$. We need to use as well the relationship $\mathrm{Erfc}(x) = 2\Phi\left(-\sqrt{2}x\right)$.

Moreover, if we have $m = 0$, the above density function reduces again to (11).

3.2 The moments of the occupation time

We can use the result of Theorem 1 for computing the moments of the occupation time. The result is based on the well known Cauchy integral formula that allows to compute the derivative of all orders at a point for an analytic function.

The moments of a random variable can be obtained from the characteristic function deriving with respect to μ and then setting $\mu = 0$, through the following well known relationship:

$$\begin{array}{l} \mathbf{E}_{0,x}\left[\tau^n(t,x;l,u,m)\right] \equiv M_n(x,t) = \\ = (-1)^n \left.\frac{\partial\Omega^n(t,\mu,x;l,u,m)}{\partial\mu^n}\right|_{\mu=0} + \\ +t^n \times \Pr_{0,x\in(l,u)}\left(\sup_{0\leq s\leq\tau} m*s+W(s)<u;\ \inf_{0\leq s\leq t} m*s+W(s)>l\right) \end{array} \tag{14}$$

Theorem 1 gives us an expression for the Laplace transform respect to time t of the function:

$$\Omega(t,\mu,x;l,u,m) = e^{-mz-\frac{m^2}{2}\tau}\mathcal{L}^{-1}\left[\omega(\gamma,\mu,x;l,u,m);\gamma\to t\right]$$

and the n-th order derivatives $\Omega^{(n)}$and $\omega^{(n)}$ can be computed using the Cauchy integral formula using the expression for ω. We have indeed:

Theorem 2 (**Cauchy integral formula**) *Let ω be analytic everywhere within and on a simple closed contour C, taken in the positive sense. If μ_0 is any point interior to C, then:*

$$\omega(\gamma,\mu_0,x;l,u,m) = \frac{1}{2\pi i}\int_C \frac{\omega(\gamma,\mu,x;l,u,m)}{\mu-\mu_0}d\mu$$

and in general the n-th order derivative with respect to μ is given by:

$$\omega^{(n)}(\gamma,\mu_0,x;l,u,m) = \frac{n!}{2\pi i}\int_C \frac{\omega(\gamma,\mu,x;l,u,m)}{(\mu-\mu_0)^{n+1}}d\mu\blacklozenge$$

Using this result and a numerical procedure for inverting the univariate Laplace transform, we can compute all the moments of the random variable occupation time. In order to calculate the value of the contour integral, we can choose as path the unitary circle with center at the origin. Moreover, we have $\mu_0 = 0$. So we obtain:

$$\begin{aligned}
\omega^{(n)}(\gamma, 0, x; l, u, m) &= \frac{n!}{2\pi i}\int_C \frac{\omega(\gamma, \mu, x; l, u, m)}{\mu^{n+1}} d\mu \\
&= \frac{n!}{2\pi i}\int_0^{2\pi} \frac{\omega\left(\gamma, e^{i\theta}, x; l, u, m\right)}{e^{(n+1)i\theta}} ie^{i\theta} d\theta \\
&= \frac{n!}{2\pi}\int_0^{2\pi} \omega\left(\gamma, e^{i\theta}, x; l, u, m\right) e^{-ni\theta} d\theta
\end{aligned} \tag{15}$$

Then the strategy for computing the moments is the following:

1. numerically compute the integral (15);

2. numerically compute the inverse respect to γ of the function $\varpi^{(n)}$and obtain $\Omega^{(n)}$,

3. then compute the nth-moment using formula 14.

We have performed the first step using a Legendre quadrature formula with 100 nodes, Press et al. [21] page 150 and following, whilst the second step has been done using two different numerical procedures: the Euler algorithm as in Abate and Whitt [1] and the univariate version of the Padé approximant method as described in Singhal and Vlach [22]. The results of the two methods agree up to the seventh digit and the computation of each moment requires less than one second. Takacs [24] has obtained, using a combinatorial approach, a recurrence relationship for the moments of the occupation time of the Brownian motion without drift in terms of the moments of the local time of the BM. An advantage of the method discussed in the present context is that we can compute separately the different moments and this fact can reduce enormously the computational time.

4 The numerical inversion

In this section we discuss the problem of the numerical inversion. From a computational point of view it is convenient to distinguish the problem of finding the density function from the problem of pricing the corridor option. Indeed, in the first case we like to use two single univariate inversions, whilst in the second it is better to use a multivariate inversion.

This choice is also due to the fact that up to now, relatively little attention has been given to inversion of multidimensional transforms, so in order to limit the programming effort a possibility is to obtain the density function using well tested univariate inversion routines as described in the following two steps:

1) Numerically find the inverse Laplace transform of the function $\Omega(t, \mu, x; l, u, m)$. We have solved this problem using the Crump's [7] method implemented in the IMSL-library subroutine FLINV[3].

2) Using the numerically computed $\Omega(t, \mu, x; l, u, m)$, find the function $f_\tau(\tau, t, x; U, L, m)$ through a further numerical Fourier transform inversion. We have computed this inverse

[3]This method is ranked among the most accurate available numerical inversion techniques in the Davies and Martin [9] comparison.

using the Fast Fourier Transform and this allows us to reduce greatly the computational time. Moreover we obtain simultaneously the entire density function, whilst if we use a different procedure (e.g. a bidimensional Laplace inversion) we need to repeat as many times as the number of points at which we desire the density function. Moreover, having the entire density function allows us to compute very complex derivative products which depend on the occupation time. The two steps described above require separate expressions for the real and imaginary part of the function $\Omega(t, \mu, x; l, u, m)$ at which we can apply the FFT. These expressions can be obtained from the Author on request.

In Figure 1, we represent the density function of the occupation time obtained using the two-step univariate procedure. In Table 1 we compare the price of the corridor bond obtained using the density coming from the double inversion described above and the analytical formula for the corridor bond as from section 2.

[FIGURE 1 HERE]

The expected value has been calculated through a simple trapezoidal rule. In the numerical inversion, we have to choose the Fourier transform maximum frequency, so from the table we can see that increasing it we obtain a great accuracy, but with the cost of increasing the computational time[4].

Initial Price	Analytical Sol.	n=2048	n=1024	n=512	n=256
80	0.04609	-0.02%	0.05%	0.25%	0.72%
85	0.08149	0.00%	0.08%	0.27%	0.68%
90	0.13134	0.02%	0.10%	0.27%	0.64%
95	0.19606	0.03%	0.10%	0.26%	0.60%
100	0.27463	0.03%	0.10%	0.24%	0.54%
105	0.30959	0.08%	0.15%	0.30%	0.60%
110	0.25770	0.01%	0.07%	0.21%	0.51%
115	0.18058	0.02%	0.08%	0.23%	0.56%
120	0.12478	0.00%	0.07%	0.23%	0.59%
125	0.08509	-0.01%	0.06%	0.23%	0.61%
average time	1"	170"	104"	56"	32"

Table 1: Price of the corridor bond using the analytical solution and difference percentage respect to the prices obtained integrating the density function obtained by the two step inversion (r=0.05,$\sigma = 0.2$, $L = 100$, $U = 110$, $t = 1yr$). n denotes the number of chosen points (a power of 2) in the FFT.

Using the results in Theorem 1, we can obtain the double Laplace transform of the price of the corridor option. Indeed, we observe that we can write the undiscounted price of the corridor option with strike K and residual live t as:

$$C(t, K; x) = \int_K^t (\tau - K) f_\tau(\tau, t, x; u, l, m) \, d\tau + (t - K)^+ \times \mathrm{Pr}_{0, x \in (l,u)} [\tau(t, x; u, l) = t]$$

and with some simple passage, we obtain that $C(t, K; x)$ is given by:

$$E_{0,x}[\tau(t, x; u, l)] - K(1 - \mathrm{Pr}_{0,x}[\tau(t, x; u, l) = 0]) + \int_0^K (K - \tau) f_\tau(\tau, t, x; u, l, m) \, d\tau$$

[4] All the calculations have been performed on a Pentium 133 machine.

where $E_{0,x}[\tau(t,x;u,l)]$ is the expected value of the r.v. $\tau(t,x;u,l)$ and is given by (2). If we consider now the Laplace transform with respect to K of the third term exploiting the convolution property, we obtain:

$$\begin{aligned}&\mathcal{L}\left[\int_0^K (K-\tau) f_\tau(\tau,t,x;U,L,m)\,d\tau; K\to\mu\right] = \\ &= \mathcal{L}[K;K\to\mu]\,\mathcal{L}[f_\tau(\tau,t,x;U,L,m);K\to\mu] \\ &= \frac{1}{\mu^2}\Omega(t,\mu,x;l,u,m)\end{aligned}$$

and then we have:

$$\begin{aligned}&C(t,K;x) = \\ &= E_{0,x}[\tau(t,x;u,l)] - K(1-\Pr_{0,x}[\tau=0]) + \mathcal{L}^{-1}\left[\frac{\Omega(t,\mu,x;l,u,m)}{\mu^2};\mu\to K\right] \\ &= E_{0,x}[\tau(t,x;u,l)] - K(1-\Pr_{0,x}[\tau=0]) + \mathcal{L}^{-1}\left[\frac{\omega(\gamma,\mu,x;l,u,m)}{\mu^2};\mu\to K,\gamma\to t\right]\end{aligned}$$

So the price of the corridor option is given by the double Laplace inverse[5] of the quantity $\omega(\gamma,\mu,x;l,u,m)/\mu^2$.

If $w(\gamma,\mu) := \omega(\gamma,\mu,x;l,u,m)/\mu^2$ is the double Laplace transform, the inverse Laplace transform $W(t,K)$ is obtained applying the inversion formula in two variables:

$$W(t,K) = \left(\frac{1}{2\pi j}\right)^2 \int_{c_1-j\infty}^{c_1+j\infty}\int_{c_2-j\infty}^{c_2+j\infty} e^{\gamma t}e^{\mu K} w(\gamma,\mu)\,d\gamma d\mu$$

We have considered two methods for numerically computing this quantity: a) the Fourier-series method firstly introduced for multidimensional transform inversion by Choudhury et al. (CLW) [6], b) the Padé approximation as suggested in Singhal et al. [22].

The Fourier-series inversion procedure[6], formula (2.11) in CLW [6], consists essentially in an enhancement of the Euler algorithm in Abate and Whitt [1] based on truncating the inversion integral and applying the trapezoidal rule. The idea consists in damping the function to be inverted multiplying it by a two dimensional decaying exponential function and then approximating the damped function by a periodic function constructed by aliasing. The inversion formula is then the two-dimensional Fourier series of the periodic function. The computation of the series can be greatly accelerated by the use of the Euler algorithm for alternating series[7]. The CLW algorithm allows a simultaneous control of the aliasing error and the round-off error coming from multiplying large numbers by small ones. CLW show indeed that both errors can be controlled choosing in a appropriate way four different constants A_1, A_2, l_1 and l_2. For the fact that the price of the corridor option has an upper bound in the price of the corridor bond, in our case the aliasing error can be bounded by $C\left(e^{-A_1}+e^{-A_2}\right)$,

[5] We can use here the expressions given in Theorem 1, once we have substituted in all expressions the quantity $\gamma - i\mu$ with the quantity $\gamma+\mu$. This is due to the fact that here we are using a double Laplace transform, whilst in Theorem 1 we have used a Fourier transform and a Laplace transform. Moreover for numerical purposes, it is convenient to divide in (8) and (9) the numerator and the denominator by $\sinh(a\pi)$ and to use the fact that $\tanh(a\pi/2) = (\cosh(a\pi)-1)/\sinh(a\pi) = \sinh(a\pi)/(\cosh(a\pi)+1)$

[6] We remark that the Crump algorithm and the FFT used in the computation of the density function are closely related to the Fourier-series method used in the multidimensional inversion, compare Crump [7] and Abate and Whitt [1].

[7] The implementation of the Euler algorithm requires the selection of two parameters, m and n. The Authors suggest $n=38$ and $m=11$, whilst in our case we have seen that a better choice consists in setting $n=20$ and $m=20$.

where $C = K\left(1 - \Pr_{0,x\in(l,u)}[\tau = 0]\right)$, so that to fix a bound on this error implies to fix A_1 and A_2. The roundoff error depends on the quantity $\exp\left(A_1/2l_1 + A_2/2l_2\right)/\left(4l_1l_2tK\right)$ that is decreasing when l_1 and l_2 increase. There is however a trade-off between error control and computation time, since this increases proportionally to the product of $l_1$and l_2, and increasing A_1 and A_2 requires an increase in l_1 and l_2. The authors suggest to use $A_1 = A_2 = 20$ and $l_1 = l_2 = 2$. We remark that as explained in Abate and Whitt [1], pag. 75-76, this inversion method is not accurate when we have a not bounded density or with very high peaks as in our case (compare figure 1). For this reason it is more appropriate to apply this method to the calculation of the corridor option price and avoid its use in the computation of the density function[8].

The method proposed by Singhal et al. [22] is very accurate if the function is smooth as in our case. The idea consists in approximating the functions e^z appearing in the Laplace inversion formula by a Padé rational function. Then, if $w(\gamma, \mu)$ is the double Laplace transform, the inverse Laplace transform $W(t, K)$ can be computed as:

$$W(t, K) = \frac{1}{tK}\sum_{i=1}^{M_1}\sum_{j=1}^{M_2} r_{1i}r_{2j}w\left(\frac{z_{1i}}{t}, \frac{z_{2j}}{K}\right) \tag{16}$$

where z_{1i} and z_{2j} are the poles of the Padé approximations to $e^{\gamma t}$ and $e^{\mu K}$, and r_{1i} and r_{2j} are the corresponding residues. M_1 and M_2 are the degree of the denominator in the Padé approximations. This inverse requires to choose the degree of the numerator and denominator of the Padé approximants. From some numerical experiments, we have observed that in order to obtain a good accuracy and avoid roundoff errors[9] the degree of the denominator has to be chosen greater than 13 and not larger than 18 . The degree of the numerator has to be chosen not greater than 4 or 5. This choice gives us an agreement to the seventh digit between the numerical computed inverse and the analytical formula for the hurdle option. Moreover this inversion can be performed very quickly (less than 1 second) and with a limited programming effort, once we have computed the poles and the residues of the Padé approximants with programs like Mathematica or Maple and we have stored them. However, respect to the previous method, particular care has to be used, because there is no way to tell how accurate the Padé approximant it is[10].

In table 2 we compare the three methods and the MonteCarlo simulation method, reporting as well the computational time and the standard error of the MonteCarlo estimate. We remark that the MonteCarlo method suffers for the intrinsic discreteness of the simulation so we do not know if the process has crossed or not the barriers and we cannot compute exactly the time spent inside the barriers during each step of the simulation. The MC simulation has been performed using 50000 simulation and using the antithetic variate technique, Boyle [5]. We have tried also to obtain a greater reduction in the standard error using as control variate the price of the corridor bond, but this technique performed poorly as K increases and we do not report the results here.

We can remark how the different methods agree in general to the third digit and sometimes more. In this respect, the Padé and the Fourier method appears the most preferable mainly

[8] Indeed from some numerical experiments applied to the simplest case of one barrier and zero drift, we have seen that in order to obtain a good result we need to set l_1 and l_2 at least equal to 7. We had the same problem with the Padé inversion. So in the computation of the density function, the bidimensional inversion appears inefficient compared to the two-step unidimensional inversion .

[9] An easy check for avoiding roundoff errors is that the sum of the residues has to be equal to 0.

[10] Press et al. [21]: " It is a powerful technique, but in the end still mysterious technique", pag. 202.

for their accuracy (up to the seventh digit) and the very low computational time that they require. The MC method appears very time consuming.

	Crump+FFT	Fourier	Fourier	Padè	MC+AV
K=0.2		$n=20, m=20$	$n=100, m=20$		
$x=90$	0.0462793	0.0463038	0.0463038	0.0463039	0.046606; 0.265
$x=95$	0.0792503	0.0792444	0.0792444	0.0792445	0.078995; 0.308
$x=100$	0.1247528	0.1247227	0.1247228	0.1247226	0.124951; 0.518
$x=105$	0.1470881	0.1469239	0.1469239	0.1469239	0.147282; 0.665
$x=110$	0.1161007	0.1161262	0.1161262	0.1161261	0.115834; 0.358
$x=115$	0.0735259	0.0735554	0.0735554	0.0735555	0.073820; 0.311
$x=120$	0.0456788	0.0457253	0.0457253	0.0457254	0.046075; 0.271
average CPU	170"	8"	39"	<1"	337"
K=0.4					
$x=90$	0.0100981	0.0101457	0.0101457	0.0101459	0.010328; 0.123
$x=95$	0.0213042	0.0213357	0.0213358	0.0213361	0.021146; 0.175
$x=100$	0.0400273	0.0400375	0.0400376	0.0400375	0.040250; 0.27
$x=105$	0.0504331	0.0503482	0.0503483	0.0503485	0.050574; 0.376
$x=110$	0.0372272	0.0372753	0.0372754	0.0372753	0.037048; 0.227
$x=115$	0.0202404	0.0202948	0.0202948	0.0202951	0.020376; 0.176
$x=120$	0.0107088	0.0107697	0.0107697	0.0107699	0.010868; 0.285
average CPU	170"	8"	38"	<1"	337"
K=0.6					
$x=90$	0.0008576	0.0009014	0.0009014	0.0009019	0.000927; 0.030
$x=95$	0.0026490	0.0026893	0.0026893	0.0026899	0.002592; 0.053
$x=100$	0.0067578	0.0067873	0.0067874	0.0067831	0.006865; 0.092
$x=105$	0.0094893	0.0094617	0.0094618	0.0094619	0.009545; 0.131
$x=110$	0.0062664	0.0063189	0.0063191	0.0063188	0.006258; 0.087
$x=115$	0.0026123	0.0026664	0.0026664	0.0026670	0.002687; 0.056
$x=120$	0.0010299	0.0010822	0.0010822	0.0010826	0.001093; 0.034
average CPU	170"	8"	37"	<1"	337"

Table 2: Price of the corridor option with parameters r=0.05, σ =0.2, L=100, U=110, t=1. In the Crump+FFT we have used 2048 sampling points. In the Fourier method we have set l_1=l_2=2 and A_1=A_2=20, whilst n and m are important for the use of the Euler algorithm that requires a total of $m+n+1$ terms. In the Padè approximation the degree of the denominator has been set to 18 and the degree of the numerator to 4. The poles and the residues have been preliminarly calculated in Mathematica 3.0 with the functions Pade[], NResidue[] and NSolve[]. In the MonteCarlo column (50000 simulations with 1200 steps and with antithetic variate) appears also the 1000xstandard error.

We observe that in order to calculate the Greeks of the contract we can simply calculate the derivatives of the double Laplace transform and invert them. So the numerical routines used for finding the price can be adapted in a simple way to the calculations of the sensitivities, whilst in the MC method the calculation of the Greeks is usually reputed to be inaccurate. In figure 2 we present the delta of the corridor option varying strike and underlying index.

[FIGURE 2 HERE]

Greater details can be found in Fusai and Tagliani [11], where there is also a comparison between continuous and discrete time monitoring of the underlying.

Finally, the following figures illustrate the behavior of expected value, standard deviation, skewness and kurtosis of the occupation time. With regard to the behavior of the expected occupation time respect to the starting level of the Brownian Motion, Figure 3, we observe the bell-shaped form: the expected value is at the highest when we are inside the barriers and as we move faraway it decreases. In the case of zero drift the shape is symmetric around the mid-point of the interval. The expected fraction of time spent inside the barriers, Figure 4, is a decreasing function of time if the index starts inside the barriers (as time passes it is more likely to move out), otherwise it is increasing and then decreasing (as time passes it is more likely to move in and then out again). The particular shape of the standard deviation, Figure 5, always respect to the starting point of the Brownian Motion is due to the fact that when the index starts exactly on the barriers it continues in moving in and out of the interval: so the standard deviation is at a maximum when we are on the barriers. The remaining figure illustrates the behavior of the standard deviation of the fraction of time spent inside the barriers (Figure 6).

[FIGURES 3-6 HERE]

5 Conclusions

In this paper we have studied a generalization of the Lévy arc-sine law providing an expression for the Laplace Transform of the characteristic function. We have used the Feynman-Kac equation and solved the related PDE. A possible extension is related to the derivation of the joint density of the occupation time and the final level of the Brownian Motion and it can be done easily following the same procedure. We have also discussed how to invert numerically the double transform in order to obtain the density function and price corridor derivatives and we have shown that using simple to implement numerical procedures the task can be accomplished without difficulty. We have also shown how to compute the moments of the occupation time using the Cauchy integral formula and the expression for the characteristic function.

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A Solution of the PDE

In order to solve the PDE (4):

$$-\frac{\partial v(t,x)}{\partial t}+\frac{1}{2}\frac{\partial^2 v(t,x)}{\partial^2 x}+m\frac{\partial v(t,x)}{\partial x}+i\mu\mathbf{1}_{(l<x<u)}v(t,x)=0$$

with initial condition:

$$v(0,x)=1,\ \forall x\in(-\infty;+\infty)$$

and boundary conditions:

$$v(t,\pm\infty)=1,\ \forall t>0$$

we consider the following transformation:

$$v(t,x)=e^{\alpha x+\beta t}h(t,x)$$

and setting $\alpha=-m$ and $\beta=-\frac{m^2}{2}$, we get the following PDE for the function $h(t,x)$:

$$-\frac{\partial h(t,x)}{\partial t}+\frac{1}{2}\frac{\partial^2 h(t,x)}{\partial^2 x}+i\mu\mathbf{1}_{(l<x<u)}h(t,x)=0 \tag{17}$$

with initial condition $h(0,x)=e^{-\alpha x}$. We can make a second transformation defining $z=(x-l)/(u-l)$ and introducing a new function:

$$y(t,z)=h(t,(u-l)z+l)$$

we get the following PDE for the function $y(t,z)$:

$$-\frac{\partial y(t,z)}{\partial t}+c^2\frac{\partial^2 y(t,z)}{\partial^2 z}+i\mu\mathbf{1}_{(0<z<1)}y(t,z)=0 \tag{18}$$

where $c^2 = \frac{1}{2(u-l)^2}$, and $y(0,z) = e^{-\alpha((u-l)z+l)}$.

In order to solve (18) we can distinguish three cases, $z < 0$, $0 < z < 1$, and $z>1$ and require a continuous and differentiable solution at these boundary points. As consequence, we can guarantee that the characteristic function as well is continuous and differentiable at $x = l$ and at $x = u$. So we can solve the PDE (18) imposing Neumann boundary conditions at $z = 0$ and at $z = 1$:

$$\begin{array}{l} \left.\frac{\partial y(t,z)}{\partial z}\right|_{z=0-} = L(t) = \left.\frac{\partial y(t,z)}{\partial z}\right|_{z=0+} \\ \left.\frac{\partial y(t,z)}{\partial z}\right|_{z=1-} = U(t) = \left.\frac{\partial y(t,z)}{\partial z}\right|_{z=1+} \end{array} \tag{19}$$

where $L(t)$ and $U(t)$ are unknown functions to be chosen in order to have continuity of the solution at $z = 0$ and at $z = 1$,i.e.:

$$\begin{array}{l} y(t,z)|_{z=0-} = y(t,z)|_{z=0+} \\ y(t,z)|_{z=1-} = y(t,z)|_{z=1+} \end{array} \tag{20}$$

In the case $z < 0$ and $z > 1$, to solve (18) with boundary conditions (19) amounts to solve the heat equation in a semi-infinite region with Neumann boundary condition. The solution can be found in Zauderer [28] (page 268, eq. 5.121). In the case $0 < z < 1$, considering the new transformation $g(t,z) = e^{-i\mu t} y(t,z)$, we get that $g(t,z)$ satisfies the heat equation in a finite strip with Neumann boundary $e^{-i\mu t} L(t)$ and $e^{-i\mu t} U(t)$ at $z = 0$ and $z = 1$, respectively. The solution in this case can be found in Necati [19] (page 62, eq.2-73a). In conclusion the function $y(t,z)$ can be expressed in terms of the unknown functions $L(t)$ and $U(t)$ in the following way:

$$y(t,z) = \\ = \begin{cases} 1_{(z>1)} \left(\begin{array}{l} e^{-\alpha u} \int_0^{+\infty} [G(z-1-\xi,t) + G(z-1+\xi,t)] e^{-\alpha(u-l)\xi} d\xi + \\ -2c^2 \int_0^t G(z-1,t-\theta) U(\theta) d\theta \end{array} \right) \\ 1_{(0<z<1)} \left(\begin{array}{l} e^{i\mu t} \frac{e^{-\alpha l} - e^{-\alpha u}}{\alpha(u-l)} + c^2 \int_0^t e^{i\mu(t-\theta)} (U(\theta) - L(\theta)) d\theta + \\ +2c^2 \sum_{n=1}^{+\infty} y_n(t) \cos n\pi z + \sum_{n=1}^{+\infty} e^{(i\mu-\lambda_n)t} \phi_n \cos n\pi z \end{array} \right) \\ 1_{(z<0)} \left(\begin{array}{l} e^{-\alpha l} \int_0^{+\infty} [G(-z-\xi,t) + G(-z+\xi,t)] e^{\alpha(u-l)\xi} d\xi + \\ +2c^2 \int_0^t G(-z,t-\theta) L(\theta) d\theta \end{array} \right) \end{cases} \tag{21}$$

where:

$$\begin{aligned} G(x,t) &= \frac{e^{-\frac{x^2}{4c^2 t}}}{\sqrt{4\pi c^2 t}} \\ \phi_n &= 2 \int_0^1 e^{-\alpha((u-l)\xi+l)} \cos(n\pi\xi) d\xi \\ y_n(t) &= \int_0^t e^{(i\mu-\lambda_n)(t-\theta)} ((-1)^n U(\theta) - L(\theta)) \\ \lambda_n &= (n\pi c)^2 \end{aligned}$$

In order to find the unknown functions $U(t)$ and $L(t)$ we now require the continuity of the function $y(t,z)$ at $z=0$ and $z=1$, i.e. we impose conditions (20). However it is more convenient to transform the continuity conditions in the following way:

$$\begin{aligned}
\lim_{z\to 1+} y(t,z) + \lim_{z\to 0-} y(t,z) &= \lim_{z\to 1-} y(t,z) + \lim_{z\to 0+} y(t,z)\\
\lim_{z\to 1+} y(t,z) - \lim_{z\to 0-} y(t,z) &= \lim_{z\to 1-} y(t,z) - \lim_{z\to 0+} y(t,z)
\end{aligned}$$

and introducing the functions $D(t) := U(t) - L(t)$ and $S(t) := U(t) + L(t)$, we obtain:

$$\lim_{z\to 1+} y(t,z) + \lim_{z\to 0-} y(t,z) =$$
$$= e^{-\alpha u+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(\alpha\frac{\sqrt{t}}{\sqrt{2}}\right) + e^{-\alpha l+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right) - 2c^2\int_0^t \frac{1}{\sqrt{4\pi c^2(t-\theta)}} D(\theta)\,d\theta$$

$$\lim_{z\to 1+} y(t,z) - \lim_{z\to 0-} y(t,z)$$
$$= e^{-\alpha u+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(\alpha\frac{\sqrt{t}}{\sqrt{2}}\right) - e^{-\alpha l+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right) - 2c^2\int_0^t \frac{1}{\sqrt{4\pi c^2(t-\theta)}} S(\theta)\,d\theta$$

$$\lim_{z\to 1-} y(t,z) + \lim_{z\to 0+} y(t,z) =$$
$$= 2e^{i\mu t}\frac{e^{-\alpha l}-e^{-\alpha u}}{\alpha(u-l)} + 2c^2\int_0^t e^{i\mu(t-\theta)} D(\theta)\,d\theta + \sum_{n=1}^{+\infty} e^{(i\mu-\lambda_n)t}\phi_n\left(1+(-1)^n\right)$$
$$+2c^2\sum_{n=1}^{+\infty}\left(1+(-1)^n\right)\int_0^t e^{(i\mu-\lambda_n)(t-\theta)} D(\theta)\,d\theta$$

$$\lim_{z\to 1-} y(t,z) - \lim_{z\to 0+} y(t,z) =$$
$$= 2c^2\sum_{n=1}^{+\infty}\left(1-(-1)^n\right)\int_0^t e^{(i\mu-\lambda_n)(t-\theta)} S(\theta)\,d\theta + \sum_{n=1}^{+\infty} e^{(i\mu-\lambda_n)t}\phi_n\left((-1)^n-1\right)$$

The determination of the functions $D(t)$ and $S(t)$ requires to solve respect to them the following integral equations:

$$\left\{\begin{array}{l}
e^{-\alpha u+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(\frac{\alpha\sqrt{t}}{\sqrt{2}}\right) + e^{-\alpha l+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right) - 2c^2\int_0^t \frac{1}{\sqrt{4\pi c^2(t-\theta)}} D(\theta)\,d\theta =\\
2e^{i\mu t}\frac{e^{-\alpha l}-e^{-\alpha u}}{\alpha(u-l)} + 2c^2\int_0^t e^{i\mu(t-\theta)} D(\theta)\,d\theta + \sum_{n=1}^{+\infty} e^{(i\mu-\lambda_n)t}\phi_n\left(1+(-1)^n\right) +\\
2c^2\sum_{n=1}^{+\infty}\left(1+(-1)^n\right)\int_0^t e^{(i\mu-\lambda_n)(t-\theta)} D(\theta)\,d\theta
\end{array}\right. \tag{22}$$

$$\left\{\begin{array}{l}
e^{-\alpha u+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(\frac{\alpha\sqrt{t}}{\sqrt{2}}\right) - e^{-\alpha l+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right) - 2c^2\int_0^t \frac{1}{\sqrt{4\pi c^2(t-\theta)}} S(\theta)\,d\theta =\\
2c^2\sum_{n=1}^{+\infty}\left(1-(-1)^n\right)\int_0^t e^{(i\mu-\lambda_n)(t-\theta)} S(\theta)\,d\theta + \sum_{n=1}^{+\infty} e^{(i\mu-\lambda_n)t}\phi_n\left((-1)^n-1\right)
\end{array}\right. \tag{23}$$

We remark that these integral equations involve separately the functions $S(t)$ and $D(t)$. We solve them using the Laplace transform. We call $s(\gamma)$ and $d(\gamma)$ the Laplace transforms, with respect to the time variable t, of the functions $S(t)$ and $D(t)$:

$$s(\gamma) \;:\; = \mathcal{L}[S(t); t\to\gamma] = \int_0^t e^{-\gamma t} S(t)\,dt$$

$$d(\gamma) \quad : \quad = \mathcal{L}[D(t); t \to \gamma] = \int_0^t e^{-\gamma t} D(t)\, dt$$

Laplace transforming (22) and (23), we obtain two linear equations to be solved separately for each function $s(\gamma)$ and $d(\gamma)$:

$$\begin{cases} \frac{e^{-\alpha u}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)} + \frac{e^{-\alpha l}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)} - \frac{2c^2}{\sqrt{4c^2\gamma}} d(\gamma) = \\ 2\frac{1}{\gamma - i\mu}\frac{e^{-\alpha l}-e^{-\alpha u}}{\alpha(u-l)} + 2c^2\frac{1}{\gamma-i\mu} d(\gamma) + \frac{1}{\pi^2 c^2}\sum_{n=1}^{+\infty}\frac{(1+(-1)^n)}{n^2+a^2}\phi_n + \\ +d(\gamma)\frac{2c^2}{\pi^2c^2}\sum_{n=1}^{+\infty}\frac{(1+(-1)^n)}{n^2+a^2} \end{cases} \tag{24}$$

$$\begin{cases} \frac{e^{-\alpha u}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)} - \frac{e^{-\alpha l}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)} - \frac{2c^2}{\sqrt{4c^2\gamma}} s(\gamma) = \\ +s(\gamma)\frac{2c^2}{\pi^2c^2}\sum_{n=1}^{+\infty}\frac{(1-(-1)^n)}{n^2+a^2} + \frac{1}{\pi^2c^2}\sum_{n=1}^{+\infty}\frac{((-1)^n-1)}{n^2+a^2} \end{cases}$$

Using the following summation series formula in Gradshteyn and Ryzhik[12], page 40 as formula 1.445.1:

$$\sum_{n=1}^{\infty}\frac{n\sin(nx)}{n^2+a^2} = \frac{\pi}{2}\frac{\sinh[a(\pi-x)]}{\sinh[a\pi]}; 0 < x < 2\pi \tag{25}$$

formula 1.445.2:

$$\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2+a^2} = \frac{\pi}{2a}\frac{\cosh[a(\pi-x)]}{\sinh[a\pi]} - \frac{1}{2a^2}; 0 < x \leq 2\pi \tag{26}$$

formula 1.445.3:

$$\sum_{n=1}^{\infty}\frac{(-1)^n\cos(nx)}{n^2+a^2} = \frac{\pi}{2a}\frac{\cosh[ax]}{\sinh[a\pi]} - \frac{1}{2a^2}; -\pi \leq x \leq \pi \tag{27}$$

and formula 1.445.4:

$$\sum_{n=1}^{\infty}\frac{(-1)^n n\sin(nx)}{n^2+a^2} = -\frac{\pi}{2}\frac{\sinh[ax]}{\sinh[a\pi]}; -\pi < x < \pi \tag{28}$$

where $a\pi = \sqrt{(\gamma - i\mu)/c^2}$, we obtain:

$$\begin{cases} \frac{e^{-\alpha u}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)} + \frac{e^{-\alpha l}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)} - \frac{1}{c}\int_0^1 \frac{\cosh(a\pi(1-x))+\cosh(a\pi x)}{\sqrt{\gamma-i\mu}\sinh(a\pi)} e^{-\alpha((u-l)x+l)}dx = \\ c\left(\frac{1}{\sqrt{\gamma}} + \frac{1}{\sqrt{\gamma-i\mu}}\frac{\cosh(a\pi)+1}{\sinh(a\pi)}\right) d(\gamma) \end{cases}$$

$$\begin{cases} \frac{e^{-\alpha u}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)} - \frac{e^{-\alpha l}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)} - \frac{1}{c}\int_0^1 \frac{\cosh(a\pi x)-\cosh(a\pi(1-x))}{\sqrt{\gamma-i\mu}\sinh(a\pi)} e^{-\alpha((u-l)x+l)}dx = \\ +c\left(\frac{1}{\sqrt{\gamma}} + \frac{1}{\sqrt{\gamma-i\mu}}\frac{\cosh(a\pi)-1}{\sinh(a\pi)}\right) s(\gamma) \end{cases} \tag{29}$$

We observe:

$$\frac{1}{c}\int_0^1 \frac{\cosh(a\pi(1-x))+\cosh(a\pi x)}{\sqrt{\gamma-i\mu}\sinh(a\pi)}e^{-\alpha((u-l)x+l)}dx =$$
$$= \frac{\left(\sqrt{\gamma-i\mu}\left(e^{-\alpha u}+e^{-\alpha l}\right)\sinh(a\pi)+\frac{\alpha}{\sqrt{2}}\left(e^{-\alpha u}-e^{-\alpha l}\right)(\cosh(a\pi)+1)\right)}{\sqrt{\gamma-i\mu}\sinh(a\pi)\left(\gamma-i\mu-\frac{\alpha^2}{2}\right)}$$

$$\frac{1}{c}\int_0^1 \frac{\cosh(a\pi x)-\cosh(a\pi(1-x))}{\sqrt{\gamma-i\mu}\sinh(a\pi)}e^{-\alpha((u-l)x+l)}dx =$$
$$= \frac{\left(\sqrt{\gamma-i\mu}\left(e^{-\alpha u}-e^{-\alpha l}\right)\sinh(a\pi)+\frac{\alpha}{\sqrt{2}}\left(e^{-\alpha u}+e^{-\alpha l}\right)(\cosh(a\pi)-1)\right)}{\sqrt{\gamma-i\mu}\sinh(a\pi)\left(\gamma-i\mu-\frac{\alpha^2}{2}\right)}$$

so substituting these expressions in (29) and solving respect to the quantities $cd(\gamma)/\sqrt{\gamma}$ and $cs(\gamma)/\sqrt{\gamma}$ we obtain (8) in Theorem 1.

From equation (21), we get as well the Laplace transform of the function $y(t,z)$ when $z=0$ and when $z=1$. Then the expressions for the characteristic function when $x=l$ and when $x=u$, are:

$$\begin{aligned} v(t,u) &= e^{-mu-\frac{m^2}{2}t}y(t,1) \\ v(t,l) &= e^{-ml-\frac{m^2}{2}t}y(t,0) \end{aligned} \tag{30}$$

A.1 Solution of the PDE with Dirichlet boundary condition

In order to find the expression for the function $v(t,x)$ for a generic value of x, we can now solve the PDE (4) in three different regions ($x<l$, $l<x<u$ and $u<x$) using as Dirichlet boundary conditions at $x=l$ and $x=u$ the known values in (30). This, after the same transformation as before, amounts to solve:

$$-\frac{\partial y(t,z)}{\partial t}+c^2\frac{\partial^2 y(t,z)}{\partial^2 z}+i\mu\mathbf{1}_{(0<z<1)}y(t,z)=0 \tag{31}$$

where $c^2=\frac{1}{2(u-l)^2}$, and $y(0,z)=e^{-\alpha((u-l)z+l)}$, in three different regions using Dirichlet boundary conditions at $z=0$ and at $z=1$ the known values of $y(t,0)$ and $y(t,1)$.

A.1.1 Case $x<l$ and $x>u$

In this case, we have the heat equation with Dirichlet boundary conditions and the solution can be found in Zauderer [28], page. 265, eq. 5.105:

$$y(t,x)=$$
$$\begin{cases} \mathbf{1}_{(x>u)}\left(\begin{array}{l} \int_0^{+\infty}\left[\frac{e^{-\frac{(x-u-\zeta)^2}{2t}}}{\sqrt{2\pi t}}-\frac{e^{-\frac{(x-u+\zeta)^2}{2t}}}{\sqrt{2\pi t}}\right]e^{-\alpha(\zeta+u)}d\zeta+ \\ +\int_0^t\frac{(x-u)}{\sqrt{2\pi(t-\theta)^3}}e^{-\frac{(x-u)^2}{2(t-\theta)}}e^{-\alpha u-\beta\theta}q(\theta)\,d\theta \end{array}\right) \\ \mathbf{1}_{(x<l)}\left(\begin{array}{l} \int_0^{+\infty}\left[\frac{e^{-\frac{(l-x-\zeta)^2}{2t}}}{\sqrt{2\pi t}}-\frac{e^{-\frac{(l-x+\zeta)^2}{2t}}}{\sqrt{2\pi t}}\right]e^{-\alpha(l-\zeta)}d\zeta+ \\ +\int_0^t\frac{(l-x)e^{-\frac{(l-x)^2}{2(t-\theta)}}}{\sqrt{2\pi(t-\theta)^3}}e^{-\alpha l-\beta\theta}p(\theta)\,d\theta \end{array}\right) \end{cases}$$

and then:

$$
v(t,x) = e^{\alpha x+\beta t} y(t,x)
$$

$$
\begin{cases}
\mathbf{1}_{(x>u)} \left(\begin{array}{l} e^{\alpha x+\beta t} \int_0^{+\infty} \left[\frac{e^{-\frac{(x-u-\zeta)^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(x-u+\zeta)^2}{2t}}}{\sqrt{2\pi t}} \right] e^{-\alpha(u+\zeta)} d\zeta + \\ + e^{\alpha(x-u)} \int_0^t \frac{(x-u)e^{-\frac{(x-u)^2}{2(t-\theta)}}}{\sqrt{2\pi (t-\theta)^3}} e^{\beta(t-\theta)} q(\theta)\, d\theta \end{array} \right) \\
\mathbf{1}_{(x<l)} \left(\begin{array}{l} e^{\alpha x+\beta t} \int_0^{+\infty} \left[\frac{e^{-\frac{(l-x-\zeta)^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(l-x+\zeta)^2}{2t}}}{\sqrt{2\pi t}} \right] e^{-\alpha(l-\zeta)} d\zeta + \\ + e^{-\alpha(l-x)} \int_0^t \frac{(l-x)e^{-\frac{(l-x)^2}{2(t-\theta)}}}{\sqrt{2\pi (t-\theta)^3}} e^{\beta(t-\theta)} p(\theta)\, d\theta \end{array} \right)
\end{cases} \tag{32}
$$

where $q(t)$ and $p(t)$ are the known values of the characteristic function at $x = u$ and $x = l$: $q(t) := v(t,u)$ and $q(t) := v(t,l)$.

Using some algebra and comparing with BS [4] (formula 1.2.4 page 198), we can remark that when $x > u$:

$$
\begin{aligned}
&= e^{\beta t} \int_0^{+\infty} \left(\frac{e^{-\frac{(x-u-\zeta)^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(x-u+\zeta)^2}{2t}}}{\sqrt{2\pi t}} \right) e^{\alpha(x-u-\zeta)} d\zeta = \\
&= 1 - \left[\tfrac{1}{2} \operatorname{Erfc}\left(\tfrac{x-u+mt}{\sqrt{2t}} \right) + \tfrac{e^{-2m(x-u)}}{2} \operatorname{Erfc}\left(\tfrac{x-u-mt}{\sqrt{2t}} \right) \right] \\
&= \Pr_{0,x\in(u,+\infty)} \left(\inf_{0\le s\le t} ms + W(s) > u \right)
\end{aligned} \tag{33}
$$

where we have used the fact that $\beta = -m^2/2$ and $\alpha = -m$. Similarly, when $x < l$ it can be shown with some algebra and using formula 1.1.4 page 197 in BS [4] that:

$$
\begin{aligned}
&e^{\alpha x+\beta t} \int_0^{+\infty} \left[\frac{e^{-\frac{(l-x-\zeta)^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(l-x+\zeta)^2}{2t}}}{\sqrt{2\pi t}} \right] e^{-\alpha(l-\zeta)} d\zeta = \\
&= 1 - \left[\tfrac{1}{2} \operatorname{Erfc}\left(\tfrac{l-x-mt}{\sqrt{2t}} \right) + \tfrac{e^{2m(l-x)}}{2} \operatorname{Erfc}\left(\tfrac{l-x+mt}{\sqrt{2t}} \right) \right] \\
&= \Pr_{0,x\in(-\infty,l)} \left(\sup_{0\le s\le t} ms + W(s) < l \right)
\end{aligned} \tag{34}
$$

In order to find the expression for $\Omega(t,\mu,x;u,l,m)$ in Theorem 1, we can use equation (30) and substituting it in (32), we obtain:

$$
\Omega(t,\mu,x;u,l,m) = \begin{cases}
\mathbf{1}_{(x>u)} e^{\alpha x+\beta t} \int_0^t \frac{(x-u)e^{-\frac{(x-u)^2}{2(t-\theta)}}}{\sqrt{2\pi (t-\theta)^3}} y(\theta,1)\, d\theta \\
\mathbf{1}_{(x<l)} e^{\alpha x+\beta t} \int_0^t \frac{(l-x)e^{-\frac{(l-x)^2}{2(t-\theta)}}}{\sqrt{2\pi (t-\theta)^3}} y(\theta,0)\, d\theta
\end{cases}
$$

and then if we consider the Laplace transform of the integrals we obtain the expression for $\omega(\gamma,\mu,x;l,u,m)$ in Theorem 1.

We now show how to find the density function of the occupation time given in equation (10) in Theorem 1. Using in (32) the fact that:

$$
\begin{aligned}
q(t) &= v(t,u;l,u) = \int_0^t e^{i\mu\theta} f_{\tau}(\theta,t,u)\, d\theta \\
p(t) &= v(t,l;l,u) = \int_0^t e^{i\mu\theta} f_{\tau}(\theta,t,l)\, d\theta
\end{aligned} \tag{35}
$$

we can observe that, for $x > u$, we have:

$$e^{\alpha(x-u)} \int_0^t \frac{(x-u)e^{-\frac{(x-u)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} e^{\beta(t-\tau)} q(\tau)\, d\tau =$$

$$= e^{\alpha(x-u)} \int_0^t \frac{(x-u)e^{-\frac{(x-u)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} e^{\beta(t-\tau)} \left(\int_0^\tau e^{i\mu\theta} f_\tau(\theta,\tau,u)\, d\theta \right) d\tau$$

$$= e^{\alpha(x-u)} \int_0^t e^{i\mu\theta} \left(\int_\theta^t \frac{(x-u)e^{-\frac{(x-u)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} e^{\beta(t-\tau)} f_\tau(\theta,\tau,u)\, d\tau \right) d\theta$$

and then:

$$v(t,x;l,u) =$$

$$= 1 \times \mathrm{Pr}_{0,x\in(u,+\infty)} \left(\inf_{0\le s\le t} ms + W(s) > u \right) +$$

$$+ e^{\alpha(x-u)} \int_0^t e^{i\mu\theta} \left(\int_\theta^t \frac{(x-u)e^{-\frac{(x-u)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} e^{\beta(t-\tau)} f_\tau(\theta,\tau,u)\, d\tau \right) d\theta$$

and so comparing with (3) the density function of the occupation when $x > u$ can be expressed in terms of the density function when $x = u$. Similarly, for $x < l$, we obtain:

$$v(t,x;l,u) =$$

$$= 1 \times \mathrm{Pr}_{0,x\in(-\infty,l)} \left(\sup_{0\le s\le t} ms + W(s) < l \right) +$$

$$+ e^{-\alpha(l-x)} \int_0^t e^{i\mu\theta} \left(\int_\theta^t \frac{(l-x)e^{-\frac{(l-x)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} e^{\beta(t-\tau)} f_\tau(\theta,\tau,l)\, d\tau \right) d\theta$$

Comparing these expressions with (3) the density function of the occupation when $x < l$ can be expressed in terms of the density function when $x = l$.

A.1.2 Case $l < x < u$

In this case the PDE (4) becomes:

$$-\frac{\partial v(t,x)}{\partial t} + \frac{1}{2}\frac{\partial^2 v(t,x)}{\partial^2 x} + m\frac{\partial v(t,x)}{\partial x} + i\mu v(t,x) = 0 \tag{36}$$

and has to be solved in the finite region $l < x < u$, with boundary conditions:

$$\begin{aligned} v(t,u) &= q(t) \\ v(t,l) &= p(t) \end{aligned}$$

and initial condition:

$$v(0,x) = 1$$

If we consider the transformation $y(t,z) = e^{-\alpha((u-l)z+l)-kt} v(t,(u-l)z+l)$, we get for the function $y(t,z)$ the heat equation in a finite region $0 < z < 1$,:

$$-\frac{\partial y(t,z)}{\partial t} + c^2\frac{\partial^2 y(t,z)}{\partial^2 z} = 0 \tag{37}$$

with boundary conditions:

$$y(t,1) = e^{-\alpha u - kt} q(t)\,;\; y(t,0) = e^{-\alpha l - kt} p(t)$$

and initial condition $y(0,z) = e^{-\alpha((u-l)z+l)}$. The solution can be found in Necati [19] (page 62, eq.2-73a). Then the expression for characteristic function is given by:

$$\begin{aligned} &v(t,x) = \\ &e^{i\mu t} 2 e^{\alpha x - \frac{\alpha^2}{2} t} \int_0^1 \left[\sum_{n=1}^{\infty} e^{-(cn\pi)^2 t} \sin(n\pi z) \sin(n\pi \xi) \right] e^{-\alpha(\xi(u-l)+l)} d\xi + \\ &+ e^{\alpha x + kt} \sum_{n=1}^{\infty} w_n(t) \sin\left(n\pi \left(\tfrac{x-l}{u-l}\right)\right) \end{aligned}$$

where:

$$w_n(t) = 2n\pi c^2 \int_0^t e^{-\lambda_n (t-s)} \left[e^{-\alpha l - ks} p(s) - (-1)^n e^{-\alpha u - ks} q(s) \right] ds \tag{38}$$

Using the properties of the theta function, compare Kevorkian [17] at page 25-26, and comparing with the expression in BS [4] (formula 1.15.4, page. 211) we obtain:

$$\begin{aligned} &2 e^{-mx - \frac{\alpha^2}{2} t} \int_0^1 \left[\sum_{n=1}^{\infty} e^{-(cn\pi)^2 t} \sin(n\pi z) \sin(n\pi \xi) \right] e^{m(\xi(u-l)+l)} d\xi = \\ &= \Pr_{0, x\in(l,u)} \left(\sup_{0\le s\le t} ms + W(s) < u;\; \inf_{0\le s\le t} ms + W(s) > l \right) \end{aligned} \tag{39}$$

In order to find the expression for $\Omega(t,x;u,l,m)$ when $x \in (l,u)$ in Theorem 1, we can use equation (30) and substituting it in (38), we obtain:

$$w_n(t) = 2n\pi c^2 \int_0^t e^{-\lambda_n (t-s) - i\mu s} \left[y(s,0) - (-1)^n y(s,1) \right] ds$$

and then:

$$\begin{aligned} &\Omega(t,\mu,x;u,l,m) = \\ &= e^{\alpha x + \beta t} \sum_{n=1}^{\infty} 2n\pi c^2 \sin\left(n\pi\left(\tfrac{x-l}{u-l}\right)\right) \int_0^t e^{-(\lambda_n - i\mu)(t-s)} \left(y(s,0) - (-1)^n y(s,1)\right) ds \end{aligned}$$

If we consider the Laplace transform of the series we get:

$$\begin{aligned} &\omega(\gamma,\mu,x;l,u,m) = \\ &= \frac{1}{\pi^2 c^2} \sum_{n=1}^{\infty} \frac{2n\pi c^2}{n^2 + \frac{\gamma - i\mu}{\pi^2 c^2}} \sin\left(n\pi\left(\tfrac{x-l}{u-l}\right)\right) \left(\mathcal{L}\left[y(t,0); t\to\gamma\right] - (-1)^n \mathcal{L}\left[y(t,1); t\to\gamma\right]\right) ds \end{aligned}$$

and using the summation formulas (25) and (26) we obtain the expression in Theorem 1.

We now show how to find the density function of the occupation time given in equation (10) in Theorem 1. Substituting in expression (38) the functions $q(t)$ and $p(t)$ as given in (35), we have:

$$\begin{aligned} &e^{\alpha x + kt} w_n(t) = \\ &= -2n\pi c^2 \int_0^t e^{\left(k - (n\pi c)^2\right)(t-s)} \times \\ &\times \left[(-1)^n e^{-\alpha(u-x)} \int_0^s e^{i\mu\theta} f_\tau(\theta,s,u) - e^{\alpha(x-l)} \int_0^s e^{i\mu\theta} f_\tau(\theta,s,l)\, d\theta \right] ds \end{aligned}$$

With a change of variable, $(\xi = t - s, \nu = t + \tau - s = \xi + \tau)$, and using the fact that $k = -\alpha^2/2 + i\mu$, we get:

$$= -2n\pi c^2 \int_0^t e^{-\left(\frac{\alpha^2}{2}+(n\pi c)^2\right)\xi} \times$$
$$\times \int_\xi^t \left[(-1)^n e^{-\alpha(u-x)} e^{i\mu\theta} f_\tau(\theta-\xi, t-\xi, u) - e^{\alpha(x-l)} e^{i\mu\theta} f_\tau(\theta-\xi, t-\xi, l)\right] d\theta d\xi$$
$$= 2n\pi c^2 \int_0^t e^{i\mu\theta} \times$$
$$\times \int_0^\theta e^{-\left(\frac{\alpha^2}{2}+(n\pi c)^2\right)\xi} \left(e^{\alpha(x-l)} f_\tau(\theta-\xi, t-\xi, l) - (-1)^n e^{-\alpha(u-x)} f_\tau(\theta-\xi, t-\xi, u)\right) d\xi d\theta$$

So for a generic starting point $x \in (l, u)$ we have:

$$v(t, x; l, u) =$$
$$= e^{i\mu t} \Pr_{0, x\in(l,u)} \left(\sup_{0\le s\le t} ms + W(s) < u; \inf_{0\le s\le t} ms + W(s) > l \right) +$$
$$+ \int_0^t e^{i\mu\theta} \left[\sum_{n=1}^\infty 2n\pi c^2 \sin\left(n\pi\left(\tfrac{x-l}{u-l}\right)\right) \int_0^\theta e^{-\left(\frac{\alpha^2}{2}+n^2\pi^2c^2\right)\xi} \times \right.$$
$$\left. \times \left(e^{\alpha(x-l)} f_\tau(\theta-\xi, t-\xi, l) - (-1)^n e^{-\alpha(u-x)} f_\tau(\theta-\xi, t-\xi, u)\right) d\xi \right] d\theta$$

and so we recognize in the square brackets inside the integral the density function of the occupation when $l < x < u$, as shown in Theorem 1.

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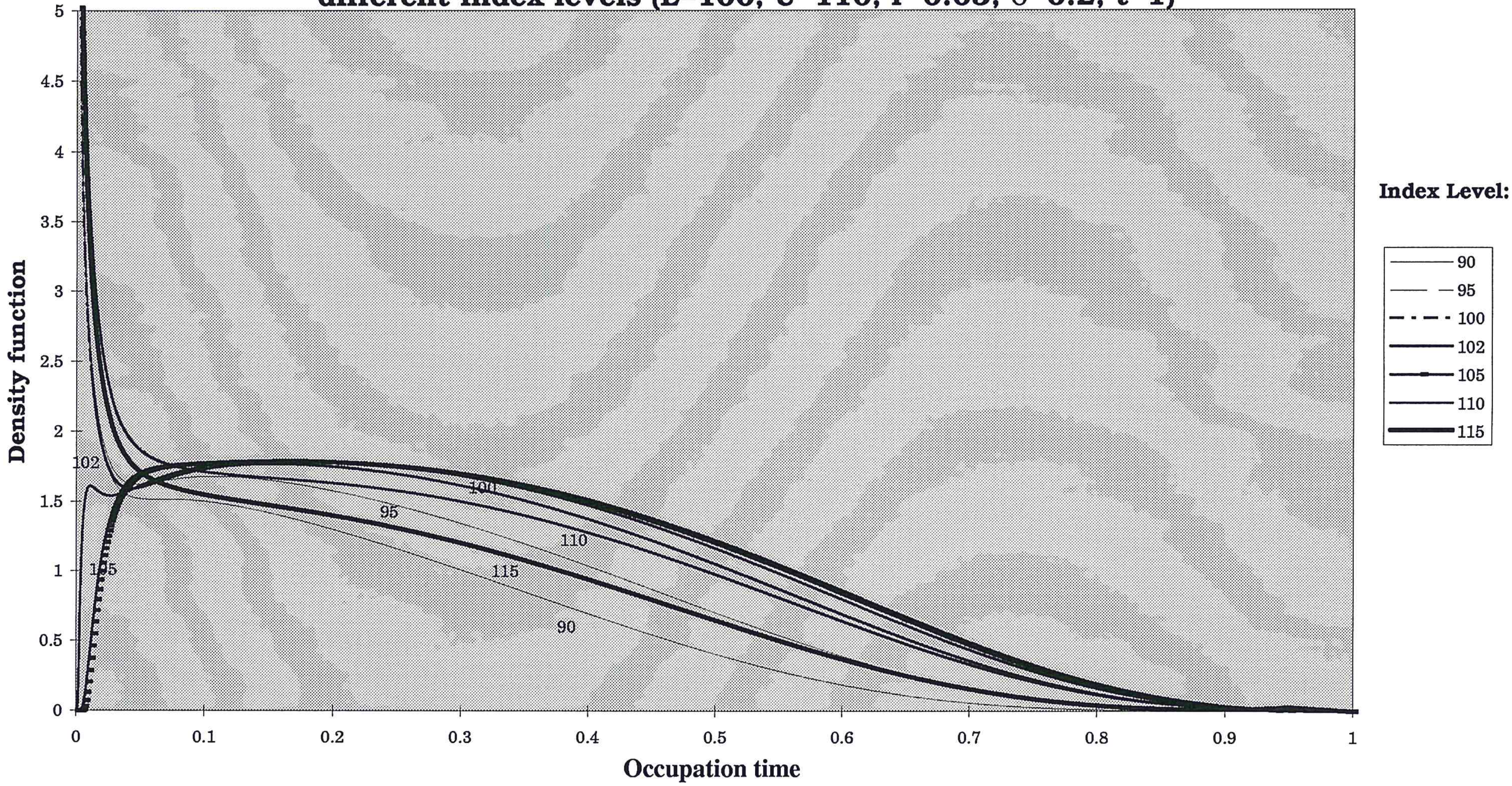
Figure 1: Density of the occupation time for different index levels (L=100, U=110, r=0.05, σ=0.2, t=1)



Price of the corridor option for different strike prices and index levels: l=100, u=110, σ=0.2, r=0.05, t=1

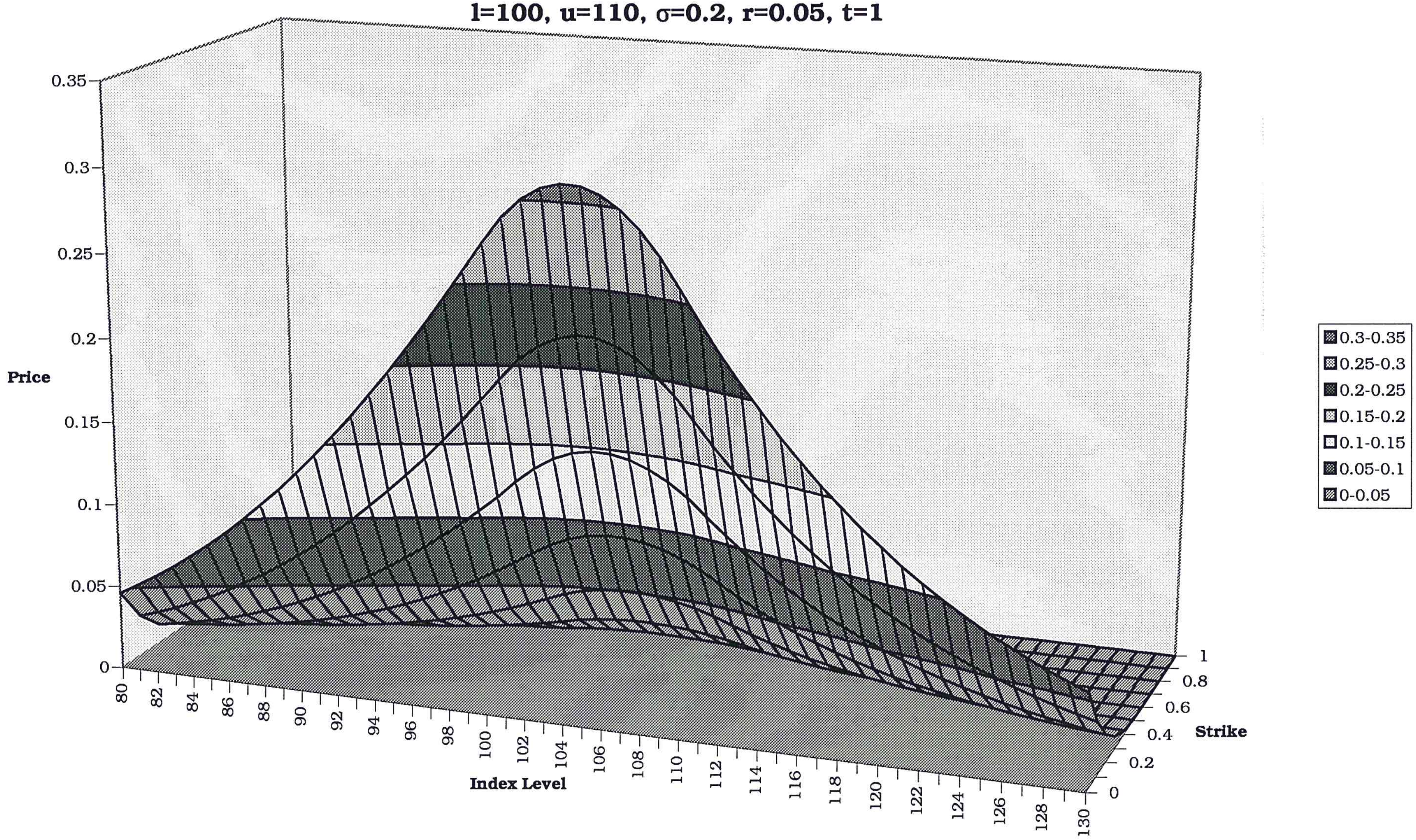


Figure 2: Delta of the corridor option for different strike prices and index levels (L=100, U=110, s=0.2, r=0.05, t=1)

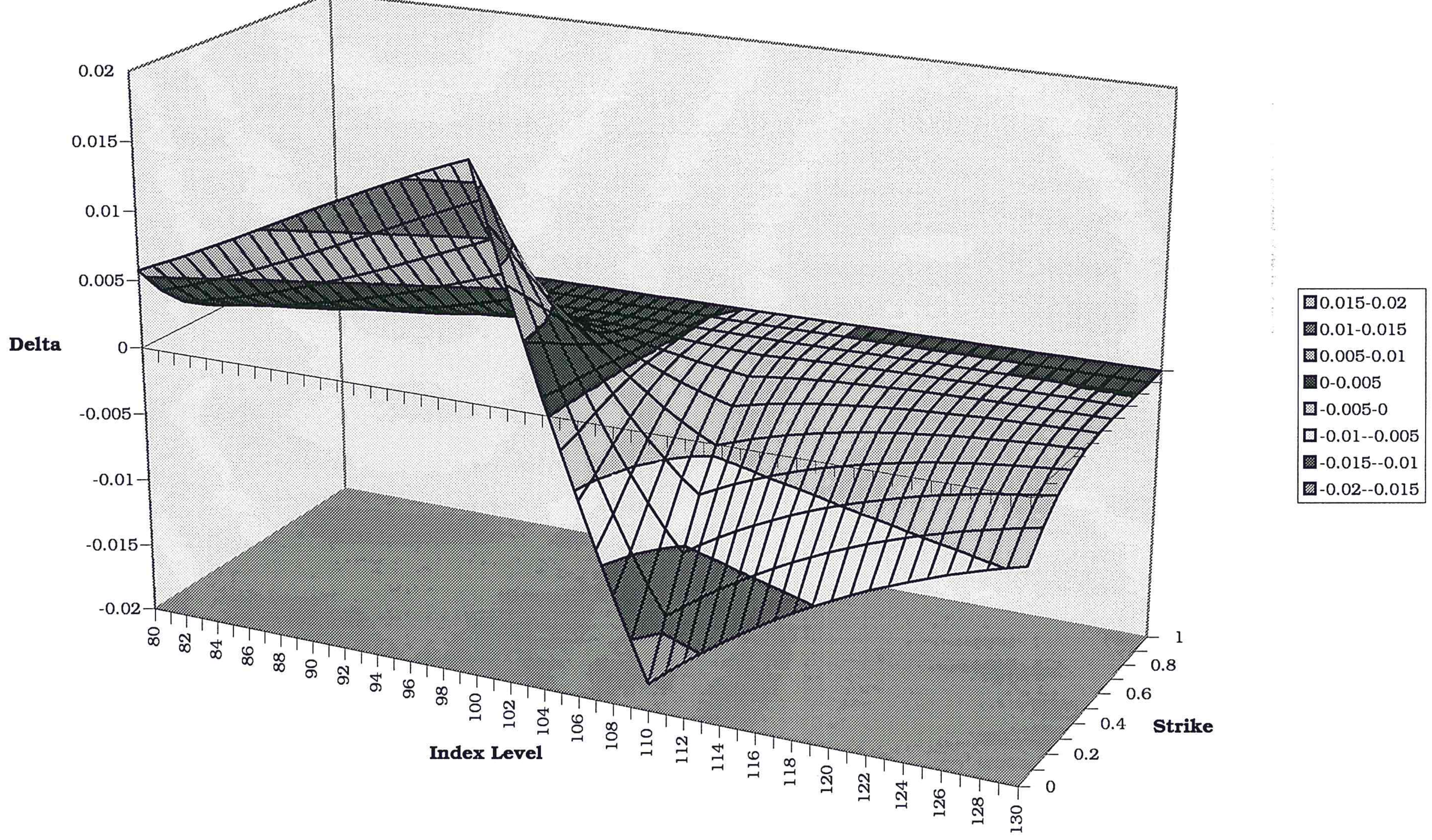


Figure 3: Occupation time: expected value
l=-1, u=1, t=1

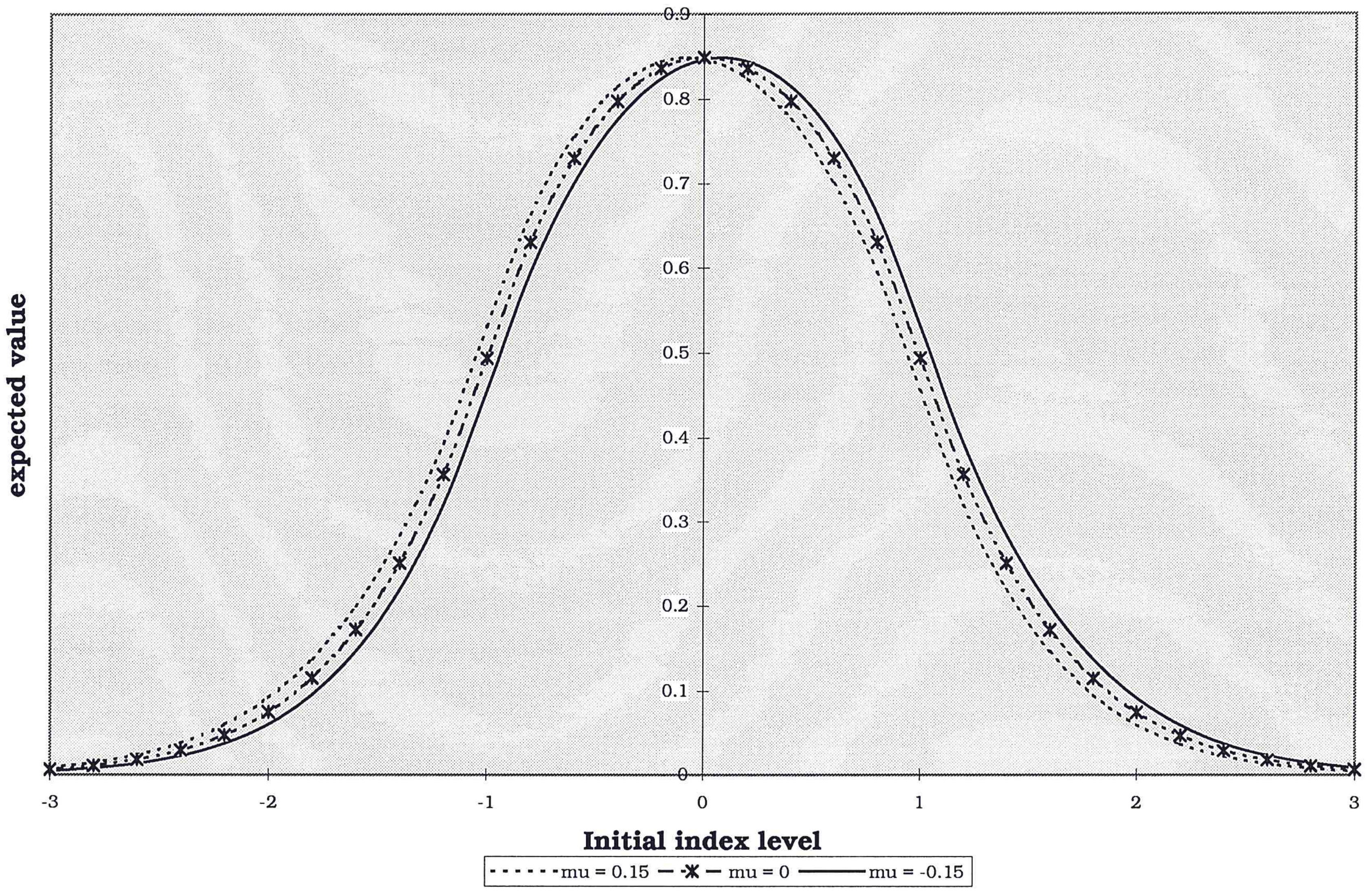


Figure 4: Occupation time: fraction of time spent inside the barriers
l=-1, u=1, drift 0.15

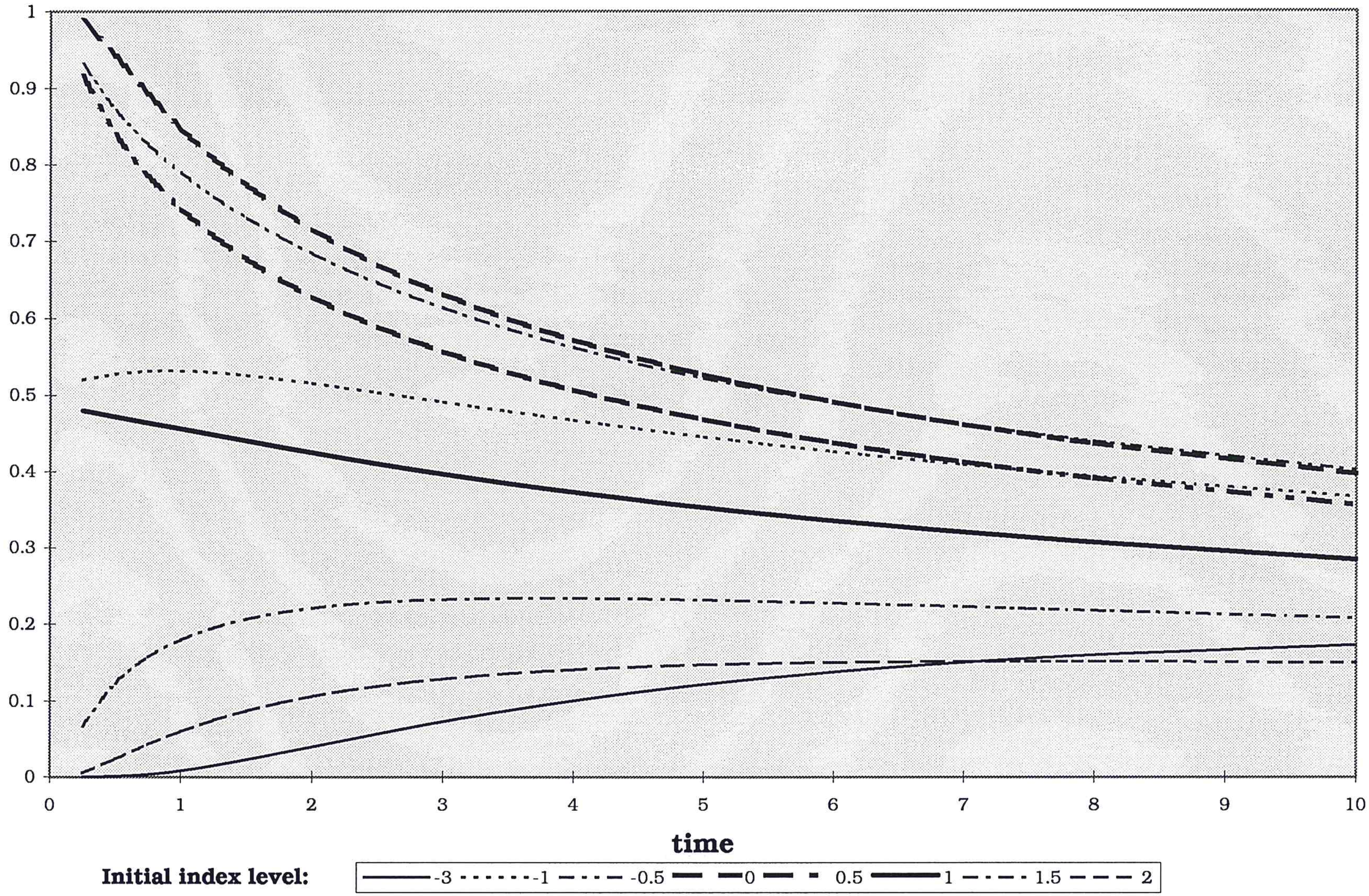


Figure 5: Occupation time: standard deviation
l=-1, u=1, t=1

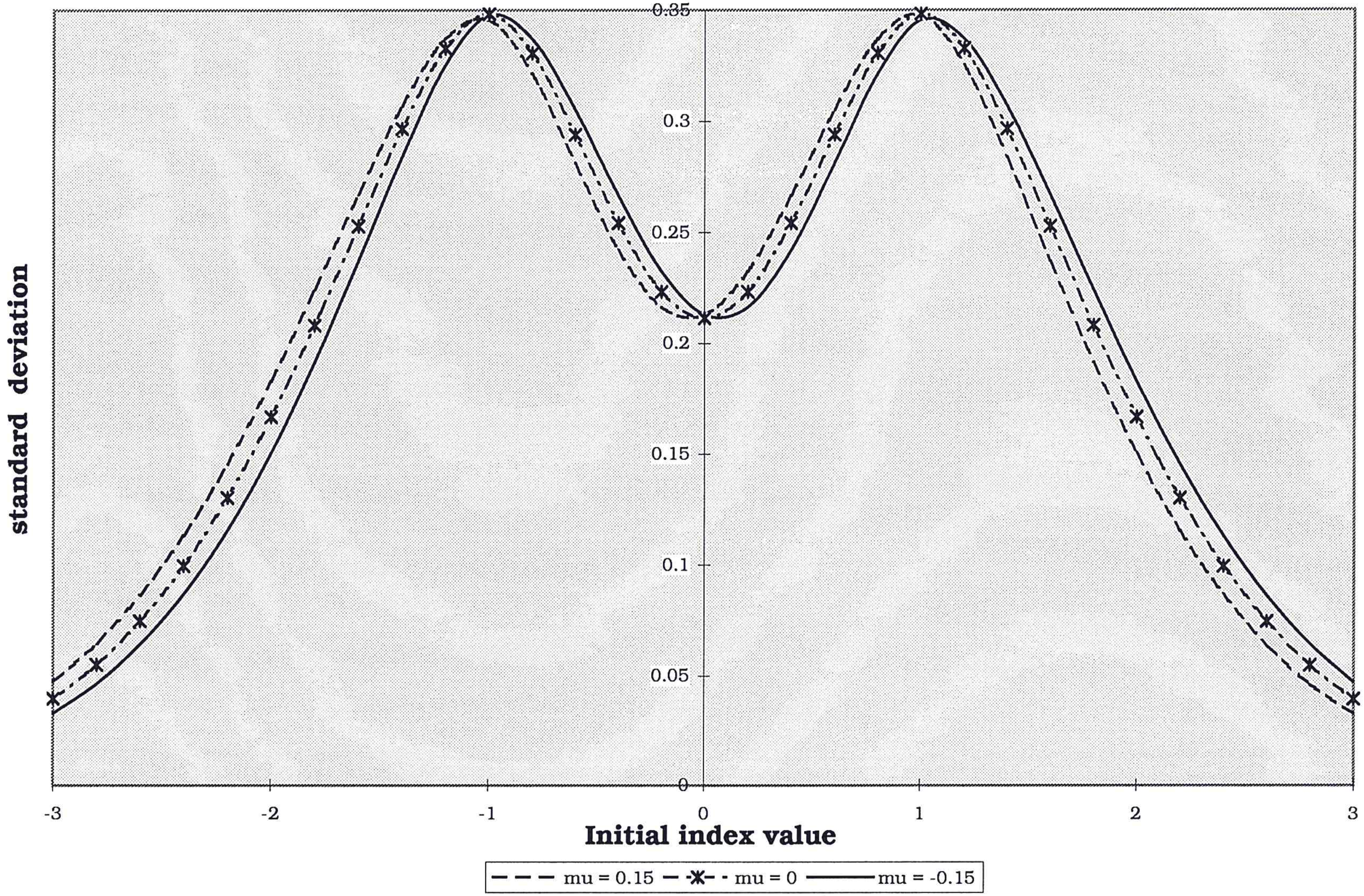


Figure 6: Occupation time: standard deviation of the fraction of time spent inside the barriers (l=0, u=1, drift = 0.15)