FINANCIAL OPTIONS RESEARCH CENTRE

University of Warwick

A Computational Framework for Contingent Claim Pricing and Hedging under Time Dependent Asset Processes

Russell Grimwood
and
Les Clewlow

September 2000

Financial Options Research Centre
Warwick Business School
University of Warwick
Coventry
CV4 7AL
Phone: (0)24 76 524118

FORC Preprint: 2000/109

A Computational Framework for Contingent Claim Pricing and Hedging under Time Dependent Asset Processes *†

Russell Grimwood ‡ Les Clewlow §

The Financial Options Research Centre
Warwick Business School
University of Warwick
Coventry CV4 7AL
United Kingdom

January 1999

Abstract

This paper proposes a general computational framework for pricing exotic options in the presence of volatility smiles. The paper extends and unifies Dupire's (1994) work on implied trinomial trees with fully explicit, fully implicit and Crank Nicolson finite difference methods. We investigate the computational efficiency of this new framework for pricing exotic options. An American barrier option is priced as an example of applying the framework to complex exotic options.

Key Words: Finite difference, implied trees and volatility smiles.

JEL classification: G13

---

*This paper forms a chapter in the book *Computational Finance 1999*, edited by Y. Abu-Mostafa, B. LeBaron, A. Lo and A. Weigend and published by MIT Press (2000) ISBN 0-262-51107-X.

†We would like to thank Dr Silio Aparicio, Professor Stewart Hodges, Dr Stewart Perry, George Skiadopoulos, Dr Christopher Strickland and Daniel Wilcox for their helpful discussions. All errors remain our own.

‡Corresponding author. Research fellow at the Financial Options Research Centre. Funding for the author was provided by the UK government Engineering and Physical Sciences Research Council grant GR/L13025. Phone +44 (0)24 7652 2701, Fax +44 (0)24 7652 4167 and email `forcrg@wbs.warwick.ac.uk`

§Associate research fellow at the Financial Options Research Centre and a principal of Lacima Consultants. Email `Les_Clewlow@compuserve.com`

1 A Computational Framework for Contingent Claim Pricing and Hedging under Time Dependent Asset Processes

Russell Grimwood and Les Clewlow

This work proposes a general computational framework for pricing exotic options in the presence of volatility smiles. The work extends and unifies Dupire's work on implied trinomial trees with fully explicit, fully implicit and Crank Nicolson finite difference methods. We investigate the computational efficiency of this framework with the example of pricing an American barrier option.

1.1 Introduction: Trees Consistent with Observed Market Volatilities

The objective of this work is to outline a new robust algorithm which is capable of pricing exotic options in a finite difference framework consistent with market volatilities. A current popular approach to pricing exotic options consistent with market volatilities is to use a recombining implied tree. Barle and Cakici [Barle and Cakici 1995] review three methods in the recent literature which take different approaches to this problem: Rubinstein [Rubinstein 1994], Derman-Kani [Derman and Kani 1994] and Dupire [Dupire 1994]. Rubinstein's approach uses only the terminal set of option maturities observed in the market and so does not incorporate all available information. Derman-Kani build a binomial tree[1] which has the disadvantage of not being able to keep a fixed asset level due to the limited number of degrees of freedom[2]. Dupire uses a trinomial tree; this permits him to specify different local volatilities and drifts at each node. Unlike Rubinstein's method the procedure suggested by Dupire uses all the observed European[3] option prices and not just the terminal values, thus it incorporates more market information. Because of its advantages we use a trinomial tree based on the method sketched out by Dupire as our starting point. Once an implied tree for a given set of input options is built, it can be used repeatedly for pricing options other than those from the input set. The tree has to be rebuilt only when the volatility smile evolves, however as the smile shape changes daily the frequency of rebuilding the tree must be controlled carefully by the practitioner.

The work is organised as follows: In Section 1.2, we review the work of Dupire

1. They were later to use trinomial trees themselves, Derman-Kani-Chriss [Derman et al 1996]

2. Their algorithm reproduces the volatility smile accurately but fails if interest rates are high.

3. We can use American style options if we use a model which explicitly takes account of the early exercise premium. Skiadopoulos, Hodges and Clewlow [Skiadopoulos et al 1998] use Barone-Adesi and Whaley's [Barone-Adesi and Whaley 1987] quadratic approximation model. Neil Chriss [Chriss 1996] uses another iterative procedure each step of which demands the valuation of an American option on a binomial tree.

on implied trinomial trees. We propose an improvement to Dupire's sequential method for calculating the state prices and transition probabilities, utilising matrix algebra, which is more robust and less susceptible to build up of numerical errors. Section 1.3 shows how the implied trinomial tree model can be extended to the θ-method to solve the underlying partial differential equation (PDE). The θ-method has nested within it the fully explicit, fully implicit and Crank-Nicolson finite difference methods. The computational efficiency and accuracy of these methods are compared in Section 1.4 for the example of pricing American barrier options. Finally we present our conclusions in Section 1.5.

1.1.1 The Scope of the Pricing Framework

Our framework is an extension to that proposed by Hull and White [Hull and White 1993]. They have a procedure to price a general path dependent option in a Cox, Ross and Rubinstein [Cox et al 1979] binomial tree. Our extension uses finite difference methods and takes into account information about the market's expectation of future volatility. The framework can handle American and European style path dependent derivatives as long as certain conditions are satisfied[4]. For an illustration of how to apply Hull and White's procedure to Asian and look-back options in trinomial trees see Clewlow and Strickland [Clewlow and Strickland 1998]. The framework can be viewed as an engine which can be used for the pricing, hedging and risk management of a portfolio of exotic options based on a common underlying asset like the S&P 500 or FTSE 100 indices.

1.2 Constructing Implied Trinomial Trees

1.2.1 The Tree Framework

The trinomial tree[5] was constructed in the natural logarithm of the underlying asset price x and time t. Where the underlying asset price was represented by S and therefore $x = ln(S)$, with time steps Δt_i and underlying price steps $\Delta x_{i,j}$. The node (i, j) in the tree corresponds to the time $t_i = \sum_{i=0}^{i} \Delta t_i$ from today and to

4. Two conditions must be satisfied: firstly that the payoff from the derivative depends on only a single function, F, of the path followed by the underlying stochastic variable, and secondly that the value of F at time $t + \Delta t$ can be calculated from the value of F at time t and the value of the underlying asset at time $t + \Delta t$.

5. The procedure for building the implied tree outlined in the following subsections is based on Chapter 5 of Clewlow and Strickland [Clewlow and Strickland 1998].

the asset price $S_{i,j} = S_{0,0} \exp\left(\sum_{k=0}^{j} \Delta x_{i,k}\right)$ where $\Delta x_{i,1}$ was the distance between the central asset price at $(i,0)$ and at $(i,1)$ and where $S_{0,0}$ was the initial value of the underlying asset, i.e. the value at the root $(0,0)$ of the tree. The underlying asset price was calculated for the whole of the trinomial tree in one forward sweep. Figure 1.1 shows the construction of an implied trinomial tree with N time steps and $2Nj+1$ asset price steps at maturity. For clarity we simplify this general framework by fixing the values of $\Delta x_{i,j}$ and Δt_i to be constant, i.e. Δx and Δt. The asset price at node (i,j) is then given by $S_{0,0}e^{j\Delta x}$. We then use the relationship between the time steps and the asset price steps recommended by Hull and White [Hull and White 1990][6],

$$\Delta x = \sigma_{max}\sqrt{3\Delta t}, \tag{1.1}$$

where the value of σ_{max} is the highest implied volatility of any of the European options in the tree[7]. If the market smiles are not too large it should be possible to use the relationship without generating negative transition probabilities[8].

1.2.2 The Diffusion Process

The implied tree is a discrete approximation of the following stochastic differential equation,

$$\frac{dS}{S} = (r(t) - \delta(S,t))\,dt + \sigma(S,t)dz, \tag{1.2}$$

$r(t)$ is the expected risk neutral rate of return, $\delta(S,t)$ is the dividend yield rate, dz is an increment in a Wiener process with a mean of zero and a variance equal to dt and $\sigma(S,t)$ is the local volatility function which is dependent on both the underlying price and time. The functional form of $\sigma(S,t)$ is inferred by requiring that option prices, calculated from the tree, fit the smile[9]. For ease of exposition

6. Hull and White point out that there are some theoretical advantages to choosing the above relationship: it ensures both stability and convergence. A stable procedure is one where the results are relatively insensitive to roundoff and other small computational errors. Convergence ensures that as Δt, $\Delta S \to 0$ the estimated value of the derivative security converges on its true continuous time value.

7. This is the value of σ_{max} which Dupire uses to control the openness of the tree; by using the maximum implied volatility we ensure that the asset price steps are large enough to allow for the local variance of the underlying process. This prevents negative transition probabilities.

8. Avoiding negative transition probabilities is one of the most difficult aspects of successfully implementing an implied trinomial tree for contingent claim pricing.

9. However, the form of $\sigma(S,t)$ is not unique as an infinite number of processes can produce the same probability density.

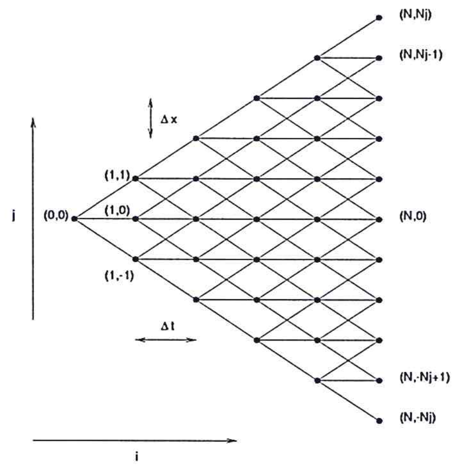


Figure 1.1
The implied trinomial tree structure.

through out this work we will assume that the interest rate is constant and the dividend yield is zero.

1.2.3 Interpolation of Option Prices

Out-of-the-money (OTM) European call prices were used to construct the upper half of the trinomial tree and OTM European put prices to construct the lower half[10]; the central row of nodes were arbitrarily chosen to be call options. Each node of the tree was labelled with an option price $C(i\Delta t, K)$, the value of which corresponded to today's value (i.e. with an underlying asset price at $S_{0,0}$) of a European option, with maturity of $i\Delta t$ and a strike price of K.

As in the papers by Dupire, Derman-Kani and Rubinstein it has been assumed that market option prices exist for all the maturities and strike prices corresponding to every node in the tree; in reality this is not the case. What we require is a smooth function of option prices against strike prices and maturity, so that for any strike price and maturity an option price can be recovered. To get smooth functions one must interpolate the available option prices. Shimko [Shimko 1993] states that the

10. Aparicio and Hodges [Aparicio and Hodges 1996] recommended using OTM and at-the-money (ATM) calls and puts because this avoids the measurement problems which arise with in-the-money (ITM) options. ITM calls and puts have a high delta relative to their option prices therefore their implied volatilities are sensitive (have large confidence intervals) to synchronisation differences between the underlying asset and option markets.

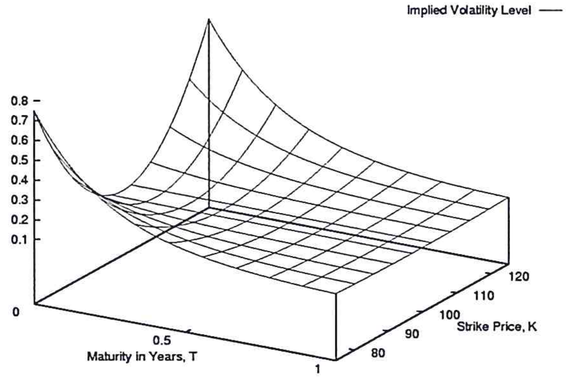


Figure 1.2
Functional form representation of a volatility smile.

prices recovered from interpolation are very sensitive to the choice of interpolation method and that smoother functions can be recovered by interpolating the implied volatility figures rather than option prices themselves. The Black-Scholes formula is used to translate back and forth between interpolated implied volatilities and option prices. The option prices for the whole tree need only be computed once.

For simplicity (and as we are primarily interested in the properties of the algorithm) we have used a mathematical function to approximate the implied volatility surface. Figure 1.2 shows a function which is quadratic across strike prices with the function curvature decreasing with increasing time to maturity[11].

1.2.4 Calculating the Implied State Prices

With each node of the trinomial tree we associate a state price $Q_{i,j}$ this is the price today of an instrument which pays 1 unit of cash if the state (i,j) is reached, and zero otherwise. To calculate the state price at each node in the tree we use a method

11. See Taylor and Xu [Taylor and Xu 1993] for a theoretical and empirical discussion of the magnitude of implied volatility smiles.

which involves varying the strike price K. The state prices at the ith time step are related to the price of a European call and its payoff by the following equation,

$$C(i\Delta t, K) = \sum_{j=-i}^{i} Q_{i,j} \max(S_{i,j} - K, 0). \tag{1.3}$$

The same equation applies for the puts, except we make the relevant change to the option payoff, $\max(K - S_{i,j}, 0)$. For a set of strike prices $K_l = S_{i,l}$, where $l = -i+1, \dots, i-1$, Equation 1.3 gives a set of linear equations which we solve to determine the state prices[12]. Writing Equation 1.3 in more compact matrix and vector form,

$$\bar{c_i} = \bar{S_i} \times \bar{q_i}, \tag{1.4}$$

where $\bar{c_i}$ is a vector of call and put prices for each node in the column at the ith time step, $j = 0, \dots, i$ are calls and $j = -1, \dots, -i$ are puts, $\bar{S_i}$ is the matrix of option payoffs. The payoffs are calculated by starting at the top (bottom) of a column of nodes and setting the strike price in the call (put) payoff function to match that of the underlying asset value at the node below (above) $S_{i,j-1}$ ($S_{i,j+1}$), this collapses onto a single node in the continuous time framework. Finally we have a vector of state prices $\bar{q_i}$ which we wish to determine. Rewriting Equation 1.4 we obtain an expression for the state prices vector,

$$\bar{q_i} = \bar{S_i}^{-1} \times \bar{c_i}, \tag{1.5}$$

which is equal to the inverse of the payoff matrix multiplied by the option prices vector. The corresponding matrix calculation is repeated at each time step to retrieve all the state prices in the tree. The solution to Equation 1.5 was determined using LU decomposition - see Press, Teukolsky, Vetterling and Flannery [Press et al 1995].

Dupire uses a sequential method to solve for the state prices in Equation 1.3. Starting at the nodes on the edge of the tree he moves towards the centre solving for the state prices. This whole process is then repeated at every time step.

1.2.5 Calculating the Transition Probabilities

Except at maturity, each node (i,j) has three branches: up, middle and down which have associated transition probabilities $p_{u,i,j}$, $p_{m,i,j}$ and $p_{d,i,j}$. The three unknown probabilities are determined by making use of three local no-arbitrage

12. State prices can be also be recovered directly from butterfly spreads.

relationships[13]. Two of these are standard backward relationships, which link the value of a claim at a node to its value at the immediate successors. The first is the price of a Δt maturity pure discount bond (i.e. a bond which pays 1 unit of cash at Δt),

$$1 = p_{u,i,j} + p_{m,i,j} + p_{d,i,j}. \tag{1.6}$$

The second relationship is the price of the underlying asset $S_{i,j}$,

$$S_{i,j} = S_{i+1,j+1}p_{u,i,j} + S_{i+1,j}p_{m,i,j} + S_{i+1,j-1}p_{d,i,j}. \tag{1.7}$$

The third relationship makes use of a technique used by Jamshidian [Jamshidian 1991], called forward induction. Here the state price of a node is related to the state prices of its immediate predecessors,

$$Q_{i+1,j} = Q_{i,j-1}p_{u,i,j-1} + Q_{i,j}p_{m,i,j} + Q_{i,j+1}p_{d,i,j+1}. \tag{1.8}$$

To determine the transition probabilities we can again use matrix algebra. However for transition probabilities this is more complicated than for the state prices as we have to solve three sets of equations rather than one. Figure 1.3 shows the matrix form of a set of linear equations equivalent to the sequential Equations 1.6, 1.7 and 1.8. This can be rewritten more compactly with matrix and vector notation,

$$\bar{v_i} = \bar{M_i} \times \bar{p_i}, \tag{1.9}$$

where the first vector is $\bar{v_i}$, the sparse matrix is $\bar{M_i}$ and the transition probability vector is $\bar{p_i}$. Rearranging this we get an expression in terms of the transition probability vector such that,

$$\bar{p_i} = \bar{M_i}^{-1} \times \bar{v_i}. \tag{1.10}$$

We use LU decomposition once more to solve Equation 1.10 which has a band diagonal structure. For band diagonal matrices it is more efficient to use LU decomposition than to invert the matrix and then to store the inverted matrix in memory[14]. This calculation is repeated for each column of nodes to determine the transition probabilities for the whole tree.

---

13. For clarity we will use the bank account as numeraire to avoid cluttering our equations with discount factors.

14. An inverted band diagonal or tridiagonal matrix will not be a sparse matrix and it will therefore require more memory space. Sparse matrices like the ones we will encounter throughout this work can be held efficiently in memory by just storing the non-zero elements and their co-ordinates - see again Press *et al.*

$$
\begin{pmatrix} Q_{i,i} \\ S_{i-1,i-1} \\ 1 \\ Q_{i,i-1} \\ S_{i-1,i-2} \\ 1 \\ Q_{i,i-2} \\ \vdots \\ Q_{i,-i+1} \\ 1 \\ S_{i-1,-i+1} \\ Q_{i,-i} \end{pmatrix}
= \begin{pmatrix}
Q_{i-1,i-1} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
S_{i,i} & S_{i,i-1} & S_{i,i-2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & Q_{i-1,i-1} & 0 & Q_{i-1,i-2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & S_{i,i-1} & S_{i,i-2} & S_{i,i-3} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & Q_{i-1,i-1} & 0 & Q_{i-1,i-2} & 0 & Q_{i-1,i-3} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & Q_{i-1,-i+2} & 0 & Q_{i-1,-i+1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & S_{i,-i+2} & S_{i,-i+1} & S_{i,-i} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & Q_{i-1,-i+1}
\end{pmatrix}
\times
\begin{pmatrix} p_{u,i-1,i-1} \\ p_{m,i-1,i-1} \\ p_{d,i-1,i-1} \\ p_{u,i-1,i-2} \\ p_{m,i-1,i-2} \\ p_{d,i-1,i-2} \\ p_{u,i-1,i-3} \\ \vdots \\ p_{d,i-1,-i+2} \\ p_{u,i-1,-i+1} \\ p_{m,i-1,-i+1} \\ p_{d,i-1,-i+1} \end{pmatrix}
$$

Figure 1.3
Representation of the system of linear equations used to recover the transition probabilities in an implied trinomial tree; the band diagonal structure of the matrix is clear.

Dupire uses an sequential method to calculate the implied transition probabilities working from edge of the tree down a column of nodes. The transition probabilities calculated using the sequential method depend on the values of the previous set of transition probabilities. A problem can therefore arise with the build up of roundoff errors. Dupire suggests it is better to work from the two edges of the tree towards the middle (rather than from the top all the way to the bottom or vice versa) because this helps limit the error size. Despite this the sequential method produces errors which grow in size towards the centre of the tree. In contrast the LU decomposition method from Press *et al* which we choose to use to solve the system of linear equations uses partial pivoting[15] of the matrix rows which prevents the build up of roundoff errors caused by the finite precision of computer calculations. It is partial pivoting which gives this method superior stability in the presence of roundoff error compared to a naive sequential method without partial pivoting. Figure 1.4 shows the relative error in the solution of a set of linear equations. The flat line is a comparison between LU decomposition and Gauss-Jordan elimination both with

15. Partial pivoting involves the interchange of matrix rows so that the highest magnitude element is in the diagonal position.

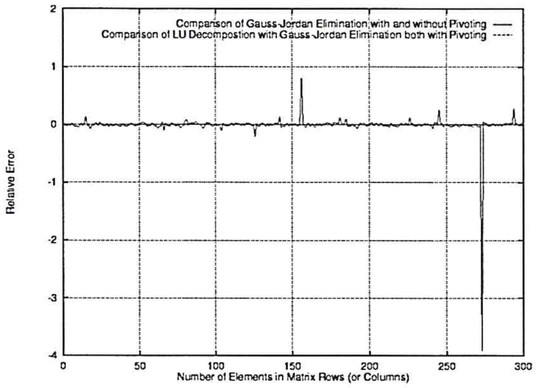


Figure 1.4
Roundoff errors in the solutions to a set of linear equations caused by using a method which does not use pivoting.

pivoting. It demonstrates that both methods produce numerically similar results as the relative errors are not visible at the scale shown. However, if we compare the Gauss-Jordan method with and without pivoting the instability of solving a set of linear equations without pivoting becomes clear. Large relative errors become visible which would produce errors in the transition coefficients. In Dupire's method this could cause negative transition probabilities.

Once the transition probabilities have been calculated we can price a target exotic option in a framework consistent with the market implied volatilities. The target option's payoff at maturity is determined for each level of the underlying asset $S_{N,j}$ at the final set of nodes (N, j). We then discount these option prices back to the next time step using the transition probabilities. This process is repeated for each time step until we have recovered the option's price today at $t = 0$. Computing standard dynamic hedge sensitivities proceeds along the usual lines for lattices.

1.3 The θ-Method: Recovering Implied Transition Coefficients Using Finite Difference

Starting with the Black-Scholes PDE,

$$\frac{\partial C(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} + rS\frac{\partial C(S,t)}{\partial S} = rC(S,t), \tag{1.11}$$

we use the natural logarithm of the underlying asset price i.e., $S = ln(x)$ and $W(x,t) = C(S,t)$ to transform the PDE so that we have constant coefficients for the partial derivatives,

$$\frac{1}{2}\sigma^2 \frac{\partial^2 \mathrm{W}(x,\,t)}{\partial x^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial \mathrm{W}(x,\,t)}{\partial x} + \frac{\partial \mathrm{W}(x,\,t)}{\partial t} = r\,\mathrm{W}(x,\,t). \tag{1.12}$$

In finite difference methods we replace the partial derivatives with difference equations[16]. Depending on whether the difference equations are centred around time step $i+1$, i or $i+\frac{1}{2}$ determines whether the finite difference method is fully explicit, fully implicit[17] or Crank-Nicolson[18] respectively, see Figure 1.5. These three finite difference methods are nested within the θ-method. The Black-Scholes difference equation for the θ-method is,

$$\begin{aligned}
1/2\,&\frac{\sigma^2\left(W_{i+1,j+1} - 2\,W_{i+1,j} + W_{i+1,j-1}\right)}{(\Delta x)^2}(1-\theta)\\
&+ 1/2\,\frac{\left(r - 1/2\,\sigma^2\right)\left(W_{i+1,j+1} - W_{i+1,j-1}\right)}{\Delta x}(1-\theta)\\
&+ \frac{W_{i+1,j} - W_{i,j}}{\Delta t} - rW_{i+1,j}\,(1-\theta)\\
&= 1/2\,\frac{\sigma^2\left(W_{i,j+1} - 2\,W_{i,j} + W_{i,j-1}\right)}{(\Delta x)^2}\theta
\end{aligned}$$

---

16. For the partial derivative with respect to time we use a forward difference equation,

$$\frac{\partial W(x,t)}{\partial t} \approx \frac{W_{i+1} - W_i}{\Delta t} + O(\Delta t), \tag{1.13}$$

for the first order partial derivative with respect to asset price we use a central difference equation,

$$\frac{\partial W(x,t)}{\partial x} \approx \frac{W_{j+1} - W_{j-1}}{2\Delta x} + O((\Delta x)^2), \tag{1.14}$$

and for the second order partial derivative with respect to asset price we use a symmetric central difference equation,

$$\frac{\partial W(x,t)}{\partial x^2} \approx \frac{W_{j+1} - 2W_j + W_{j-1}}{(\Delta x)^2} + O((\Delta x)^2). \tag{1.15}$$

The $O(\cdot)$ terms in the above difference equations represent the rest of the Taylor series expansion. Henceforth we will truncate the expansions by not including these terms. The smaller the Δt and Δx terms become (i.e., the finer the discretization) the more accurate our approximations become, that is to say the truncation errors become smaller.

17. The explicit and implicit finite difference frameworks were first used in option pricing by Brennan and Schwartz [Brennan and Schwartz 1978]

18. The Crank Nicolson method was first introduced into the contingent claim pricing literature by Courtadon [Courtadon 1982]

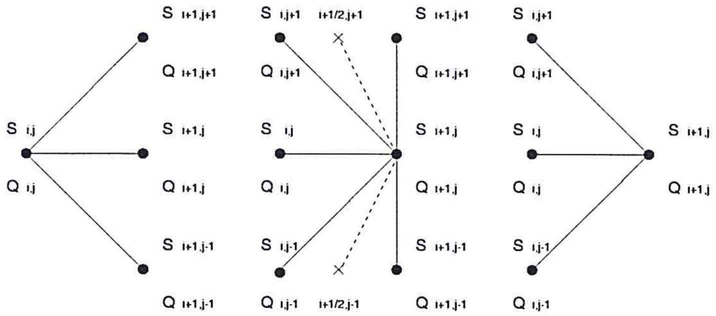


Figure 1.5
Fully explicit element centred around $i+1, j$, Crank-Nicolson element centred around $i+\frac{1}{2}, j$ and a fully implicit element centred around i, j.

$$+\,1/2\,\frac{\left(r-1/2\,\sigma^2\right)\left(W_{i,j+1}-W_{i,j-1}\right)}{\Delta x}\theta+rW_{i,j}\theta, \tag{1.16}$$

where $0 \leq \theta \leq 1$. Wilmott, Dewynne and Howison [Wilmott et al 1993] point out that this can be thought of as a θ weighted average of the explicit and fully implicit finite difference methods. When $\theta = 0$, it gives the explicit method, when $\theta = \frac{1}{2}$ it gives the Crank-Nicolson method and when $\theta = 1$ it gives the implicit method. The fully explicit finite difference method has the disadvantage that it is only stable and convergent with the imposition of the restriction $\Delta x \geq \sigma\sqrt{3\Delta t}$. The restriction implies that in order to have the large number of asset price steps necessary for accurate prices ridiculously small time steps are required. The accuracy of the fully explicit and fully implicit methods is $O((\Delta x)^2 + \Delta t)$ whereas the Crank-Nicolson method has a superior accuracy of $O\left((\Delta x)^2 + \left(\frac{\Delta t}{2}\right)^2\right)$. As it has superior convergence and is unconditionally stable our exposition will continue using the Crank-Nicolson method as our example.

Setting $\theta = \frac{1}{2}$ in equation 1.16 gives,

$$\begin{aligned}
&1/4\,\frac{\sigma^2\left(W_{i+1,j+1}-2\,W_{i+1,j}+W_{i+1,j-1}+W_{i,j+1}-2\,W_{i,j}+W_{i,j-1}\right)}{(\Delta x)^2}\\
&\quad+1/4\,\frac{\left(r-1/2\,\sigma^2\right)\left(W_{i+1,j+1}-W_{i+1,j-1}+W_{i,j+1}-W_{i,j-1}\right)}{\Delta x}\\
&\quad+\frac{W_{i+1,j}-W_{i,j}}{\Delta t}-r\left(1/2\,W_{i+1,j}+1/2\,W_{i,j}\right)=0,
\end{aligned} \tag{1.17}$$

which can be rewritten as,

$$
\begin{aligned}
&\left(1/4\,\frac{\Delta tr}{\Delta x}-1/8\,\frac{\Delta t\sigma^2}{\Delta x}+1/4\,\frac{\Delta t\sigma^2}{(\Delta x)^2}\right)W_{i+1,j+1}\\
&\quad+\left(1-1/2\,\frac{\Delta t\sigma^2}{(\Delta x)^2}-1/2\,\Delta tr\right)W_{i+1,j}\\
&\quad+\left(1/4\,\frac{\Delta t\sigma^2}{(\Delta x)^2}+1/8\,\frac{\Delta t\sigma^2}{\Delta x}-1/4\,\frac{\Delta tr}{\Delta x}\right)W_{i+1,j-1}\\
&\quad+\left(1/4\,\frac{\Delta tr}{\Delta x}-1/8\,\frac{\Delta t\sigma^2}{\Delta x}+1/4\,\frac{\Delta t\sigma^2}{(\Delta x)^2}\right)W_{i,j+1}\\
&\quad+\left(-1/2\,\Delta tr-1/2\,\frac{\Delta t\sigma^2}{(\Delta x)^2}-1\right)W_{i,j}\\
&\quad+\left(1/4\,\frac{\Delta t\sigma^2}{(\Delta x)^2}+1/8\,\frac{\Delta t\sigma^2}{\Delta x}-1/4\,\frac{\Delta tr}{\Delta x}\right)W_{i,j-1}=0.
\end{aligned}
\tag{1.18}
$$

If we separate the terms at time step i and $i+1$ onto either side of the equality, write the coefficients of W as $P_{u,i+1,j}$, $P_{m,i+1,j}$ and $P_{d,i+1,j}$ and use the bank account as numeraire we get,

$$
\begin{aligned}
&P_{u,i+1,j}W_{i,j+1}+P_{m,i+1,j}W_{i,j}+P_{d,i+1,j}W_{i,j-1}\\
&\quad=-P_{u,i+1,j}W_{i+1,j+1}-(P_{m,i+1,j}-2)W_{i+1,j}\\
&\quad-P_{d,i+1,j}W_{i+1,j-1}.
\end{aligned}
\tag{1.19}
$$

The RHS of the above Equation 1.19 is made up of known option prices (we know the option's payoff at maturity $i=N$) and (once we have calculated them) known transition coefficients $P_{u,i+1,j}$, $P_{m,i+1,j}$ and $P_{d,i+1,j}$; the RHS can therefore be considered as a known constant, $Z_{i+1,j}$,

$$P_{u,i+1,j}W_{i,j+1}+P_{m,i+1,j}W_{i,j}+P_{d,i+1,j}W_{i,j-1}=Z_{i+1,j}. \tag{1.20}$$

To calculate the transition coefficients, consistent with the market implied volatilities, for the Crank-Nicolson method we must reformulate Equations 1.6, 1.7 and 1.8. The one period bond price for Δt maturity pure discount bond gives us,

$$1=P_{u,i+1,j}+P_{m,i+1,j}+P_{d,i+1,j}. \tag{1.21}$$

For the fully explicit finite difference method the transition coefficients can be interpreted as probabilities whereas with the implicit methods there is no such interpretation. From Equation 1.21 they must sum to unity but they are not constrained to $\in [0, 1]$. The price of the underlying asset $S_{i+1,j}$ is related to its immediate predecessors by,

$$S_{i+1,j} = \frac{S_{i,j+1} + S_{i+1,j+1}}{2} P_{u,i+1,j} + \frac{S_{i,j} + S_{i+1,j}}{2} P_{m,i+1,j} + \frac{S_{i,j-1} + S_{i+1,j-1}}{2} P_{d,i+1,j}. \tag{1.22}$$

Finally the state price[19] of a node $(i + 1/2, j)$ is given by,

$$\frac{Q_{i,j} + Q_{i+1,j}}{2} = Q_{i+1,j+1} P_{d,i+1,j+1} + Q_{i+1,j} P_{m,i+1,j} + Q_{i+1,j-1} P_{u,i+1,j-1}. \tag{1.23}$$

The values of the state prices are computed in the same manner as for the trinomial tree. The above set of linear equations cannot be solved sequentially and we must instead determine the whole vector of transition coefficients at each time step by matrix algebra. The system represented in Figure 1.6 can rewritten using more compact notation,

$$\bar{v_i} = \bar{M_i} \times \bar{p_i}, \tag{1.24}$$

where the first vector is $\bar{v_i}$, the band diagonal matrix is $\bar{M_i}$ and the transition coefficient vector is $\bar{p_i}$. Rearranging this we get an expression in terms of the transition coefficient vector such that,

$$\bar{p_i} = \bar{M_i}^{-1} \times \bar{v_i}. \tag{1.25}$$

LU decomposition is used to solve Equation 1.25. This calculation is repeated for each time step to determine the transition coefficients for the whole grid. Once the transition coefficients have been recovered they can be used with the option's boundary conditions to determine the option price by backward induction.

19. Jamshidian states that the Green's function satisfies two fundamental differential equations, the Kolmogorov backward equation and the Fokker-Plank forward equation. In discrete time finance the Green's function has the interpretation of state prices. We can therefore relate state prices to each other in both forward and backward equations.

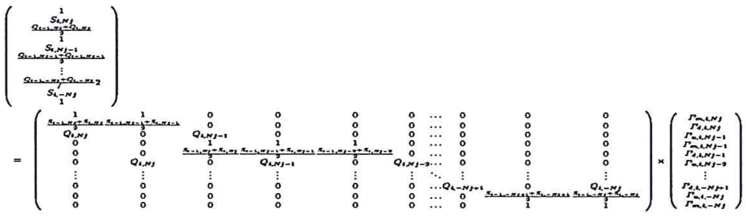


Figure 1.6
Representation of the system of linear equations used to recover the transition coefficients in a Crank Nicolson finite difference framework; the band diagonal structure of the matrix is clear.

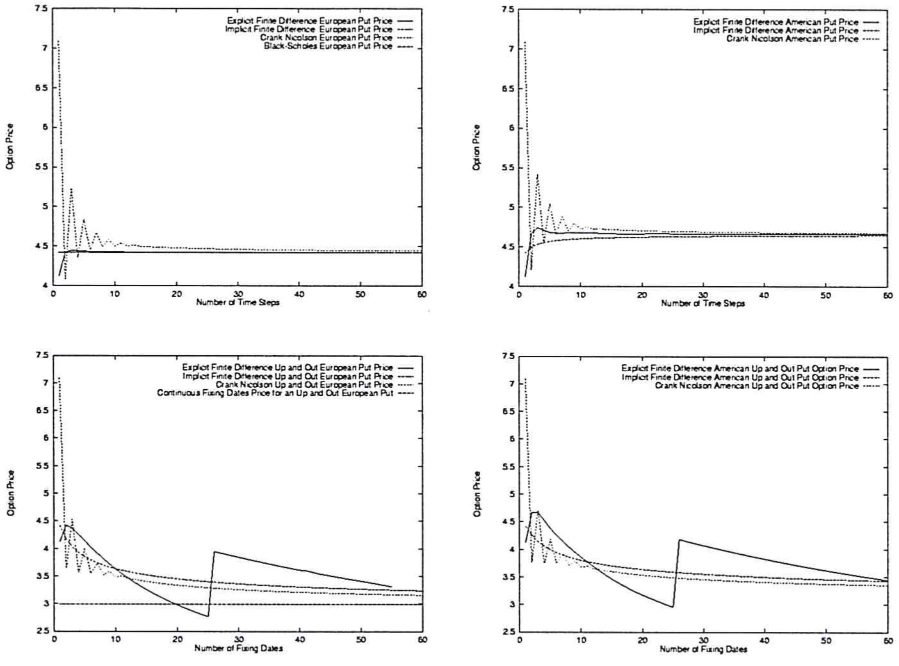
1.4 Results

1.4.1 Introduction

We examine the convergence properties of prices derived from the fully explicit, fully implicit and Crank Nicolson finite difference methods. Specifically we look at a European put, an American put, an up-and-out European put and an up-and-out American put both with constant volatility and in the presence of a volatility smile. For the European put we compare the prices with the Black-Scholes price and for the up-and-out European put we compare the prices with the price from an analytical formula due to Hull [Hull 1993].

In all the price convergence plots the Crank Nicolson method has oscillatory behaviour for small N, i.e. large Δt. Although the fully implicit and Crank Nicolson methods are unconditionally stable there are other conditions related to the size of Δx and Δt which need to be satisfied to prevent spurious oscillation; see Zvan, Vetzal and Forsyth [Zvan et al 1998] for a fuller discussion of this point. Suffice it to say, that it is because Δt is large for small N that we find oscillations in the Crank Nicolson method.

Table 1.1
The table of figures show the convergence properties for the finite difference methods for the following contracts: a European put, an American put, an up-and-out European put and an up-and-out American put. The contracts have the following specifications: $S = 100, K = 100, T = 0.5, r = 0.05$, where there is a barrier it is at a level $H = 105$ and the volatility is constant at $\sigma = 0.2$.



1.4.2 Option Prices with Constant Volatility Across Strike Prices and Maturities

The first figure in Table 1.1 shows that all three finite difference methods converge towards the Black-Scholes solution as we would expect. The second figure shows again that all three methods converge towards a common price which dominates the European price reflecting the early exercise premium associated with American options.

For the European and American barrier option plots the x-axis represents the number of fixing dates (i.e., the monitoring times). For this analysis we have used the same number of time steps as fixing dates to highlight the different convergence properties of the three methods. However, to achieve penny accuracy one would need to have more time steps than fixing dates. Both the third and forth figures demonstrate the well known feature that for down-and-out options the price of the option decreases as the number of fixing dates is increased, as there is a higher likelihood that if the barrier is crossed (even temporarily) it will be on a fixing date and therefore the option will be knocked out. The plots also demonstrate the fact that barrier options are always worth less than their vanilla counterparts as there is always a chance that they will be knocked out or fail to be knocked in.

The explicit finite difference line shows the characteristic saw tooth shape convergence which we associate with pricing barrier options in trinomial trees. Boyle and Lau [Boyle and Lau 1994], Derman-Kani-Ergener and Bardhan [Derman et al 995b] and Heynen-Kat [Heynen and Kat 1997] have all suggested methods for avoiding or correcting this saw tooth convergence[20].

The fully implicit and Crank Nicolson methods converge smoothly and asymptotically towards the price with a continuous number of fixing dates, this price is shown for the up-and-out European put where an analytical formula exists. The accuracy of the Crank Nicolson method is superior in these plots to the implicit method because of its faster convergence $O((\Delta t)^2)$ rather then $O(\Delta t)$.

1.4.3 Option Prices with Non-Constant Volatilities Across Strike Prices and Maturities

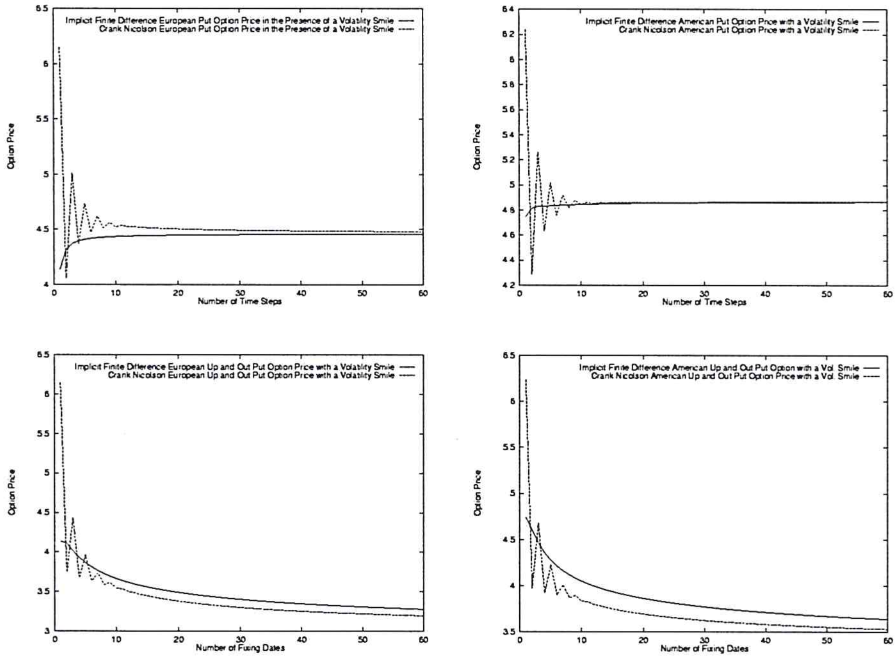
The four figures in Table 1.2 show the same option contracts as above except in the presence of a volatility smile. For this size of volatility smile the explicit finite difference method fails. This is because the size of the asset price step Δx is controlled (to insure stability) by Equation 1.1 which relates the size of Δx to the maximum implied volatility σ_{max}. If Δx is large for stability reasons then a conflict can arise with our accuracy requirements[21]. The resulting prices for the explicit finite difference method were very noisy and unusable.

All the contracts priced in the presence of a volatility smile have prices which dominate the contracts with constant volatility. This is expected as the minimum level of volatility in the smile was 20% which was the level for the constant volatility

20. The uneven convergence is the product of the barrier level and the position of the nearest node. For the fully explicit finite difference method the size of the asset price step Δx and therefore the position of the nodes in asset-space are related to the size of the time step Δt via Equation 1.1. In our plot Δt decreases in size as N increases and therefore the position of the nearest node changes relative to the fixed barrier level, H. As the node approaches the barrier the price of the option becomes more accurate reaching its most accurate when the node level is equal to the barrier level, conversely as the node moves away from the barrier level the accuracy deteriorates, finally the next node becomes the closest to the barrier at which point the price jumps.

21. Remember for the explicit finite difference method the accuracy is $O((\Delta x)^2)$.

Table 1.2
The table of figures show the convergence properties for the finite difference methods for the following contracts: a European put, an American put, an up-and-out European put and an up-and-out American put. The contracts have the following specifications: $S = 100, K = 100, T = 0.5$, $r = 0.05$ and where there is a barrier it is at a level $H = 105$. The contracts are priced in the presence of a volatility smile with a minimum value of $\sigma = 0.2$.



contracts. The higher overall level of volatility of the contracts with the volatility smile gives them a higher price.

1.4.4 Computational Efficiency and Accuracy

If we have a portfolio of exotic options on a common underlying asset, for example the S&P 500 then we can think of the pricing framework as having two stages. The first stage building a grid of transition coefficients (calibrating the model to the volatility smile); this is the pricing engine and has a one off run time cost[22]. The second stage is using the transition coefficients together with the target option's boundary conditions to calculate the price, this stage must be repeated for each different option. The time for building the grid of coefficients is $\approx 100s$ for $N = 60$ and $Nj = 50$ on a Sun UltraSPARC workstation. The time for pricing an American barrier option once the coefficients have been calculated is $\approx 3.70s$ for Crank Nicolson method and $\approx 3.67s$ for fully implicit finite difference method.

The accuracy of the framework has already been demonstrated by checking its convergence to Black-Scholes prices; the example given was for a European put with constant volatility.

Once the model has been calibrated the explicit finite difference method requires the least number of computer operations as it does not use a routine for solving band diagonal matrices[23] to determine option prices at the preceding time step. However, because it is not unconditionally stable it requires more time steps (larger N) to achieve the resolution in Δx which is needed for penny accuracy. The Crank Nicolson method requires additional arithmetic operations per time step over the fully implicit method because of the need to calculate the vector of $Z_{i+1,j}$ values from option prices and transition coefficients.

---

22. The coefficients only have to be recomputed when the implied volatility smile changes shape beyond some defined tolerance level or time period.

23. When fully implicit and Crank Nicolson finite difference methods are conventionally implemented with fixed value coefficients, a routine which takes advantage of the tridiagonal nature of the linear set of equations can be employed. We tried solving the linear set of equations for the option prices using a tridiagonal routine rather than a general routine for solving band diagonal matrices as the tridiagonal routine requires less memory and fewer arithmetic operations. However, the tridiagonal routine does not use pivoting and therefore it can sometimes fail when the matrix is non-singular (i.e., it can encounter a zero pivot); we found this to occur for $N >\approx 20$. For the tridiagonal algorithm, to avoid a zero pivot we need $|P_{m,j}| > |P_{u,j}| + |P_{d,j}|$ where $j = 1, \ldots, 2Nj + 1$; this is ***diagonal dominance***. Although this condition is satisfied near the centre of the tridiagonal matrix (and for conventional implementation of the implicit and Crank Nicolson methods) it is not satisfied near the top and bottom of the tridiagonal matrix where a down or an up movement may dominate a middle movement. The band diagonal routine uses pivoting unlike the tridiagonal routine and so should only fail if the matrix is singular.

1.5 Conclusions

In this work we have outlined three finite difference methods for pricing and hedging contingent claims in the presence of volatility smiles. Our findings suggest that it is difficult to use the explicit finite difference method when volatility smiles are very pronounced. Hence, although the fully implicit and Crank Nicolson methods require more computer operations they offer more promising results for pricing contingent claims in the presence of realistic volatility smiles. The fully implicit method requires slightly fewer computer operations than the Crank Nicolson method however the Crank Nicolson method has faster convergence. The choice between the Crank Nicolson method or the fully implicit finite difference method is problem dependent.

The work has also highlighted the importance of using pivoting when solving linear equations in order to prevent instability due to the build up of roundoff errors.

The example of pricing an American barrier option using fully implicit and Crank Nicolson finite difference methods demonstrates their superiority over unadjusted trinomial tree and fully explicit finite difference methods which have very poor convergence properties for barrier options.

References

Aparicio, S. and S. Hodges. 1996. Estimating implied distributions and issues in static hedging. Proceedings of the 9th Annual Financial Options Research Centre Conference at the University of Warwick.

Barle, S. and N. Cakici. 1995. Growing a smiling tree. *Risk*, 8:76–81.

Barone-Adesi, G. and R. Whaley. 1987. Efficient analytic approximations of American option values. *The Journal of Finance*, 42:301–320.

Boyle, P. and S. Lau. 1994. Bumping up against the barrier with the binomial method. *The Journal of Derivatives*, 1:6–14.

Brennan, M. and E. Schwartz. 1978. Finite difference methods and jump processes arising in the pricing of contingent claims: A synthesis. *Journal of Financial and Quantitative Analysis*, 13:462–474.

Chriss, N. 1996. Transatlantic trees. *Risk*, 9.

Clewlow, L. and C. Strickland. 1998. *Implementing Derivative Models*. John Wiley and Sons Ltd.

Courtadon, G. 1982. A more accurate finite difference approximation for the valuation of options. *Journal of Financial and Quantitative Analysis*, XVII(5).

Cox, J., S. Ross, and M. Rubinstein. 1979. Option pricing: A simplified approach. *Journal of Financial Economics*, 7:229–264.

Derman, E. and I. Kani. 1994. The volatility smile and its implied tree. Technical report, Goldman Sachs, Quantitative Strategies Research Notes.

Derman, E., I. Kani, and N. Chriss. 1996. Implied trinomial trees of the volatility smile. Technical report, Goldman Sachs, Quantitative Strategies Research Notes.

Derman, E., I. Kani, D. Ergener, and I. Bardhan. 1995b. Enhanced numerical methods for options with barriers. Quantitative Strategies Research Notes, Goldman Sachs.

Dupire, B. 1994. Pricing with a smile. *Risk*, 7:18–20.

Heynen, R. and H. Kat. 1997. *Exotic Options the State of the Art*, chapter 6: Barrier Options. International Thomson Business Press.

Hull, J. 1993. *Options, Futures, and Other Derivative Securities*, chapter 16, pages 418–420. Prentice-Hall, second edition.

Hull, J. and A. White. 1990. Valuing derivative securities using the explicit finite difference method. *Journal of Financial and Quantitative Analysis*, 25(1):87–100.

Hull, J. and A. White. 1993. Efficient procedures for valuing European and American path dependent options. *The Journal of Derivatives*, pages 21–31.

Jamshidian, F. 1991. Forward induction and construction of yield curve diffusion models. *Journal of Fixed Income.*

Press, W., S. Teukolsky, W. Vetterling, and B. Flannery. 1995. *Numerical Recipes in C: The Art of Scientific Computing.* Cambridge University Press, second edition.

Rubinstein, M. 1994. Implied binomial trees. *The Journal of Finance*, 69(3):770–818.

Shimko, D. 1993. Bounds of probability. *Risk*, 6:33–37.

Skiadopoulos, G., S. Hodges, and L. Clewlow. 1998. The dynamics of smiles. FORC Preprint, Financial Options Research Centre, University of Warwick.

Taylor, S. and X. Xu. 1993. The magnitude of implied volatility smiles: Theory and empirical evidence for exchange rates. FORC Preprint, Financial Options Research Centre, University of Warwick.

Wilmott, P., J. Dewynne, and S. Howison. 1993. *Option Pricing: Mathematical Models and Computation.* Oxford Financial Press.

Zvan, R., K. Vetzal, and P. Forsyth. 1998. Swing low swing high. *Risk*, 11:71–75.