Evolving Systems of Financial Returns: AutoRegressive Conditional Beta

George A Christodoulakis[†] Stephen E Satchell[‡]

[†]Department of Investment & Risk Management
City University Business School, London

[‡]Trinity College & Faculty of Economics
University of Cambridge

01 July, 2000

Abstract

We characterize a multi-factor model of financial asset returns with endogenously shocked time-varying beta coefficients in a multivariate framework, in which asset returns and the stochastic common factors are jointly determined. A Cholesky decomposition of the conditional covariance matrix guarantees its positive definiteness at each point in time and is expressed in terms of conditional variances and betas. Variances are allowed to follow any process from the ARCH family and betas follow special discrete time stochastic processes conditioned on the available information set, called ARCBeta processes. We derive sufficient conditions for the existence of strictly stationary solutions to the composite asset conditional variances and covariances. The model is capable of generating time variation and persistence in betas and such stylized facts as comovement between factor betas and variances as well as between betas on the same factor. We present empirical evidence from the EU11 stock markets as an evolving system. The proposed framework has relevant implications for asset pricing, risk management and can be used for out-of-sample forecasting.

GEL Classification: C22, C32, C5, G12,G15, Keywords: Beta, Evolution, Factor Models, Pricing, Stationarity, Forecasting

1 Introduction[1]

Multifactor models constitute a major class of models describing the dependences and dynamics of financial asset returns. In this paper we characterize a multifactor model as an evolving system, driven by endogenously shocked random processes for factor beta coefficients. Given the strong evidence in the literature on multivariate non-normality, a key question concerns the sources of such a phenomenon in connection to the evolution of the entire covariance structure of financial asset returns. Thorough understanding of the distributional and intertemporal characteristics of multivariate data would lead to more accurate pricing and hedging of financial risks.

The literature has addressed these issues with the development of multifactor models under multivariate ARCH-type disturbances, see for example Ng et al (1990), Engle et all (1992), Diebold and Nerlove (1989), thus accommodating the evolution of the covariance structure over time. Yet, it can be shown that time varying beta coefficients can lead to similar multiple heteroscedasticity structures and there is extensive empirical evidence supporting the time variation of factor coefficients as documented later in the text.

In modelling the dynamics of a covariance structure as a function measurable on the information set available at time t, our principal objective would be to ensure that such a matrix would be at each point in time positive semi-definite. Further, its dynamics should be parsimoniously represented involv-

[1]The authors would like to thank Long Chen, Petros Dellaportas, Antonis Demos, Gordon Gemmill, Soosang Hwang, John Knight, Kate Phylaktis, Zacharias Psaradakis and Mark Salmon, for helpful comments, participants of the 1st GARP Research Conference in Financial Risk Managenet, London June 2000, the EURO XVII annual meeting, Budapest July 2000, the Athenian Policy Forum meeting, Athens August 2000, the Northern Finance Association annual meeting, Waterloo Canada September 2000, as well as finance seminar participants at the AUEB Greece and the City University Business School, London.

ing the smallest possible number of unknown parameters. The first objective has proven to be a very difficult task and various alternative parametrizations of a multivariate GARCH have been proposed to secure the non-negative definiteness of the covariance matrix. These involve "constrained" optimization in the sense that complicated parameter restrictions have to be imposed in the estimation to ensure the covariance positive semi-definiteness. Such an approach is computationally expensive and difficult to implement. On the other side, an "unconstrained" optimization would involve parametrization of the covariance matrix in such a way that would force positive definiteness leaving parameters unrestricted. Such approaches have been proposed within the Statistics literature for the modelling of longitudinal and other types of data.

In this paper we will attempt an unconstrained modelling of the covariance matrix originating from a multifactor model of stock returns. We adopt the Cholesky decomposition of a covariance matrix proposed by Pourahmadi (1999a,b). This ensures the positive definiteness of the covariance matrix at each time and allows for a meaningful statistical interpretation of its parameters in terms of time-varying conditional variances and conditional factor betas. The latter are modeled as unrestricted ARCH-like processes, leading to interesting dynamic results for the entire conditional covariance matrix which we characterize in detail.

2 Stylized Facts and Literature Review

The introduction of the seminal work of Sharpe (1964) and Lintner (1965) on the Capital Asset Pricing Model and Ross (1976) on the Arbitrage Pricing Theory has sparked several streams of studies examining the empirical regularities and the implications of those models.. The interpretation of factor beta coefficient as a measure of the price of risk and its central role for financial decision making has motivated a sequence of papers addressing its intertemporal properties and a number of empirical stylized facts have been reported in the literature.

Numerous studies in the early seventies, as Black, Jensen and Scholes (1973), Blume and Friend (1973), Fama and MacBeth (1973) and Klemovsky and Martin (1975) to mention but a few, report significant **time variation** of factor beta coefficients. A common finding is the "**regression tendencies**" of factor betas, in the sense that betas tend to regress towards the mean over time. This stationary behaviour was initially attributed to selection bias but Blume (1975) presents theoretical and empirical arguments that the observed regression tendencies are due to "real non-stationarities in the underlying values of betas" and that the selection bias is not of dominant importance. The mean-reverting behaviour of factor betas has also been documented in recent studies such as Bos and Newbold (1984), Collins, Ledolter and Rayburn (1987) and Rockinger and Urga (1999). Also, a **co-movement** between the factor beta coefficient and the factor conditional variance has been found in studies such as Schwert and Seguin (1990), Koutmos, Lee and Theodossiou (1994) and Episcopos (1996). The sign of the co-movement seems to change for different data sets and time periods. This phenomenon raises our intuition that there may exist common or correlated shocks between the factor volatility and beta coefficient, an issue that will also be addresses later in the text. Also, a recent stream of papers presents evidence for deterministic **systematic variation** of factor betas. Studies such as Ferson and Harvey (1993), Ferson and Korajczyc (1995) as well as Bekaert and Harvey (1997) and Christopherson, Ferson and Tamer (1999) present significant relationships between economic macro- and micro-structure variables and varying betas.

The literature has used various techniques to capture the temporal characteristics of beta coefficients. Direct parametrization methodologies could possibly be classified into four main categories. First, many studies have modeled beta coefficients as Hildreth-Houck (1968) random parameters in that beta consists of a constant plus a noise term. This approach was followed among others by Fabozzi and Francis (1978), Chen and Keown (1981), Chen (1982) and Brooks, Faff and Lee (1992) who for monthly US and Australian equity returns find strong evidence against constant betas. A second, more recent strand of papers models beta coefficients as latent AR(1) or Random Walk processes using Kalman Filtering techniques. Bos and Newbold (1984), Collins, Lodolter and Rayburn (1987) as well as more recent studies such as Rockinger and Urga (1999), Hall, Urga and Zalewska-Mitura (1998) present empirical evidence supporting the time varying and mean reverting behaviour of betas using equity data for frequencies ranging from daily to monthly. A detailed examination of this approach is given by Wells (1995). Also, a related model is a doubly shocked process adopted by Ohlson and Rosenberg (1982) and Collins, Ledolter and Rayburn (1987) which augments the Hildreth-Houck random coefficient with an additional autoregressive noise to capture persistence in beta variation. Using weekly US equity data on specific stocks and portfolios they report results suggesting that roughly one quarter of the variation of beta is autocorrelated. A third class of papers models beta coefficients as functions of exogenous macro- or micro-economic variables to capture regime shifts. This approach does not consider endogenous dynamics in betas but presents significant association between betas and variables such as dividend yields, interest rates, market capitalization and credit rating, see for example Ferson and Harvey (1993), Ferson and Korajczyk (1995), Bekaert and Harvey (1995, 1997) as well as Kryzanowski, Lalancette and To (1997) and Christopherson, Ferson and Turner (1999) for recent results. Also Connor and Linton (2000) construct a characteristic-based factor model of stock returns in which factor betas are smooth non-linear functions of observed security characteristics. They perform joint estimation of betas and returns by combining non-parametric kernel methods and parametric non-linear regression. Finally, a number of studies model conditional betas as an inverse function of factor conditional volatility. This is a simple approach followed for example by Schwert and Seguin (1990), Koutmos et al (1994) and Episcopos (1996) who in most of the cases report a significant positive relationship between conditional beta and factor volatility.

Indirect approaches to modelling, consider the beta coefficient as the ratio of the asset and factor conditional covariance over the factor conditional variance and model the numerator and the denominator of the ratio separately.. The vector of all the unknown parameters is then jointly estimated using GMM (see Mark (1988), Harvey (1995)) or Quasi Maximum Likelihood techniques (see Hall, Miles and Taylor (1989)). Empirical results using foreign exchange, emerging market equity and UK equity data respectively strongly support the time variation of factor betas.

3 AutoRegressive Conditional Beta Model

We construct an unconstrained parametrization of a $(N+K) \times (N+K)$ covariance matrix by introducing the Cholesky decomposition of a positive definite matrix. Pourahmadi (1999a,b) introduced this approach to study the dynamics of longitudinal data by modelling the Cholesky decomposition of the inverse covariance matrix. The corner stone of such an approach is that a symmetric matrix[2] Ω is positive definite if and only if there exists a unique upper (lower) triangular matrix M with units on its principal diagonal and a unique diagonal matrix Σ with positive elements such that

$$M\ \Omega\ M^{'} = \Sigma$$

As we shall see, such a decomposition admits a meaningful interpretation in that it allows to model the elements of Ω in terms of variances and beta coefficients. Positive variances is sufficient to ensure the positive definiteness of the matrix while leaving betas unrestricted. It is now a matter of model design to make these elements measurable with respect to the generated $\sigma-$ field.

A typical multifactor model would regress a vector of N financial asset excess returns on K common factors. Under the classical linear model assumptions one would obtain OLS estimates of betas; such an approach, while highly popular among practitioners, ignores the stochastic properties of the factors by assuming they are determined exogenously. Allowing for stochastic factors determined within the system would lead clearly to biased OLS estimates. A first step in our modelling strategy will allow for asset returns and the factors to be jointly determined within the system. We will then examine how their joint stochastic properties can play a role for the evolution of the whole covariance structure.

Let $\mathbf{y}_t$ be an $N \times 1$ vector of asset excess returns generated by the following process

$$\begin{aligned} \mathbf{y}_t &= \boldsymbol{\mu}_{y,t} + \sum_{j=1}^{k} \boldsymbol{\beta}_{j,t}\ e_{j,t} + \boldsymbol{\varepsilon}_t \qquad (1) \\ x_{j,t} &= \mu_{x_j,t} + e_{j,t} \quad \text{for} \quad j = 1, ..., k \end{aligned}$$

[2] As a notational convention, we will use capital letters to denote matrices, lower case boldface letters for vectors and regular lower case letters for scalar variables.

and

$$\begin{pmatrix} \boldsymbol{\varepsilon}_t \\ \mathbf{e}_t \end{pmatrix} \mid I_{t-1} \sim D\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_{\varepsilon,t} & 0 \\ 0 & \Sigma_{e,t} \end{pmatrix}\right)$$

where $x_{j,t}, \boldsymbol{\beta}_{j,t}$ is the j-th factor and $N \times 1$ vector of conditional betas respectively, and $\boldsymbol{\mu}_{i,t}$, for $i = y, x_j, j = 1...k$, are $N \times 1$ vectors of conditional means. The covariance matrices $\Sigma_{\varepsilon,t}$, $\Sigma_{e,t}$ are diagonal with dimensions $N \times N$ and $K \times K$ respectively, but conditionally time-varying. We can express the system more compactly in partitioned matrix notation as

$$\begin{bmatrix} \underset{(N\times 1)}{\boldsymbol{\varepsilon}_t} \\ \underset{(K\times 1)}{\mathbf{e}_t} \end{bmatrix} = \begin{bmatrix} \underset{(N\times N)}{I_N} & \underset{(N\times K)}{-B_t} \\ \underset{(K\times N)}{0} & \underset{(K\times K)}{I_K} \end{bmatrix} \begin{bmatrix} \underset{(N\times 1)}{\mathbf{y}_t - \boldsymbol{\mu}_{y,t}} \\ \underset{(K\times 1)}{\mathbf{x}_t - \boldsymbol{\mu}_{x,t}} \end{bmatrix} \tag{2}$$

where $\mathbf{x}_t$ is the $K \times 1$ vector of common factors and B_t the $N \times K$ matrix of factor beta coefficients. Provided that B_t is conditionally known, the conditional covariance structure corresponding to equation (2) is

$$\begin{bmatrix} \Sigma_{\varepsilon,t} & 0 \\ 0 & \Sigma_{e,t} \end{bmatrix} = \begin{bmatrix} I_N & -B_t \\ 0 & I_K \end{bmatrix} \begin{bmatrix} \Omega_{yy,t} & \Omega_{yx,t} \\ \Omega_{xy,t} & \Omega_{xx,t} \end{bmatrix} \begin{bmatrix} I_N & -B_t \\ 0 & I_K \end{bmatrix}' \tag{3}$$

where Ω_{yy} and Ω_{xx} are the asset excess return and factor covariance matrix respectively and $\Omega_{yx} = \Omega_{xy}'$ include the covariances between the N assets and K factors. This is the Cholesky decomposition of Ω, the joint covariance matrix of N assets and K factors, in terms of the diagonal matrix Σ with positive elements as the conditional variances of asset idiosyncratic and factor shocks and the matrix M, the off-diagonal block of which corresponds to minus the factor beta coefficients $(-B_t)$. The matrices Σ_ε, Σ_e and B_t are allowed to be time-varying but is conditionally known. Solving with respect to Ω we obtain

$$\begin{bmatrix} \Omega_{yy,t} & \Omega_{yx,t} \\ \Omega_{xy,t} & \Omega_{xx,t} \end{bmatrix} = \begin{bmatrix} \Sigma_{\varepsilon,t} + B_t\Sigma_{e,t}B_t' & B_t\Sigma_{e,t} \\ \Sigma_{e,t}B_t' & \Sigma_{e,t} \end{bmatrix} \tag{4}$$

As an example, let the number of assets be $N = 2$ and the number of factors $K = 3$. Now $\Omega_{xx,t}$ will have as its diagonal elements the conditional variance of the j-th factor shock, $\sigma^2_{e_j,t}$, $j = 1, 2, 3$ and zero elsewhere, $\Omega_{yy,t}$ will be composite

$$\Omega_{yy,t} = \begin{bmatrix} \sigma^2_{\varepsilon_1,t} & 0 \\ 0 & \sigma^2_{\varepsilon_2,t} \end{bmatrix} + \begin{bmatrix} \beta_{11,t} & \beta_{21,t} \\ \beta_{12,t} & \beta_{22,t} \\ \beta_{13,t} & \beta_{23,t} \end{bmatrix}' \begin{bmatrix} \sigma^2_{e1,t} & 0 & 0 \\ 0 & \sigma^2_{e2,t} & 0 \\ 0 & 0 & \sigma^2_{e3,t} \end{bmatrix} \begin{bmatrix} \beta_{11,t} & \beta_{21,t} \\ \beta_{12,t} & \beta_{22,t} \\ \beta_{13,t} & \beta_{23,t} \end{bmatrix}$$

which in vech form can be written as

$$\text{vech}\left(\Omega_{yy,t}\right) = \begin{bmatrix} \sigma^2_{\varepsilon_1,t} \\ 0 \\ \sigma^2_{\varepsilon_2,t} \end{bmatrix} + \begin{bmatrix} \beta^2_{11,t} & \beta^2_{12,t} & \beta^2_{13,t} \\ \beta_{11,t}\beta_{21,t} & \beta_{12,t}\beta_{22,t} & \beta_{13,t}\beta_{23,t} \\ \beta^2_{21,t} & \beta^2_{22,t} & \beta^2_{23,t} \end{bmatrix} \begin{bmatrix} \sigma^2_{e_1,t} \\ \sigma^2_{e_2,t} \\ \sigma^2_{e_3,t} \end{bmatrix}$$

Under this specification the variance of an asset excess return will be decomposed into its idiosyncratic variance plus a time-varying combination of the factor conditional variances. The latter will appear as a common component but with different time-varying combinations, in all asset variances and covariances. The off-diagonal block of (4) represents covariances between asset excess returns and the factors.

The specification of (1), (2) and (3) provides a full multivariate framework for the joint generating process of N asset excess returns and K common factors. The Cholesky decomposition of the conditional covariance matrix allows the unrestricted modelling of its elements in terms of conditional variances and factor betas which can be measurable with respect to the available information set. Under the ARMA-ARCH modelling philosophy σ^2_t's and β_t's belong to the available information set I_{t-1} as they are assumed to be functions of available information. It remains to specify how the conditional covariance matrices $\Sigma_{\varepsilon,t}$, $\Sigma_{e,t}$ and the beta coefficient matrix B_t evolve over time. A key issue is that the matrix Σ in the Cholesky decomposition should remain diagonal with positive elements. An immediate solution, which we adopt in the following, would be that each individual asset idiosyncratic and factor conditional variance follow uncorrelated univariate GARCH processes, although other solutions can be considered. The evolution of beta coefficients will be analysed in detail in the following section. However, if σ_t^2's and β_t's are latent variables in that they are shocked by idiosyncratic innovations, the model can be seen as a Stochastic Volatility-type of model. In this paper we shall follow the first approach assuming that all variables are measurable with respect to I_{t-1}.

Rewriting equation (1) but with constant betas $\boldsymbol{\beta}$ one whould obtain a typical factor model. Such a heteroscedastic structure, as pointed out by Engle et al (1990) and Ng et al (1992) admits a further interpretation. Considering that $e_{j,t} = h_{j,t}^{\frac{1}{2}} v_{j,t}$ it can be seen as a time-varying beta process in which $\boldsymbol{\beta}_{j,t}^* = \boldsymbol{\beta}_j h_{j,t}^{\frac{1}{2}}$ and the factors $v_{j,t}$ exhibit zero mean and unit variance. Both interpretations lead to exactly the same conditional covariance structure and therefore are observationally equivallent. However, the implied evolution of betas in this structure is restricted to be proportional to conditional factor volatility and most importantly it is restricted to exhibit either positive or negative sign according to the sign of $\boldsymbol{\beta}_j$.

3.1 ARCBeta Processes

From the specification of asset excess return generating process in (2) we have that for asset i

$$\varepsilon_{i,t}^* = y_{i,t} - E\left(y_{i,t} \mid I_{t-1}\right) = \sum_{j=1}^{k} \beta_{ij,t}\, e_{j,t} + \varepsilon_{i,t} \tag{5}$$

for $i = 1, ..., N$ and

$$\beta_{i,j,t} = E\left(\frac{\varepsilon_{i,t}^* e_{j,t}}{\sigma_{j,t}^2} \mid I_{t-1}\right)$$

Joint modelling of the first and second conditional moments in (2) make the elements of $\Sigma_{\varepsilon,t}$ and $\Sigma_{e,t}$ known at time t, thus providing a history of such past errors as

$$\frac{\varepsilon^*_{i,t-1}e_{j,t-1}}{\sigma^2_{e_j,t-1}}, \frac{\varepsilon^*_{i,t-2}e_{j,t-2}}{\sigma^2_{e_j,t-2}}, \cdots$$

that belong to the information set I_{t-1}. It is now natural to make conditional beta coefficient $\beta_{ij,t}$ measurable with respect to this minimal information. One possible functional form is

$$\begin{aligned} \beta_{ij,t} &= E\left(\frac{\varepsilon^*_{i,t}e_{j,t}}{\sigma^2_{e_j,t}} \mid I_{t-1}\right) = \alpha_{ij,0} + \alpha_{ij,1}\xi_{ij,t-1} + \cdots + \alpha_{ij,p}\xi_{ij,t-p} \quad (6) \\ \text{for } t &= 0, \pm 1, ... \end{aligned}$$

where $\xi_{ij,t} = \frac{\varepsilon^*_{i,t}e_{j,t}}{\sigma^2_{e_j,t}}$ for asset $i = 1, ..., N$ and factor $j = 1, ..., k$, which can be called a ARCBeta process of order p. The term $\varepsilon^*_{i,t}$ as shown above contains shocks form all k factors as well as the idiosyncratic shock of asset i. Because of independence, the conditional beta of asset i on factor j is effectively shocked by squared innovations on factor j only so that $\xi_{ij,t} = \beta_{ij,t}\left(\frac{e_{j,t}}{\sigma_{e_{j,t}}}\right)^2$, making equation (6) be an unrestricted ARCH-like process. Thus, if the j-th factor innovation $\frac{e_{j,t}}{\sigma_{e_{j,t}}} = v_{j,t} \overset{iid}{\sim} D(0,1)$ we have that $E\left(\xi_{ij,t}|I_{t-1}\right) = \beta_{ij,t}$. Linearity for the conditional beta formulation has also been used by Shanken (1990), Ferson and Harvey (1991, 1993) and Harvey (1995). Alternative specifications of (6) can be considered to accommodate specific stylized facts regarding the evolution of $\beta_{ij,t}$, such as asymmetric responses of betas to positive or negative shocks. The latter approach is proposed by Braun, Nelson and Sunier (1995), who build a bivariate framework for the market model of excess asset returns and let β_t be generated by an asymmetric process in the spirit of (6). They apply this to portfolios of US stocks for monthly returns from July 1926 to December 1990. Although they report significant time variation for β_t, their empirical evidence does not support asymmetric effects, however such results are period- and data-specific.

As a standard result in time series analysis, stability of the ARCBeta stochastic difference equation should guarantee covariance stationarity of $\beta_{ij,t}$ as a random variable. We summarize the relevant results in the following straightforward theorem.

Theorem 1 *The ARCBeta process of order p will be stable if and only if all the roots of the associated characteristic polynomial are less than one in modulus. Its steady-state mean is*

$$E\left(\beta_{ij,t}\right) = \frac{\alpha_{ij,0}}{1 - \alpha_{ij,1} - \ldots - \alpha_{ij,p}} \qquad \textit{(Th.1a)}$$

Proof: see appendix.

Following standard results, an infinite order polynomial structure, $\Xi(L)$, for an ARCBeta process under stability can always be written as the ratio of two finite order polynomials, $\Xi(L) = \frac{A(L)}{B(L)}$, provided that $B(L)$ has all its roots outside the complex unit circle, and thus

$$\beta_{ij,t} B(L) = a_0 + \xi_t A(L)$$

where

$$B(L) = 1 - b_1 L - \cdots - b_q L^q \quad \text{and} \quad A(L) = a_1 L + \cdots + a_p L^p$$

providing a usual parsimonious representation of a longer memory process, called a Generalized ARCBeta(p,q) process.

As shown in Bollerslev (1986a,b) for GARCH variance processes, the autocorrelation structure of the squared residual will mimic the behaviour of an ARMA process but with a more restrictive admissable region for autocorrelations, due to the non negativity constraints on the parameters. ARCBeta processes do not carry such parameter restrictions and thus ξ_t will exhibit an autocorrelation structure of ARMA form (see Brockwell and Davis (1987)).

3.2 Estimation

The proposed model belongs to the broader class of multivariate models with time-varying first and second conditional moments. The properties of the Maximum Likelihood (ML) estimator of such models are in general unknown. We rely on the arguments of Bollerslev and Wooldridge (1992) who, under general regularity conditions, prove the consistency and asymptotic normality of the quasi ML estimator for a multivariate structure with time-varying second conditional moments. We pursue ML estimation of (1) under the assumption of conditional normality although departures from normality can also be considered as in the conventional multivariate GARCH models. For a sample size T of a vector of $N+K$ assets and common factors the Gaussian log-likelihood function will be

$$\begin{aligned}\ln L &= -\frac{T\left(N+K\right)}{2}\ln\left(2\pi\right)-\frac{1}{2}\sum_{t=1}^{T}\ln\left|\begin{pmatrix}\boldsymbol{\Sigma}_{\varepsilon,t} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Sigma}_{e,t}\end{pmatrix}\right| \\ &\quad -\frac{1}{2}\sum_{t=1}^{T}\begin{pmatrix}\boldsymbol{\varepsilon}_t\\ \mathbf{e}_t\end{pmatrix}'\left(\begin{pmatrix}\boldsymbol{\Sigma}_{\varepsilon,t} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Sigma}_{e,t}\end{pmatrix}^{-1}\begin{pmatrix}\boldsymbol{\varepsilon}_t\\ \mathbf{e}_t\end{pmatrix}\right)\end{aligned}$$

where

$$\begin{bmatrix}\underset{(N\times 1)}{\boldsymbol{\varepsilon}_t}\\ \underset{(k\times 1)}{\mathbf{e}_t}\end{bmatrix}=\begin{bmatrix}\underset{(N\times N)}{I_N} & \underset{(N\times k)}{-B_t}\\ \underset{(k\times N)}{0} & \underset{(k\times k)}{I_k}\end{bmatrix}\begin{bmatrix}\underset{(N\times 1)}{\mathbf{y}_t-\boldsymbol{\mu}_{y,t}}\\ \underset{(k\times 1)}{\mathbf{x}_t-\boldsymbol{\mu}_{x,t}}\end{bmatrix}$$

as given by (2), and the covariance matrix, (3), is diagonal due to the independence of the innovation processes. In spite of the diagonality of the covariance matrix, the log likelihood function is highly non-linear because of the presence of time-varying conditional variances as well as beta coefficients. Thus we estimate the unknown parameters by numerically maximizing the likelihood. We find more effective to start iterations using the BHHH (Berndt, Hall, Hall, Hausmann (1974)) algorithm and then switch to BFGS (Broyden

(1967)) for faster convergence. The fact that all innovation processes are independent and factors are common to all assets allows the reduction of the scale of the estimation problem from a system of $N+K$ equations to N systems $K+1$ equations. That is, instead of estimating the whole system simultaneously which for large N makes numerical maximization difficult, one could estimate a sequence of smaller systems of one asset and K factors. However in our empirical applications later, we provide estimates from the full multivariate QML optimization.

We initialize ARCBeta and GARCH processes at the OLS beta and the sample variances respectively. Also, a sensitive problem in estimating the model is the choice of initial values for the ARCBeta parameters as these are unrestricted parameters taking any sign. Because of the multivariate nature of the problem and the corresponding dimensions of the parameter space, it is not difficult that one obtains local rather than global maxima. We have found that a helpful procedure is to start estimation with constant betas and then relax the assumptions of constancy progressively, for one parameter at a time and re-estimate the model, until all parameters are left to vary. An empirical application is presented later in the text.

4 Persistence and the Steady-State

The north-west block on both sides of equation (4) provides the conditional covariance matrix of the $N \times 1$ vector of asset excess returns, which is a composite process. It aggregates the conditional variances of the relevant factors and the assets' idiosyncratic shocks, the former being adjusted by the square of the corresponding factor time-varying beta. Thus for asset i, $\varepsilon^*_{i,t}$ is given by equation (5) which is a zero-mean process with conditional variance

$$\sigma_{i,t}^2 = \sum_{j=1}^{k} \beta_{ij,t}^2 \sigma_{e_j,t}^2 + \sigma_{e_i,t}^2$$

and the conditional covariance between asset i and asset k will be

$$\sigma_{ik,t} = \sum_{j=1}^{k} \beta_{ij,t} \beta_{kj,t} \sigma_{e_j,t}^2$$

When β_{ij} is constant for all i, j, the composite conditional variance of asset i, $\sigma_{i,t}^2$ reduces to the linear aggregate of k GARCH-type factor conditional variances plus the idiosyncratic one. This case has been analyzed by Karanasos, Psaradakis and Sola (1999) who provide expressions for the unconditional moments of the process as well as conditions under which the persistence of the composite variance is higher than that of the individual variances. Also Zaffaroni (1999) studies the contemporaneous aggregation of GARCH processes under either idiosyncratic or common shocks. He finds that, unlike its components, strict stationarity, ergodicity and finite kurtosis might fail for the aggregate. He also concludes that under no conditions, for the cases examined, aggregation of GARCH induces long-memory conditional heteroscedasticity. In our framework, the time variation of factor beta coefficients complicates the analysis and it is important to establish conditions under which for each asset i the composite conditional variance and covariances with asset j are stationary processes. Such conditions will place joint restrictions on the values of the parameters of conditional variance and beta processes.

Since the idiosyncratic volatility components are allowed to follow ARCH-type processes, their stationarity and moment properties are known for many cases, see Karanasos (1999) and He and Terasvirta (1999) for recent results. We will therefore concentrate to the analysis of the composite term $\beta_{ij,t}^2 \sigma_{e_j,t}^2$

which exhibits a level of complication due to non-linearities. The fact that both $\beta_{ij,t}$ and $\sigma^2_{ej,t}$ share the same innovation process $v^2_{j,t}$ will greatly facilitate the derivation of results. For the purpose of this analysis, it is convenient to represent both the conditional volatility and beta processes as random parameter processes and then work with similar techniques as in Nicholls and Quinn (1982) and Pham (1985). In particular, the generalized ARCBeta(p_1, q_1) process for asset i on factor j can be written[3] in its state-space form as

$$\begin{bmatrix} \beta_t \\ \beta_{t-1} \\ \vdots \\ \beta_{t-p+1} \\ \xi_{t-1} \\ \vdots \\ \xi_{t-q+1} \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha_1 v^2_{t-1} + b_1 & b_2 & \dots & b_p & \alpha_2 & \dots & \alpha_q \\ 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots 1 & 0 & 0 & \dots & 0 \\ v^2_{t-1} & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots 1 & 0 \end{bmatrix} \begin{bmatrix} \beta_{t-1} \\ \beta_{t-2} \\ \vdots \\ \beta_{t-p} \\ \xi_{t-2} \\ \vdots \\ \xi_{t-q} \end{bmatrix}$$

which more compactly can be written as

$$\underset{m_1\times 1}{B_t} = \underset{m_1\times 1}{A_0} + \underset{m_1\times m_1}{A_{t-1}}\,\underset{m_1\times 1}{B_{t-1}} \tag{7}$$

where $m_1 = p_1 + q_1 - 1$. Also the GARCH(p_2, q_2) process for factor j admits a similar representation

$$\vec{\Sigma}_t = \Gamma_0 + \Gamma_{t-1}\vec{\Sigma}_{t-1} \tag{7a}$$

in which $\vec{\Sigma}_t$ is $(m_2 = p_2 + q_2 - 1) \times 1$ vector containing appropriate lags of σ^2_t and Γ_0, Γ_{t-1} contain the parameters of the process, sharing the same innovations $v^2_{j,t}$ with the generalized ARCBeta(p_1, q_1) process as shown in the previous section. Now, $\beta^2_{ij,t}$ will be the first element of

$$\begin{aligned} \text{vec}\left(B_t B_t^{'}\right) &= \text{vec}\left(A_0 A_0^{'}\right) + \left[(A_{t-1}\otimes A_0) + (A_0 \otimes A_{t-1})\right] B_{t-1} \\ &\quad + (A_{t-1}\otimes A_{t-1})\,\text{vec}\left(B_{t-1}B_{t-1}^{'}\right) \end{aligned} \tag{8}$$

[3]We drop the ij subscripts for simplicity.

or equivalently

$$\bar{B}_t = \bar{A}_0 + \bar{A}_{t-1}B_{t-1} + \dot{A}_{t-1}\bar{B}_{t-1} \tag{8a}$$

Post-multiplying the above equation by $\vec{\Sigma}^{'}_{i,t}$ and then vectoring this product we obtain

$$\begin{aligned}\mathrm{vec}\left(\bar{B}_t\vec{\Sigma}^{'}_t\right) &= \mathrm{vec}\left(\bar{A}_0\Gamma^{'}_0\right) + \left(\Gamma_{t-1}\otimes\bar{A}_0\right)\vec{\Sigma}_{t-1} + \left(\Gamma_0\otimes\bar{A}_{t-1}\right)B_{t-1}\\ &+ \left(\Gamma_{t-1}\otimes\bar{A}_{t-1}\right)\mathrm{vec}\left(B_{t-1}\vec{\Sigma}^{'}_{t-1}\right) + \left(\Gamma_0\otimes\dot{A}_{t-1}\right)\bar{B}_{t-1}\\ &+ \left(\Gamma_{t-1}\otimes\dot{A}_{t-1}\right)\mathrm{vec}\left(\bar{B}_{t-1}\vec{\Sigma}^{'}_{t-1}\right)\end{aligned} \tag{9}$$

where $\bar{B}_t$ =vec$\left(B_tB^{'}_t\right)$, $\bar{A}_0$ =vec$\left(A_0A^{'}_0\right)$, $\bar{A}_{t-1} = (A_{t-1}\otimes A_0) + (A_0\otimes A_{t-1})$ and $\dot{A}_{t-1} = (A_{t-1}\otimes A_{t-1})$. This equation is an $m_1^2m_2\times 1$ vector process with random parameters and its first element gives the $\beta^2_{ij,t}\sigma^2_{e_j,t}$ component of the composite conditional variance of asset i. If K factors are relevant for the evolution of the conditional mean equation of asset i, then K such components need to be stationary together with the idiosyncratic conditional variance process.

4.1 A Set of Sufficient Conditions

We wish to establish sufficient conditions on the parameters of A_t and Γ_t for the existence of a strictly stationary process conforming to the model described by the above equation. Since this is a composite process we will first examine convergence of its parts, that is $\vec{\Sigma}_{t-1}$, B_{t-1}, vec$\left(B_tB^{'}_t\right)$ and vec$\left(B_{t-1}\vec{\Sigma}^{'}_{t-1}\right)$ and then for vec$\left(\bar{B}_{t-1}\vec{\Sigma}^{'}_{t-1}\right)$ the first element of which is our primal interest. Results on the former will be presented in the form of lemmas which will subsequently be used in the proof of a theorem for the latter. The form of equations (7) and (7a) allows the random parameter matrices A_t, Γ_t, $\dot{A}_t$ and their kronecker products be *iid* sequances, which will facilitate subsequant calulations.

Lemma 2 **(B_t ~ARCBeta(p_1,q_1), $\vec{\Sigma}_t$ ~GARCH(p_2,q_2))** *Let X_t represent either of B_t or $\vec{\Sigma}_t$, $v_t \overset{iid}{\sim} D(0,1)$ with finite fourth moment. As in (7)*

$$X_t = \Delta_0 + \Delta_{t-1} X_{t-1} \tag{L21}$$

If $\Delta = E(\Delta_t \otimes \Delta_t)$ has all its eigenvalues within the unit circle, then

$$\sum_{i \geq 1} \prod_{j=1}^{i} \Delta_{t-j} \Delta_0$$

will be absolutely convergent almost surely, and

$$X_t = \Delta_0 + \sum_{i \geq 1} \prod_{j=1}^{i} \Delta_{t-j} \Delta_0 \tag{L22}$$

will be a strictly stationary process conforming to ($L21$).

Proof: *See Appendix.*

The above result provides a sufficient condition for the existence of a stationary solution of a process in the form of an ARCBeta(p_1,q_1) or GARCH(p_2,q_2). It is now trivial to see for a process of the form $x_t = \delta_0 + \left(\delta_1 v_{t-1}^2 + \zeta_1\right) x_{t-1}$, the above sufficient condition requires that v_t has finite fourth moment γ_4, and $\gamma_4 \delta_1^2 + \zeta_1^2 + 2\delta_1\zeta_1 < 1$ which, under Gaussian innovations ($\gamma_4 = 3$), is identical to the result of Bollerslev (1986, p 311) for the existence of the fourth moment of a GARCH(1,1) process. Of course when x_t represents an ARCBeta process its parameters can take any sign.

Corollary 3 *The steady-state x_t is given by the first element of*

$$E(X_t) = (I - E(\Delta_t))^{-1} \Delta_0$$

i.e.

$$E(x_t) = \frac{\delta_0}{1 - \delta_1 ... - \delta_p - \zeta_1 ... - \zeta_q}$$

Proof: *the result follows directly from Lemma 2 by taking expectations in (L22) and taking into account Theorem 1.*

Lemma 4 (vec$(B_t B_t^{'})$) *Let $\bar{B}_t$ be generated as in equation (8a), B_t follow an ARCBeta(p_1,q_1) process, $v_t \overset{iid}{\sim} D(0,1)$ with finite up to the eighth moments and Lemma 2 holds. If $E\left(\dot{A}_t \otimes \dot{A}_t\right)$ has all its eigenvalues within the unit circle, then $\bar{B}_t$ has a strictly stationary solution conforming to equation (8a).*

Proof: *See Appendix.*

As an example, the above sufficient condition for a strictly stationary solution of the square of an ARCBeta(1,1) process states that

$$a_1^4\gamma_8 + \left(a_1^3 b_1 + 2a_1^2 b_1^2\right)\gamma_4 + b_1^4 + b_1^3 a_1 + 4a_1^2 b_1^2 < 1$$

where γ_8, γ_4 are the eighth and fourth moments respectively of the innovation process which are required to be finite.

Corollary 5 *For the ARCBeta(1,1), the steady-state β_t^2 is given by the first element of*

$$\begin{aligned} E\left(vec\left(B_t B_t^{'}\right)\right) &= vec\left(A_0 A_0^{'}\right) + \bar{A}\left(I - A\right)^{-1} A_0 \\ &\quad + \left(I - \dot{A}\right)^{-1} \dot{A}\left(vec\left(A_0 A_0^{'}\right) + \bar{A}\left(I - A\right)^{-1} A_0\right) \end{aligned}$$

Proof: *See Appendix.*

For the gaussian ARCBeta(1,1) process we have

$$E\left(\beta_t^2\right) = \frac{a_0^2}{1 - 3a_1^2 - b_1^2 - 2a_1 b_1} \times \frac{1 + a_1 + b_1}{1 - a_1 - b_1}$$

Lemma 6 (vec$\left(B_{t-1}\vec{\Sigma}'_{t-1}\right)$) *Let B_t and $\vec{\Sigma}_t$ be generated by an ARCBeta(p_1,q_1) and a GARCH(p_2,q_2) respectively as in equations (7) and (7a), $v_t \overset{iid}{\sim} D(0,1)$ with finite up to the eighth moments and Lemma 2 hold. Then $vec\left(B_{t-1}\vec{\Sigma}'_{t-1}\right)$ has a strictly stationary solution if $E\left((\Gamma_t \otimes A_t) \otimes (\Gamma_t \otimes A_t)\right)$ has all its eigenvalues within the unit circle.*

Proof: *See Appendix.*

In terms of the ARCBeta(1,1) parameters, (a_1, b_1) and GARCH(1,1), (say c_1, d_1) the above condition states that

$$a_1^2 c_1^2 \gamma_8 + \left(2a_1^2 c_1 d_1 + 2a_1 b_1 c_1^2\right)\gamma_6 + \left(a_1^2 d_1^2 + a_1 b_1 c_1 d_1\right)\gamma_4 + b_1^2 d_1^2 < 1$$

where γ_j is the j-th moment of the common innovation process.

Corollary 7 *The steady-state $\beta_t \sigma_t^2$ is given by the first element of*

$$\begin{aligned} & E\left(vec\left(B_t \vec{\Sigma}'_t\right)\right) \\ = \; & vec\left(A_0 \Gamma'_0\right) + \Gamma_{A_0}(I-\Gamma)^{-1}\Gamma_0 + A_{\gamma_0}(I-A)^{-1}A_0 \\ & + (I-\Gamma_A)^{-1}\Gamma_A\left[vec\left(A_0\Gamma'_0\right) + \Gamma_{A_0}(I-\Gamma)^{-1}\Gamma_0 + A_{\gamma_0}(I-A)^{-1}A_0\right] \end{aligned}$$

Proof: *See Appendix.*

So far we have derived results for the ARCBeta and GARCH processes and their products. We shall now make use of those results to derive sufficient conditions for $\beta_t^2 \sigma_t^2$.

Theorem 8 (vec$\left(\bar{B}_{t-1}\vec{\Sigma}'_{t-1}\right)$) *Let $B_{t-1}, \vec{\Sigma}_{t-1}$ follow ARCBeta(p_1,q_1) and GARCH(p_2,q_2) processes respectively and v_t has finite moments up to twelfth order. $vec\left(\bar{B}_{t-1}\vec{\Sigma}'_{t-1}\right)$ has a strictly stationary solution if Lemmas (2) to (6) hold and*

$$E\left(\left(\Gamma_{t-j} \otimes \dot{A}_{t-j}\right) \otimes \left(\Gamma_{t-j} \otimes \dot{A}_{t-j}\right)\right)$$

has all its eigenvalues within the unit circle.

Proof: See Appendix.

For ARCBeta(1,1) and GARCH(1,1) parameters, the theorem states that in addition to Lemmas (2) to (6) restrictions, the processes should jointly satisfy

$$\begin{aligned} & a_1^4 c_1^2 \gamma_{12} + \left(4c_1^2 a_1^3 b_1 + 2c_1 d_1 a_1^4\right) \gamma_{10} \\ & + \left(6c_1^2 a_1^2 b_1^2 + 8c_1 d_1 b_1 a_1^3 + d_1^2 a_1^4\right) \gamma_8 \\ & + \left(4c_1^2 a_1 b_1^3 + 12c_1 d_1 a_1^2 b_1^2 + 4d_1^2 a_1^3 b_1\right) \gamma_6 \\ & + \left(c_1^2 b_1^4 + 8c_1 d_1 a_1 b_1^3 + 6d_1^2 a_1^2 b_1^2\right) \gamma_4 \\ & + \left(2c_1 d_1 b_1^4 + 4d_1^2 a_1 b_1^3\right) \gamma_2 + d_1^2 b_1^4 \\ < \; & 1 \end{aligned}$$

where γ_j denotes the j-th moment of a standard innovation process. For $b_1 = d_1 = 0$ the condition reduces to $a_1^4 c_1^2 \gamma_{12} < 1$.

Corollary 9 *The steady-state $\beta_t^2 \sigma_t^2$ is given by the first element of*

$$\begin{aligned} E\left(vec\left(\bar{B}_t \vec{\Sigma}_t^{'}\right)\right) &= vec\left(\bar{A}_0 \Gamma_0^{'}\right) + \left(I - \Gamma_{\dot{A}}\right)^{-1} vec\left(\bar{A}_0 \Gamma_0^{'}\right) \\ & + \left(I - \Gamma_{\dot{A}}\right)^{-1} E\left(\Phi_{t-k}\right) \end{aligned}$$

Proof: *See Appendix.*

A similar analysis provides sufficient conditions for the existence of a strictly stationary solutions for the composite covariance processes between assets. From the second equation of section 4, the covariance between any two assets will be composed by a number (say k) of sub-processes equal to the number of significant common factors between the two assets, thus k such components need to be stationary. We summarize the relevant results for the j-th component, when factor betas and variances follow ARCBeta(p_1,q_1) and GARCH(p_2,q_2) processes respectively, in the following theorem.

Theorem 10 *Let $B_{i,t}$, $B_{l,t}$ be ARCBeta(p_1,q_1) processes and $\vec{\Sigma}_{j,t}$ be a GARCH(p_2,q_2) process, with common innovation $v_t \sim D(0,1)$ with finite moments up to the twelfth order, and lemmas (2) to (6) hold. Then $vec\left(\bar{B}_{il,t}\vec{\Sigma}'_{j,t}\right)$ has a strictly stationary solution if*

$$E\left(\left(\Gamma_{j,t-n} \otimes \bar{A}_{li,t-n}\right) \otimes \left(\Gamma_{j,t-n} \otimes \bar{A}_{li,t-n}\right)\right)$$

has all its eigenvalues within the unit circle.

Proof: *See Appendix.*

The steady-state covariance can now be computed using Corollary 9.

The preceding analysis presents a set of sufficient conditions for the existence of a strictly stationary solution for a number of processes formed as products of time series processes. Although not necessary, they have the advantage that are easy to check. Derivation of necessay conditions is a more difficult exercise and there are only a few studies in the literature addressing this aspect, which we discuss in the following section.

4.2 On Sharper Conditions

Equations (7) and (7a) form a random parameter difference equation often described as Generalized Autoregressive process. Vervaat (1979), Brandt (1986) and Bourgerol and Picard (1992a) have studied the stationarity properties of such processes and have reached conditions that are both necessary and sufficient. Recent applications of such ideas in finance are presented by Bourgerol and Picard (1992b), Emrbechts et al (1997) who study ARCH and GARCH processes and Bond (2000) for the Semi-Variance GARCH model. In particular Bourgerol and Picard (1992b) establish necessary and sufficient conditions for Generalized Autoregressive processes with *i.i.d.* non-negative random parameter matrix. This is exactly the case of a GARCH volatility process of the form of equation (7a).

In our framework the ARCBeta Generalized Autoregressive process (7), exhibits an *i.i.d.* random parameter matrix A_t, which can be both positive and negative since ARCBeta parameters are not restricted. For this case Brandt (1986) provides general sufficient conditions and Bourgerol and Picard (1992a) present necesssary and sufficient conditions for the existence of a stationary solution by establishing the converse of Brandt's theorem. The analysis is similar to but more complicated of that applied to GARCH processes in Bourgerol and Picard (1992b). We shall present the relevant results preceded by two definitions which are necessary for the exposition.

Definition 11 *(Bourgerol and Picard (1992a), Definition 2.2)*

A non-anticipative *strictly stationary solution of (7) is a strictly stationary process* $\{B_t, t \in \mathbf{Z}\}$ *which is a solution of (7) such that, for any* $k \in \mathbf{Z}$, B_k *is independent of the random variable* $\{A_t, t > k\}$.

Definition 12 *(Bourgerol and Picard (1992a), Definition 2.3)*

An affine subspace H *of* $\mathbf{R}^d$ *is said to be* invariant *under the model (7) if* $\{A_{t=\mathbf{0}}\beta + A_{\mathbf{0}}; \beta \in H\}$ *is contained in* H *almost surely. The model (7) is called* irreducible *if* $\mathbf{R}^d$ *is the only affine invariant subspace.*

Theorem 13 *(Bourgerol and Picard (1992a), Theorem 2.5)*

Suppose that the Generalized Autoregressive model (7) with i.i.d. coefficients is irreducible and that $E\left(\ln^+ \|A_{t=\mathbf{0}}\|\right)$ *is finite. Then (7) has a non-anticipative strictly stationary solution if and only if the top Lyapounov exponent* γ *defined as*

$$\gamma = \inf\left\{E\left(\frac{1}{n+1}\ln\|A_{t=\mathbf{0}}A_{t=-1}...A_{t=-n}\|\right), n \in N\right\}$$

is strictly negative.

The theorem presents both necessary and sufficient conditions for strict stationarity, but requires that $\{A_t\}$ forms an *i.i.d.* process. This is directly applicable to an ARCBeta processes B_t such as (7). From a computational point of view, Furstenberg and Kesten (1960) show that almost surely

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \ln \|A_{t=0}A_{t=-1}...A_{t=-n}\|$$

which can be estimated using monte carlo simulations. However the application of this approach to product of processes such as $\bar{B}_t = \text{vec}\left(B_t B_t^{'}\right)$ and $\text{vec}\left(\bar{B}_t \vec{\Sigma}_t^{'}\right)$ is not straightforward and we leave this as a question for future research.

5 On the Convergence of the European Stock Markets

We illustrate the empirical relevance of our preceding theoretical arguments by representing a set of European stock market portfolios as an evolving system. Since the inception of the European Economic Community, its member states coordinate progressively their national economies with respect to both the real and monetary sectors. This intended to that they all would satisfy the Maastricht convergence criteria and eventually form the European Monetary Union. This raises questions on the evolution of integration between European stock markets and their association with global factors. We utilize the multivariate system of asset returns presented in section 3 to represent a set of European stock market portfolios (UK, France, Germany) as an evolving system with respect to three common global factors (EU, Japan, US).

5.1 Data and Empirical Model Building

We use weekly data from DataStream-calculated (see DataStream (1990)) market price indexes in US dollars, covering a period of twenty seven years from 03.04.73 to 15.03.00 (1409 obs).

The weekly return at time t for each series $P_{i,t}$ is calculated as $\ln\left(\frac{P_{i,t}}{P_{i,t-1}}\right)\times 100$. A preliminary analysis of the return series uncovers high unconditional kurtosis and strong conditional heteroscedasticity for all cases but absence of any serial correlation for the return levels.

We shall perform Maximum Likelihood (ML) estimation assuming conditional normality. This is a convenient assumption which simplifies numerical computations without harming the properties of the estimator. Bollerslev and Wooldridge (1992) show under general regularity conditions that, the Quasi ML estimator (i.e. falsely assuming normality) of parameters for models with varying first and second conditional moments will be consistent and asymptotically normal as long as the first two conditional moments are correctly specified. As possible excess kurtosis will result in understated standard errors for the parameter estimates, we employ the White (1980) heteroscedasticity-adjusted standard errors. Also we do not impose any restrictions on the estimation since, ARCBeta processes are restriction-free and GARCH processes are estimated following the arguments Nelson and Cao (1992). Numerical maximization has been performed on the GAUSS matrix language and we initialize conditional variance and beta series at their unconditional expectations.

As an empirical model building strategy, we shall first expose the data to restricted versions of the full multivariate model and then relax the parameter restrictions progressively. First we estimate the full system as a constant factor beta model, allowing for factor time-varying volatility but constant idiosyncratic volatility. Next we relax the assumption of constant betas for one portfolio only (UK) keeping the remaining betas constant and re-estimate the full model. We check the adequacy of the new, unrestricted specification, in the light of the statistical significance of the parameters, likelihood ratio tests and Bayesian information criteria. In doing so, we experiment with various lag specifications for the conditional means, variances and betas. Following we relax sequentially the constancy of betas for the second (France) and third (Germany) portfolio by repeating the above analysis at each step and allow for idiosyncratic volatility to vary as well. Finally we perform further specification tests, such as conditional moment tests introduced by Wooldridge (1990).

5.2 Results

First we estimate the model under constant factor betas and allow for time varying conditional means and factor variances. At this stage we keep idiosyncratic variances constant and reserve our right to relax this later. We estimate a number of different specifications for the constant beta model and present the final results in Table I in the appendix. For all series a constant conditional mean and a GARCH(1,1) volatility processes seems an adequate parametrization in the light of Information Criteria and Likelihood Ratio tests. The beta coefficients of the three national portfolios on the EU factor take significant values close to unity with the UK being the largest in magnitude. All betas with respect to Japan and US take significant values but close to zero.

Next we progressively relax the assumptions on constant betas and let them be driven by ARCBeta processes. We first allow the beta coefficients of the UK on the three factors to vary, keeping the rest of the beta parameters constant. Table II presents the estimation results which clearly support the unrestricted form of the model. In particular most of the ARCBeta parameters of the UK on the three factors are statistically significant and increase the Likelihood ratio value sufficiently so as to reject the null of no ARCBeta effects quite confidently. Parameters of lagged betas (b_1) take high values signifying persistence of shocks on betas, but in all cases the process seems to be stationary. However Information Criteria are more conservative and just approve the unrestricted specification. Graph I plots the estimated UK betas on the corresponding three factors. In all cases the estimated ARCBeta process implies a steady-state close enough to the constant beta estimate. The elasticity of the UK portfolio with respect to EU varies substantially taking values from unity to 1.6 and exhibits a "clustering" behavior but much smoother compared to a volatility process. Elasticities with respect to Japan and US factors are very close to zero but exhibit a statistically significant variation around it. The sample around the 780-th observation corresponds to the October 1987 crash period, in which we observe that the UK portfolio elasticity sharply increased with respect to the EU factor, sharply decreased with respect to the US factor and remained stable with respect to the Japanese market.

Second, we relax the assumption of constant French betas, re-estimate the entire model and present the results in Table III. The France to EU ARCBeta process is highly significant, exhibiting high persistence and interestingly the opposite fluctuations of those of UK to EU as Graph II shows. French beta with respect to Japan appears constant as all attempts to estimate an ARCBeta process have failed. Also, the process with respect to US presents a questionable significance as the a_1 parameter does not defer from zero and its contribution to the likelihood value is very small. Thus, likelihood ratio and information criteria are very close not to reject the null of constant French betas overall, but when we consider only the France to UK elasticity as ARCBeta and the rest constant model selection is in favor of ARCBeta.

Last, we relax the assumption of constant German betas and re-estimate the entire model. Table IV presents the results which indicate strong and significantly persistent fluctuations for all three parameter processes. Betas with respect to UK and the US behave similarly to those of the French portfolio. A new feature though is the negative sign of the ARCBeta parameter with respect to Japan. Shocks are still highly persistent and the process is stationary but estimated betas exhibit clusters of high and low values of alternate signs. Last, we relax the assumption of constant idiosyncratic variances for UK, France and Germany and re-estimate the entire system, which dramatically improves the likelihood value (Table V). As a final check, we summarize the steady-state betas implied by the full system estimation (table IV or V) in table VI in comparison with the constant estimated betas (of table I), which appear remarkably similar.

5.3 Discussion and Conclusions

The model formally presents that various variables in the system experience common shocks, thus generating common -to some extend- fluctuations. Using appropriate combinations of Corollaries 11, 13, 15 and 17 it is easy to examine the comovement between several variables of the system. For example, it is often reported in the literature that there exist a positive or negative relationship between the beta coefficient and the corresponding factor conditional variance. Under ARCBeta effects, the two variables share the same innovation process and we would expect them to exhibit some comovement the sign and the magnitude of which may depend on the respective process parameters. To see this we can easily calculate Cov $\left(\beta_{ij,t},\sigma^2_{j,t}\right)$ by combining Corollaries 11 and 15. Assuming the two variables follow ARCBeta(1,0) and ARCH(1) processes respectively and have a standard gaussian innovation process in common, we can easily see that

$$\text{Cov}\left(\beta_{ij,t},\sigma^2_{j,t}\right)=2\gamma_0 a_0\frac{a_1\gamma_1}{\left(1-3a_1\gamma_1\right)\left(1-\gamma_1\right)\left(1-a_1\right)}$$

ARCBeta parameters a_0 and a_1 are unrestricted and ARCH parameters should satisfy $\gamma_0>0$ and $\gamma_1\geqslant 0$. The denominator of the above equation presents clearly the individual and joint conditions the two processes should satisfy for this quantity to exist. Ignoring the two intercept terms a_0,γ_0 we plot Cov$\left(\beta_{ij,t},\sigma^2_{j,t}\right)$ against a range of plausible values of a_1 and γ_1 in Graph IV (see appendix). It is clear that the sign and the magnitude of the covariance depends on the trade-off between ARCH and ARCBeta effects. In particular, positive covariance generally is generated for high negative values of the ARCBeta parameter and any values for the ARCH parameter as well as for weak ARCH effects and any values for the ARCBeta parameter. Also, the interaction of the intercept terms a_0,γ_0 controls the scale of the covariance and adjusts the position of the covariance surface on the vertical axis.

As other example, we can examine the comovement between beta coefficients of different assets on a common factor, by calculating Cov $\left(\beta_{ij,t},\beta_{kj,t}\right)$. Using Lemma 14 and Corollary 15, if $\beta_{1,t}$ and $\beta_{2,t}$ both follow ARCBeta(1,0) processes with common gaussian innovations, we obtain

$$\text{Cov}\left(\beta_{1,t},\beta_{2,t}\right)=a_{10}a_{20}\frac{2a_{11}-6a_{11}^2a_{21}+4a_{11}a_{21}+3a_{21}^2a_{11}^2-1}{\left(1-3a_{11}a_{21}\right)\left(1-a_{21}\right)\left(1-a_{11}\right)}$$

Ignoring the $a_{10}a_{20}$, we plot in Graph V the covariance between betas as a function of the ARCBeta parameters a_{11} and a_{21}. For a plausible range parameter values the covariance between factor betas is positive but, both its scale and sign will also depend on the $a_{10}a_{20}$ factor.

The proposed model also implies time varying conditional correlations between assets and/or factors that are guaranteed to be less than one in absolute value. In particular, equation (4) in section 3 presents the conditional covariance matrix Ω_t to be measurable with respect to the generated sigma field as well as positive definite. Taking the decomposition $\Omega_t = S_t R_t S_t$ where $S_t = \text{diag}(\Omega_t)^{\frac{1}{2}}$, we obtain $R_t = S_t^{-1}\Omega_t S_t^{-1}$. The north west block of R_t is the asset correlation matrix which (see equation (4)) can be written as

$$R_{yy,t} = S_{yy,t}^{-1}\Sigma_{\varepsilon,t}S_{yy,t}^{-1} + S_{yy,t}^{-1}B_t\Sigma_{\varepsilon,t}B_t^{'}S_{yy,t}^{-1}$$

where $S_{yy,t} = \text{diag}(\Omega_{yy,t})^{\frac{1}{2}}$.

In this paper we have presented a multivariate framework that jointly parametrizes the evolution of N asset returns and K heteroscedastic common factors. The model guarantees the positive definiteness of the covariance matrix and is flexible enough to generate a number of reported empirical stylized facts in the finance literature. This goes beyond the standard phenomena of multivariate heteroscedasticity and non-normality and presents arguments for the time variation of factor beta coefficients, their mean-reverting properties and the co-movement between factor betas and volatilities. The model provides an explicit framework for asset variance, covariance, correlation and factor beta out-of-sample forecasting and capable of accommodating the effects of exogenous variables. We provide sufficient conditions for the existence of the implied asset covariance structure and present empirical evidence for the three major European stock market portfolios as an evolving system.

6 Appendix

6.1 Proof of Theorem 1

Consider the representation of the model

$$\begin{bmatrix} \xi_t \\ \xi_{t-1} \\ \vdots \\ \xi_{t-p+1} \end{bmatrix} = \begin{bmatrix} \alpha_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \alpha_1 & \cdots & \alpha_{p-1} & \alpha_p \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_{t-1} \\ \xi_{t-2} \\ \vdots \\ \xi_{t-p} \end{bmatrix} + \begin{bmatrix} \xi_t - \beta_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or

$$\boldsymbol{\xi}_t = A_0 + A\boldsymbol{\xi}_{t-1} + \mathbf{v}_t$$

upon recursive backward substitution for s times, we obtain

$$\boldsymbol{\xi}_t = (I + A + ... + A^{s-1})A_0 + A^s\boldsymbol{\xi}_{t-s} + \sum_{j=0}^{s-1} A^j \mathbf{v}_{t-j}$$

As $s \to \infty$, the process is stable if and only if A has all its eigenvalues within the complex unit circle, or equivalently the associated characteristic polynomial $\lambda^p - \alpha_1\lambda^{p-1} - ... - \alpha_q$ having all its roots inside the complex unit circle. Then $\lim_{s\to\infty} A^s = 0$ and

$$\boldsymbol{\xi}_t = (I - A)^{-1} A_0 + \sum_{j=0}^{\infty} A^j \mathbf{v}_{t-j}$$

The second term of the right-hand-side converges under the stability conditions and the stochastic difference equation is well-defined. Taking expectations, it follows that the steady-state $\boldsymbol{\xi}_t$ is $(I - A)^{-1} A_0$ and thus the result (Th1.a).

6.2 Proof of Lemma 2

By Chung (1974, xi, p 42)

$$\sum_{i\geq 1} E\left|\prod_{j=1}^{i} \Delta_{t-j}\Delta_0\right|_l < \infty$$

for all l (where $(\cdot)_l$ denotes the l-th element of a vector and $(\cdot)_{l,n}$ denotes the (l,n)-th elment of a matrix) implies that

$$\sum_{i\geq 1}\prod_{j=1}^{i} \Delta_{t-j}\Delta_0$$

is absolutely convergent almost surely. We are interested in its first element $(l = 1)$ as the second is trivial, that is

$$E\left|\prod_{j=1}^{i} \Delta_{t-j}\Delta_0\right|_1 = E\left|\sum_{s=1}^{2}\left(\prod_{j=1}^{i}\Delta_{t-j}\right)_{1,s}(\Delta_0)_s\right| \leq \sum_{s=1}^{2} E\left|\left(\prod_{j=1}^{i}\Delta_{t-j}\right)_{1,s}(\Delta_0)_s\right|$$

by triangle inequality. The last term of the r-h-s of the above equation is not grater than

$$\sum_{s=1}^{2}\left[E\left(\left(\prod_{j=1}^{i}\Delta_{t-j}\right)_{1,s}(\Delta_0)_s\right)^2\right]^{\frac{1}{2}}$$

by Cauchy-Schwartz inequality. Now we denote $(X \otimes Y)_{k,l:m,n}$ the product of the $(\cdot)_{k,l}(\cdot)_{m,n}$ elements of X, Y and evaluate for $s = 1, 2$

$$\begin{aligned} E\left(\prod_{j=1}^{i}\Delta_{t-j}\right)_{1,s}^{2} &= E\left(\prod_{j=1}^{i}\Delta_{t-j} \otimes \prod_{j=1}^{i}\Delta_{t-j}\right)_{1,s:1,s} \\ &= \prod_{j=1}^{i} E\left(\Delta_{t-j}\otimes\Delta_{t-j}\right)_{1,s:1,s} = \Delta^{i}_{1,s:1,s} \end{aligned}$$

by independence. This quantity will be finite if all the eigenvalues of $\Delta = E(\Delta_{t-j} \otimes \Delta_{t-j})$ are less than one in modulus. Q.E.D.

6.3 Proof of Lemma 4

From equation (number) $\text{vec}\left(B_t B_t^{'}\right) = \bar{B}_t = \bar{A}_0 + \bar{A}_{t-1} B_{t-1} + \dot{A}_{t-1} \bar{B}_{t-1}$. Upon recursive substitution k times we obtain

$$\begin{aligned} \bar{B}_t &= \bar{A}_0 + \bar{A}_{t-1} B_{t-1} + \sum_{i=1}^{k-1} \prod_{j=1}^{i} \dot{A}_{t-j} \bar{A}_0 + \sum_{i=2}^{k} \prod_{j=1}^{i-1} \dot{A}_{t-j} \bar{A}_{t-i} B_{t-i} \quad \text{(L41)} \\ &+ \prod_{j=1}^{k} \dot{A}_{t-j} \bar{B}_{t-k} \end{aligned}$$

We shall examine of each component of this process separately.

1. By Lemma 2 and independence between $\bar{A}_{t-1}$ and $A_{t-j}, j \geqslant 2$ the term $\bar{A}_{t-1} B_{t-1}$ is strictly stationary if v_t has finite fourth moment and $E\left(A_t \otimes A_t\right)$ has spectral radius less than one.

2. For $k \to \infty$, by Chung (1974, xi, p 42)

$$\sum_{i \geqslant 1} E \left| \prod_{j=1}^{i} \dot{A}_{t-j} \bar{A}_0 \right|_l < \infty$$

for all l, (where $(\cdot)_{l,n}$denotes the $l, n-$th elment of a matrix), implies that

$$\sum_{i \geqslant 1} \prod_{j=1}^{i} \dot{A}_{t-j} \bar{A}_0$$

is absolutely convergent almost surely. We establish the result for $l = 1$:

$$E \left| \left(\prod_{j=1}^{i} \dot{A}_{t-j} \bar{A}_0 \right)_1 \right| = E \left| \sum_{s=1}^{4} \left(\prod_{j=1}^{i} \dot{A}_{t-j} \right)_{1,s} \left(\bar{A}_0\right)_s \right| \leq \sum_{s=1}^{4} E \left| \left(\prod_{j=1}^{i} \dot{A}_{t-j} \right)_{1,s} \left(\bar{A}_0\right)_s \right|$$

by triangle inequality. The last term of the r-h-s of the above equation is not grater than

$$\sum_{s=1}^{4} \left[E \left(\left(\prod_{j=1}^{i} \dot{A}_{t-j} \right)_{1,s} \left(\bar{A}_0\right)_s \right)^2 \right]^{\frac{1}{2}}$$

by Caucky-Schwartz inequality. Now we denote $(X \otimes Y)_{k,l:m,n}$ the product of the $(\cdot)_{k,l}\,(\cdot)_{m,n}$ elements of X, Y and evaluate for $s = 1, 2, 3, 4$

$$\begin{aligned} E\left(\prod_{j=1}^{i} \dot{A}_{t-j}\right)^2_{1,s} &= E\left(\prod_{j=1}^{i} \dot{A}_{t-j} \otimes \prod_{j=1}^{i} \dot{A}_{t-j}\right)_{1,s:1,s} \\ &= \prod_{j=1}^{i} E\left(\dot{A}_{t-j} \otimes \dot{A}_{t-j}\right)_{1,s:1,s} = \left(E\left(\dot{A}_{t-j} \otimes \dot{A}_{t-j}\right)\right)^i_{1,s:1,s} \end{aligned}$$

which will be finite if $E\left(\dot{A}_{t-j} \otimes \dot{A}_{t-j}\right)$ has all its eigenvalues within the unit circle.

3. Letting $k \to \infty$ and by Lemma 2 the term

$$\sum_{i\geqslant 2} \prod_{j=1}^{i-1} \dot{A}_{t-j}\bar{A}_{t-i}B_{t-i} = \sum_{i\geqslant 2} \prod_{j=1}^{i-1} \dot{A}_{t-j}\bar{A}_{t-i}A_0 + \sum_{i\geqslant 2} \prod_{j=1}^{i-1} \dot{A}_{t-j}\bar{A}_{t-i} \sum_{r\geqslant i+1} \prod_{j=i+1}^{r} A_{t-j}A_0$$

Repeating the arguments of the previous step and using Lemma 2 and the independence between $\dot{A}_{t-j}, \bar{A}_{t-i}$and A_{t-j} we see easily that the above term also is absolutely convergent almost surely if $E\left(\dot{A}_{t-j} \otimes \dot{A}_{t-j}\right)$ has all its eigenvalues within the unit circle. Q.E.D.

6.4 Proof of Corollary 5

Let $k \to \infty$ in $L41$ (see proof of Lemma 4) and take expectations.. The last term of the r-h-s will vanish by stationarity and by independence we have

$$\begin{aligned} E\left(\bar{B}_t\right) &= \bar{A}_0 + E\left(\bar{A}_{t-1}\right) E\left(B_{t-1}\right) \\ &\quad + \sum_{i\geqslant 1} \prod_{j=1}^{i} E\left(\dot{A}_{t-j}\right) \bar{A}_0 \\ &\quad + \sum_{i\geqslant 2} \prod_{j=1}^{i-1} E\left(\dot{A}_{t-j}\right) E\left(\bar{A}_{t-i}\right) E\left(B_{t-i}\right) \end{aligned}$$

By Corollary 3 $E(B_{t-i}) = (I-A)^{-1} A_0$ where $A = E(A_{t-j})$, also let $\bar{A} = E(\bar{A}_{t-i})$, $\dot{A} = E\left(\dot{A}_{t-j}\right)$, then by Lemma 2

$$E(\bar{B}_t) = \bar{A}_0 + \bar{A}(I-A)^{-1} A_0 + \left(I-\dot{A}\right)^{-1} \dot{A}\left(\bar{A}_0 + \bar{A}(I-A)^{-1} A_0\right)$$

6.5 Proof of Lemma 6

Consider

$$\begin{aligned}\operatorname{vec}\left(B_t\vec{\Sigma}_t^{'}\right) &= \operatorname{vec}\left(A_0\Gamma_0^{'}\right) + (\Gamma_{t-1}\otimes A_0)\operatorname{vec}\left(\vec{\Sigma}_{t-1}^{'}\right) + (\Gamma_0\otimes A_{t-1})\operatorname{vec}(B_{t-1}) \\ &\quad + (\Gamma_{t-1}\otimes A_{t-1})\operatorname{vec}\left(B_{t-1}\vec{\Sigma}_{t-1}^{'}\right)\end{aligned}$$

Upon recursive substitution k times we obtain

$$\begin{aligned}\operatorname{vec}\left(B_t\vec{\Sigma}_t^{'}\right) &= \operatorname{vec}\left(A_0\Gamma_0^{'}\right) + (\Gamma_{t-1}\otimes A_0)\vec{\Sigma}_{t-1} + (\Gamma_0\otimes A_{t-1})B_{t-1} \\ &\quad + \sum_{r=1}^{k-1}\prod_{j=1}^{r}(\Gamma_{t-j}\otimes A_{t-j})\operatorname{vec}\left(A_0\Gamma_0^{'}\right) \\ &\quad + \sum_{r=2}^{k}\prod_{j=1}^{r-1}(\Gamma_{t-j}\otimes A_{t-j})(\Gamma_{t-r}\otimes A_0)\vec{\Sigma}_{t-r} \qquad \text{(L61)} \\ &\quad + \sum_{r=2}^{k}\prod_{j=1}^{r-1}(\Gamma_{t-j}\otimes A_{t-j})(\Gamma_0\otimes A_{t-r})B_{t-r} \\ &\quad + \prod_{j=1}^{k}(\Gamma_{t-j}\otimes A_{t-j})\operatorname{vec}\left(B_{t-k}\vec{\Sigma}_{t-k}^{'}\right)\end{aligned}$$

We proceed in steps.

1. Terms $(\Gamma_{t-1}\otimes A_0)\vec{\Sigma}_{t-1}$ and $(\Gamma_0\otimes A_{t-1})B_{t-1}$ have a strictly stationary solution by Lemma 2 and the independence of Γ_{t-1} and A_{t-1} with $A_{t-j}, \Gamma_{t-j}, j \geqslant 2$.

2. Letting $k\to\infty$ and using the same arguments as in step 2 of Lemma 3 $\sum_{r\geqslant 1}\prod_{j=1}^{r}(\Gamma_{t-j}\otimes A_{t-j})\operatorname{vec}\left(A_0\Gamma_0^{'}\right)$ will be absolutely convergent almost

surely if all the eigenvalues of $E\left(\left(\Gamma_{t-j}\otimes A_{t-j}\right)\otimes\left(\Gamma_{t-j}\otimes A_{t-j}\right)\right)$ lie within the unit circle.

3. Letting $k\rightarrow\infty$ we have $\sum_{r\geqslant 2}\prod_{j=1}^{r-1}\left(\Gamma_{t-j}\otimes A_{t-j}\right)\left(\Gamma_{t-r}\otimes A_{0}\right)\vec{\Sigma}_{t-r}$ and $\sum_{r\geqslant 2}\prod_{j=1}^{r-1}\left(\Gamma_{t-j}\otimes A_{t-j}\right)\left(\Gamma_{\mathbf{0}}\otimes A_{t-r}\right)B_{t-r}$. By Lemma 2 and the independence of A_{t-r},Γ_{t-r} and $\Gamma_{t-j},A_{t-j},j=1,..,.r-1$, both sequences will be absolutely convergent almost surely if all the eigenvalues of

$$E\left(\left(\Gamma_{t-j}\otimes A_{t-j}\right)\otimes\left(\Gamma_{t-j}\otimes A_{t-j}\right)\right)$$

lie within the unit circle. Q.E.D.

6.6 Proof of Corollary 7

Let $k\rightarrow\infty$ in L61 (see proof of Lemma 6) and take expectations. Then the last term of the r-h-s will vanish by stationarity and by independence we have

$$\begin{aligned}
E\left(\operatorname{vec}\left(B_{t}\vec{\Sigma}_{t}^{'}\right)\right) &= \operatorname{vec}\left(A_{0}\Gamma_{\mathbf{0}}^{'}\right)+E\left(\Gamma_{t-1}\otimes A_{0}\right)E\left(\vec{\Sigma}_{t-1}\right)\\
&+E\left(\Gamma_{\mathbf{0}}\otimes A_{t-1}\right)E\left(B_{t-1}\right)\\
&+\sum_{r\geqslant 1}\prod_{j=1}^{r}E\left(\Gamma_{t-j}\otimes A_{t-j}\right)\operatorname{vec}\left(A_{0}\Gamma_{\mathbf{0}}^{'}\right)\\
&+\sum_{r\geqslant 2}\prod_{j=1}^{r-1}E\left(\Gamma_{t-j}\otimes A_{t-j}\right)E\left(\Gamma_{t-r}\otimes A_{0}\right)E\left(\vec{\Sigma}_{t-r}\right)\\
&+\sum_{r\geqslant 2}\prod_{j=1}^{r-1}E\left(\Gamma_{t-j}\otimes A_{t-j}\right)E\left(\Gamma_{\mathbf{0}}\otimes A_{t-r}\right)E\left(B_{t-r}\right)
\end{aligned}$$

By Corollary 3 we have $E\left(B_{t-r}\right)=\left(I-A\right)^{-1}A_{0}$ and $E\left(\vec{\Sigma}_{t-r}\right)=\left(I-\Gamma\right)^{-1}\Gamma_{\mathbf{0}}$, (where $A=E\left(A_{t-j}\right)$, and $\Gamma=E\left(\Gamma_{t-j}\right)$), also let $E\left(\Gamma_{t-r}\otimes A_{0}\right)=\Gamma_{A_{0}}$, $E\left(\Gamma_{\mathbf{0}}\otimes A_{t-r}\right)=A_{\gamma_{0}}$ and $E\left(\Gamma_{t-j}\otimes A_{t-j}\right)=\Gamma_{A}$. If Γ_{A} has all its eigenvalues within the unit circle we have

$$
\begin{aligned}
E\left(\text{vec}\left(B_t\vec{\Sigma}_t^{'}\right)\right) &= \text{vec}\left(A_0\Gamma_{\mathbf{0}}^{'}\right) + \Gamma_{A_0}\left(I-\Gamma\right)^{-1}\Gamma_{\mathbf{0}} + A_{\gamma_0}\left(I-A\right)^{-1}A_{\mathbf{0}} \\
&+ \left(I-\Gamma_A\right)^{-1}\Gamma_A\left[\text{vec}\left(A_{\mathbf{0}}\Gamma_{\mathbf{0}}^{'}\right) + \Gamma_{A_0}\left(I-\Gamma\right)^{-1}\Gamma_{\mathbf{0}}\right] \\
&+ \left(I-\Gamma_A\right)^{-1}\Gamma_A\left[A_{\gamma_0}\left(I-A\right)^{-1}A_{\mathbf{0}}\right]
\end{aligned}
$$

6.7 Proof of Theorem 8

Upon recursive backward substitution in equation (9) we obtain

$$
\begin{aligned}
\text{vec}\left(\bar{B}_t\vec{\Sigma}_t^{'}\right) &= \text{vec}\left(\bar{A}_0\Gamma_{\mathbf{0}}^{'}\right) + \Phi_{t-1} + \sum_{i=1}^{k-1}\prod_{j=1}^{i}\left(\Gamma_{t-j}\otimes\dot{A}_{t-j}\right)\text{vec}\left(\bar{A}_0\Gamma_{\mathbf{0}}^{'}\right) \\
&+ \sum_{i=2}^{k}\prod_{j=1}^{i-1}\left(\Gamma_{t-j}\otimes\dot{A}_{t-j}\right)\Phi_{t-k} \qquad \text{(Th8.1)} \\
&+ \prod_{j=1}^{k}\left(\Gamma_{t-j}\otimes\dot{A}_{t-j}\right)\text{vec}\left(\bar{B}_{t-k}\vec{\Sigma}_{t-k}^{'}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_{t-k} &= \left(\Gamma_{t-k}\otimes\bar{A}_{\mathbf{0}}\right)\vec{\Sigma}_{t-k} + \left(\Gamma_{\mathbf{0}}\otimes\bar{A}_{t-k}\right)B_{t-k} \\
&+ \left(\Gamma_{t-k}\otimes\bar{A}_{t-k}\right)\text{vec}\left(B_{t-k}\vec{\Sigma}_{t-k}^{'}\right) + \left(\Gamma_{\mathbf{0}}\otimes\dot{A}_{t-k}\right)\bar{B}_{t-k}
\end{aligned}
$$

We proceed in steps.

1. For Φ_{t-1}, both $\left(\Gamma_{t-1}\otimes\bar{A}_{\mathbf{0}}\right)\vec{\Sigma}_{t-1}$ and $\left(\Gamma_{\mathbf{0}}\otimes\bar{A}_{t-1}\right)B_{t-1}$ are strictly stationary by Lemma 2 and the independence of Γ_{t-1}, $\bar{A}_{t-1}$ with Γ_{t-j}, $\bar{A}_{t-j}$, $j\geqslant 2$ respectively, if the conditions of Lemma 2 hold. Also, $\left(\Gamma_{t-1}\otimes\bar{A}_{t-1}\right)\text{vec}\left(B_{t-1}\vec{\Sigma}_{t-1}^{'}\right)$ is strictly stationary by Lemma 6 and the independence of $\left(\Gamma_{t-1}\otimes\bar{A}_{t-1}\right)$ with $\left(\Gamma_{t-j}\otimes\bar{A}_{t-j}\right)$, $j\geqslant 2$, if the conditions of Lemma 6 hold and so is $\left(\Gamma_{\mathbf{0}}\otimes\dot{A}_{t-1}\right)\bar{B}_{t-1}$ by Lemma 4 and the independence of $\dot{A}_{t-1}$ with $\dot{A}_{t-j}$, $j\geqslant 2$, if the conditions of Lemma 4 hold.

2. For $k \to \infty$ and following the methodology of Lemma 2, the third term of the r-h-s will be absolutely convergent almost surely if

$$E\left(\left(\Gamma_{t-j} \otimes \dot{A}_{t-j}\right) \otimes \left(\Gamma_{t-j} \otimes \dot{A}_{t-j}\right)\right)$$

has all its eigenvalues within the unit circle.

3. The term

$$\begin{aligned}
& \sum_{i=2}^{k} \prod_{j=1}^{i-1} \left(\Gamma_{t-j} \otimes \dot{A}_{t-j}\right) \Phi_{t-k} \\
= \; & \sum_{i=2}^{k} \prod_{j=1}^{i-1} \left(\Gamma_{t-j} \otimes \dot{A}_{t-j}\right) \left(\Gamma_{t-k} \otimes \bar{A}_{0}\right) \vec{\Sigma}_{t-k} \\
& + \sum_{i=2}^{k} \prod_{j=1}^{i-1} \left(\Gamma_{t-j} \otimes \dot{A}_{t-j}\right) \left(\Gamma_{0} \otimes \bar{A}_{t-k}\right) B_{t-k} \\
& + \sum_{i=2}^{k} \prod_{j=1}^{i-1} \left(\Gamma_{t-j} \otimes \dot{A}_{t-j}\right) \left(\Gamma_{t-k} \otimes \bar{A}_{t-k}\right) \operatorname{vec}\left(B_{t-k} \vec{\Sigma}_{t-k}'\right) \\
& + \sum_{i=2}^{k} \prod_{j=1}^{i-1} \left(\Gamma_{t-j} \otimes \dot{A}_{t-j}\right) \left(\Gamma_{0} \otimes \dot{A}_{t-k}\right) \bar{B}_{t-k}
\end{aligned}$$

3a. For $k \to \infty$ and using Lemma 2 and the independence between Γ_{t-k}, A_{t-k} and Γ_{t-j}, A_{t-j}, $j = 1, ...k-1$ (exactly as in step 3 of Lemmas 4 and 6) the first two terms of the r-h-s will be strictly stationary if $E\left(\left(\Gamma_{t-j} \otimes \dot{A}_{t-j}\right) \otimes \left(\Gamma_{t-j} \otimes \dot{A}_{t-j}\right)\right)$ has all its eigenvalues within the unit circle.

3b. For $k \to \infty$ and Lemmas 4 and 6 respectively, and independence as in (3a) the last two terms of the r-h-s are also strictly stationary if $E\left(\left(\Gamma_{t-j} \otimes \dot{A}_{t-j}\right) \otimes \left(\Gamma_{t-j} \otimes \dot{A}_{t-j}\right)\right)$ has all its eigenvalues within the unit circle.

6.8 Proof of Corollary 9

Let $k \to \infty$ in Th8.1 (see proof of theorem 8) and take expectations. By Theorem 8 the last term of the r-h-s will vanish and by independence we have

$$\begin{aligned} E\left(\operatorname{vec}\left(\bar{B}_t\vec{\Sigma}_t^{'}\right)\right) &= \operatorname{vec}\left(\bar{A}_0\Gamma_{\mathbf{0}}^{'}\right) + E\left(\Phi_{t-1}\right) \\ &+ \sum_{i\geqslant 1}\prod_{j=1}^{i} E\left(\Gamma_{t-j}\otimes \dot{A}_{t-j}\right)\operatorname{vec}\left(\bar{A}_0\Gamma_{\mathbf{0}}^{'}\right) \\ &+ \sum_{i\geqslant 2}\prod_{j=1}^{i-1} E\left(\Gamma_{t-j}\otimes \dot{A}_{t-j}\right) E\left(\Phi_{t-k}\right) \end{aligned}$$

Using corollaries 3, 5 and 7 and denoting $\Gamma_{\bar{A}_0} = E\left(\Gamma_{t-k}\otimes \bar{A}_{\mathbf{0}}\right)$, $A_{\bar{\gamma}_0} = E\left(\Gamma_{\mathbf{0}}\otimes \bar{A}_{t-k}\right)$, $\Gamma_{\bar{A}} = E\left(\Gamma_{t-k}\otimes \bar{A}_{t-k}\right)$, $A_{\dot{\gamma}_0} = E\left(\Gamma_{\mathbf{0}}\otimes \dot{A}_{t-k}\right)$ we have

$$\begin{aligned} & E\left(\Phi_{t-k}\right) \\ = \; & \Gamma_{\bar{A}_0}\left(I-\Gamma\right)^{-1}\Gamma_{\mathbf{0}} + A_{\bar{\gamma}_0}\left(I-A\right)^{-1}A_{\mathbf{0}} \\ & + A_{\dot{\gamma}_0}\left[\bar{A}_{\mathbf{0}} + \bar{A}\left(I-A\right)^{-1}A_{\mathbf{0}} + \left(I-\dot{A}\right)^{-1}\dot{A}\left(\bar{A}_{\mathbf{0}} + \bar{A}\left(I-A\right)^{-1}A_{\mathbf{0}}\right)\right] \\ & + \Gamma_{\bar{A}}\left[\operatorname{vec}\left(A_{\mathbf{0}}\Gamma_{\mathbf{0}}^{'}\right) + \Gamma_{A_0}\left(I-\Gamma\right)^{-1}\Gamma_{\mathbf{0}} + A_{\bar{\gamma}_0}\left(I-A\right)^{-1}A_{\mathbf{0}}\right] \\ & + \Gamma_{\bar{A}}\left[\left(I-\Gamma_A\right)^{-1}\Gamma_A\left(\operatorname{vec}\left(A_{\mathbf{0}}\Gamma_{\mathbf{0}}^{'}\right) + \Gamma_{A_0}\left(I-\Gamma\right)^{-1}\Gamma_{\mathbf{0}} + A_{\bar{\gamma}_0}\left(I-A\right)^{-1}A_{\mathbf{0}}\right)\right] \end{aligned}$$

Also denote $\Gamma_{\dot{A}} = E\left(\Gamma_{t-j}\otimes \dot{A}_{t-j}\right)$, then by Theorem 8

$$\begin{aligned} E\left(\operatorname{vec}\left(\bar{B}_t\vec{\Sigma}_t^{'}\right)\right) &= \operatorname{vec}\left(\bar{A}_0\Gamma_{\mathbf{0}}^{'}\right) + \left(I-\Gamma_{\dot{A}}\right)^{-1}\operatorname{vec}\left(\bar{A}_0\Gamma_{\mathbf{0}}^{'}\right) \\ &+ \left(I-\Gamma_{\dot{A}}\right)^{-1}E\left(\Phi_{t-k}\right) \end{aligned}$$

6.9 Proof of Theorem 10

The proof follows exactly the steps of Theorem 8. Thus we sketch the proof primarily to establish the notation presented in the text and then make available a formula for the steady-state covariance. Let $B_{i,t}$, $B_{l,t}$, and $\vec{\Sigma}_{j,t}$ be generated by

$$\begin{aligned} B_{i,t} &= A_{i,\mathbf{0}} + A_{i,t-1}B_{i,t-1} \\ B_{l,t} &= A_{l,\mathbf{0}} + A_{l,t-1}B_{l,t-1} \\ \vec{\Sigma}_{j,t} &= \Gamma_{j,\mathbf{0}} + \Gamma_{j,t-1}\vec{\Sigma}_{j,t-1} \end{aligned}$$

and let $\bar{B}_{il,t} = \text{vec}\big(B_{i,t}B'_{l,t}\big)$. Then

$$\text{vec}\left(\bar{B}_{il,t}\vec{\Sigma}'_{j,t}\right) = \text{vec}\left(\bar{A}_{il,\mathbf{0}}\Gamma'_{j,\mathbf{0}}\right) + \Psi_{t-1} + \left(\Gamma_{j,t-1} \otimes \bar{A}_{li,t-1}\right)\text{vec}\left(\bar{B}_{il,t-1}\vec{\Sigma}'_{j,t-1}\right) \tag{Th10.1}$$

where

$$\begin{aligned} \Psi_{t-1} \;=\; & \left(\Gamma_{j,t-1} \otimes \bar{A}_{li,\mathbf{0}}\right)\vec{\Sigma}'_{j,t-1} + \left(\Gamma_{j,\mathbf{0}} \otimes \bar{A}_{l,t-1}\right)B_{l,t-1} + \left(\Gamma_{j,\mathbf{0}} \otimes \bar{A}_{i,t-1}\right)B_{i,t-1} \\ & + \left(\Gamma_{j,t-1} \otimes \bar{A}_{l,t-1}\right)\text{vec}\left(B_{l,t-1}\vec{\Sigma}'_{j,t-1}\right) + \left(\Gamma_{j,t-1} \otimes \bar{A}_{i,t-1}\right)\text{vec}\left(B_{i,t-1}\vec{\Sigma}'_{j,t-1}\right) \\ & + \left(\Gamma_{j,\mathbf{0}} \otimes \bar{A}_{li,t-1}\right)\bar{B}_{il,t-1} \end{aligned}$$

and

$$\begin{aligned} \bar{B}_{il,t} &= \text{vec}\left(B_{i,t}B'_{l,t}\right) \\ \bar{A}_{il,\mathbf{0}} &= \text{vec}\left(A_{i,\mathbf{0}}A'_{l,\mathbf{0}}\right) \\ \bar{A}_{l,t-1} &= \left(A_{l,t-1} \otimes A_{i,\mathbf{0}}\right) \\ \bar{A}_{i,t-1} &= \left(A_{l,\mathbf{0}} \otimes A_{i,t-1}\right) \\ \bar{A}_{li,t-1} &= \left(A_{l,t-1} \otimes A_{i,t-1}\right) \end{aligned}$$

Recursive substitution in Th10.1 yields

$$\begin{aligned} \text{vec}\left(\bar{B}_{il,t}\vec{\Sigma}'_{j,t}\right) &= \text{vec}\left(\bar{A}_{il,\mathbf{0}}\Gamma'_{j,\mathbf{0}}\right) + \Psi_{t-1} + \sum_{i=1}^{k-1}\prod_{j=1}^{i}\left(\Gamma_{j,t-j} \otimes \bar{A}_{li,t-j}\right)\text{vec}\left(\bar{A}_{il,\mathbf{0}}\Gamma'_{j,\mathbf{0}}\right) \\ & + \sum_{i=2}^{k}\prod_{j=1}^{i-1}\left(\Gamma_{j,t-j} \otimes \bar{A}_{li,t-j}\right)\Psi_{t-k} \end{aligned}$$

$$+\prod_{j=1}^{k}\left(\Gamma_{j,t-j}\otimes\bar{A}_{li,t-j}\right)\operatorname{vec}\left(\bar{B}_{il,t-k}\vec{\Sigma}'_{j,t-k}\right)$$

which is of the same form as Th8.1. The proof proceeds as in Theorem 8 for appropriately defined matrices.

6.10 Tables

Table I : Constant Beta Factors/Constant Idiosyncratic Variances

Parameters	UK	France	Germany	EU	Japan	US
Mean						
c	.23 (2.2)	.23 (2.6)	.21 (2.5)	.23 (3.8)	.25 (3.8)	.21 (4.1)
Variance						
γ_0	1.98 (33)	4.75 (51)	2.62 (27)	.53 (5.1)	.26 (4.3)	.21 (3.3)
γ_1	-	-	-	.17 (6.1)	.12 (6.4)	.12 (6.5)
δ_1	-	-	-	.71 (18)	.85 (38)	.82 (30.1)
Beta				EU	Japan	US
UK				1.28 (44.5)	$-.085$ (-3.4)	$-.028$ $(-.94)$
France				.85 (23.5)	.06 (2.1)	.05 (1.4)
Germany				.82 (23.7)	.07 (2.5)	$-.05$ (-1.6)
Log L AIC SIC	$-17,541$ 24.86 24.76					

NOTES: Weekly data 05.04.1973 to 30.03.00 (N=1409 observations). Multivariate Quasi Maximum Likelihood estimates, heteroscedasticity robust t-statistics in brackets. $AIC = -2\left(\frac{\ln L}{N} + \frac{k}{N}\right)$, $SIC = -2\left(\frac{\ln L}{N} + \frac{k \ln N}{2N}\right)$, $LR = 2\left(\ln L_{ur} - \ln L_r\right)$

Table II : ARCBeta Factor (UK)

Parameters	UK	France	Germany	EU	Japan	US
Mean						
c	.30 (3.1)	.27 (3.2)	.26 (3.2)	.28 (4.5)	.25 (3.9)	.23 (4.7)
Variance						
γ_0	1.92 (32.5)	4.77 (50.8)	2.6 (26.6)	.53 (5.0)	.26 (4.3)	.22 (3.3)
γ_1	–	–	–	.176 (6.2)	.116 (6.3)	.128 (6.4)
δ_1	–	–	–	.71 (18.1)	.85 (38.9)	.82 (29.1)
ARCBeta				EU	Japan	US
UK a_0				.0096 (1.9)	$-.0018$ (-2.5)	$-.0014$ (-3.3)
UK a_1				.0049 (12)	.0246 (1.02)	.3262 (3.9)
UK b_1				.9872 (155)	.9508 (34.82)	.6371 (8.3)
UK $\frac{a_0}{1-a_1-b_1}$				1.2	.036	$-.038$
France				.85 (25)	.06 (2.0)	.046 (1.5)
Germany				.82 (29)	.067 (2.7)	$-.05$ (-2.37)
Log L	$-17,532$					
LR	18 [12.6]					
AIC	24.83					
SIC	24.71					

NOTES: Weekly data 05.04.1973 to 30.03.00 (N=1409 observations). Multivariate Quasi Maximum Likelihood estimates, heteroscedasticity robust t-statistics in brackets.brackets. $XX^2\,(6)$ critical value in square brackets. $AIC = -2\left(\frac{\ln L}{N} + \frac{k}{N}\right)$, $SIC = -2\left(\frac{\ln L}{N} + \frac{k \ln N}{2N}\right)$, $LR = 2\left(\ln L_{ur} - \ln L_r\right)$

Table III : ARCBeta Factors (UK, FRANCE)

Parameters	UK	France	Germany	EU	Japan	US
Mean						
c	.31 (3.2)	.28 (3.3)	.25 (3.2)	.28 (4.6)	.25 (3.9)	.23 (4.8)
Variance						
γ_0	1.92 (34.1)	4.77 (69.6)	2.6 (27)	.53 (5.0)	.26 (4.3)	.22 (3.3)
γ_1	–	–	–	.176 (6.2)	.116 (6.3)	.128 (6.4)
δ_1	–	–	–	.71 (18.1)	.85 (38.9)	.82 (29.1)
ARCBeta				EU	Japan	US
UK a_0				.0096 (1.8)	−.0014 (−1.8)	−.0013 (3.3)
UK a_1				.0044 (4)	.0405 (1.1)	.3161 (3.7)
UK b_1				.98 (70)	.93 (29)	.64 (8.1)
UK $\frac{a_0}{1-a_1-b_1}$						
FR a_0				.0077 (1.1)	.056 (2.28)	.0011 (.52)
FR a_1				−.0026 (−1.7)	–	−.0526 (−.99)
FR b_1				.9936 (131)	–	1.0299 (15.2)
FR $\frac{a_0}{1-a_1-b_1}$				.85	.056	.048
GE				.8231 (40.6)	.0669 (4.3)	−.0543 (−2.6)
Log L	−17, 527					
LR	10 (9.49)					
AIC	24.82					
SIC	24.68					

NOTES: Weekly data 05.04.1973 to 30.03.00 (N=1409 observations). Multivariate Quasi Maximum Likelihood estimates, heteroscedasticity robust t-statistics in brackets. X^2 (4) critical value in square brackets. AIC (Akaike), SIC (Schwartz)

Table IV : ARCBeta Factors (UK, FR, GE)

NOTES: Weekly data 05.04.1973 to 30.03.00 (N=1409 observations). Multivariate Quasi Maximum Likelihood estimates, heteroscedasticity robust t-statistics in brackets. $X^2(6)$ critical value in square brackets. $AIC = -2\left(\frac{\ln L}{N} + \frac{k}{N}\right)$, $SIC = -2\left(\frac{\ln L}{N} + \frac{k \ln N}{2N}\right)$, $LR = 2\left(\ln L_{ur} - \ln L_r\right)$

Table IV		UK	France	Germany	EU	Japan	US
Mean							
c		.33 (3.3)	.30 (3.4)	.28 (3.3)	.29 (4.8)	.25 (3.8)	.23 (4.6)
Variance							
γ_0		1.92 (31.1)	4.77 (48.6)	2.6 (27)	.95 (3.3)	.27 (4.4)	.23 (3.4)
γ_1		–	–	–	.05 (4.1)	.12 (6.5)	.12 (6.5)
δ_1		–	–	–	.71 (9.2)	.84 (38.3)	.82 (28.9)
ARCBeta					EU	Japan	US
UK	a_0				.0096 (1.9)	−.0020 (−2.3)	−.0014 (−3.0)
	a_1				.0049 (10)	.0190 (.75)	.3225 (3.7)
	b_1				.9872 (83)	.9534 (32.3)	.64 (7.9)
	$\frac{a_0}{1-a_1-b_1}$				1.2	−.07	−.038
FR	a_0				.007 (1.1)	.056 (2.0)	.0012 (.52)
	a_1				−.0028 (−1.8)	–	−.0517 (−.99)
	b_1				.9936 (128)	–	1.0275 (13.4)
	$\frac{a_0}{1-a_1-b_1}$				.8695	.056	.05
GE	a_0				.101 (1.7)	.1016 (2.8)	−.0106 (−1.1)
	a_1				−.0025 (2.9)	.0936 (4.1)	−.0645 (−.84)
	b_1				.9931 (14)	−1.0518 −63.6	.9339 (7.12)
	$\frac{a_0}{1-a_1-b_1}$				.87	.0571	−.0676
Log L LR		−17, 516 22 (12.6)					
AIC		24.8					
SIC		24.64					

Table V : ARCBeta Factors (UK, FR, GE) & Idiosyncratic GARCH

NOTES: Weekly data 05.04.1973 to 30.03.00 (N=1409 observations). Multivariate Quasi Maximum Likelihood estimates, heteroscedasticity robust t-statistics in brackets. $AIC = -2\left(\frac{\ln L}{N} + \frac{k}{N}\right)$, $SIC = -2\left(\frac{\ln L}{N} + \frac{k \ln N}{2N}\right)$, $LR = 2\left(\ln L_{ur} - \ln L_r\right)$

Table V		UK	France	Germany	EU	Japan	US
Mean							
c		.26 (3.1)	.29 (3.9)	.24 (3.2)	.26 (4.2)	.25 (3.7)	.23 (4.5)
Variance							
γ_0		.04 (3.1)	.06 (4.6)	.05 (2.3)	.74 (5.2)	.26 (4.3)	.25 (3.7)
γ_1		.11 (7.1)	.10 (11.2)	.06 (5.1)	.15 (5.5)	.116 (6.3)	.13 (6.6)
δ_1		.86 (45.2)	.89 (107.2)	.92 (60)	.69 (15.5)	.85 (39.7)	.81 (27.9)
ARCBeta					EU	Japan	US
UK	a_0				.0096 (1.8)	$-.0075$ (-1.8)	$-.0011$ (-2.3)
	a_1				.0049 (6.0)	$-.09$ (-2.2)	.3680 (2.88)
	b_1				.9872 (70)	.96 (14.4)	.5974 (5.2)
	$\frac{a_0}{1-a_1-b_1}$				1.2	$-.057$	$-.032$
FR	a_0				.0066 (.48)	.06 (2.28)	.0012 (.72)
	a_1				$-.0012$ ($-.59$)	$-$	$-.0416$ (-1.2)
	b_1				.9939 (72)	$-$	1.0195 (24)
	$\frac{a_0}{1-a_1-b_1}$				.8695	.06	.05
GE	a_0				.0082 (1.9)	.0756 (2.6)	$-.0052$ (-2.26)
	a_1				$-.0025$ (3.4)	.0883 (3.4)	$-.2476$ (-5.2)
	b_1				.9931 (19)	-1.0585 (-56)	1.1549 (43)
	$\frac{a_0}{1-a_1-b_1}$				.87	.039	$-.056$
Log L		$-17,114$					
AIC							
SIC							

Table VI : Constant and Steady-State Betas

	EU		*Japan*		*US*	
	Constant	*ARCBeta*	*Constant*	*ARCBeta*	*Constant*	*ARCBeta*
UK	1.28	1.22	−.085	−.060	−.028	−.030
FR	0.85	0.90	0.060	−	0.050	0.050
GE	0.82	0.86	0.070	0.04	−.050	−.050

6.11 Graphs

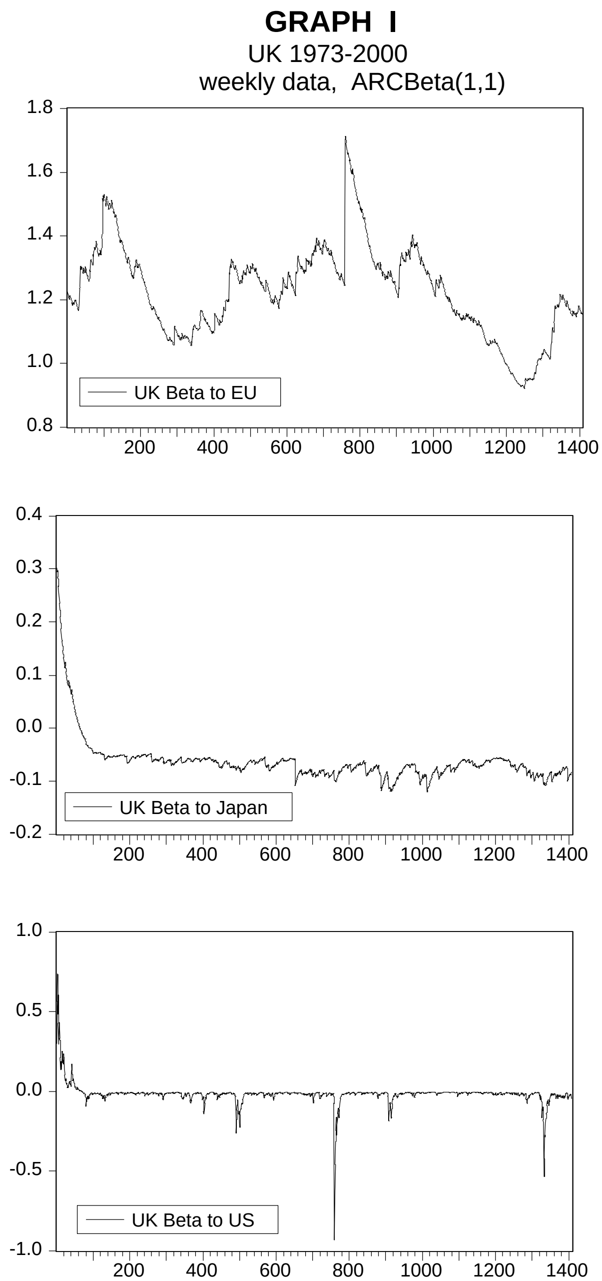

GRAPH I
UK 1973-2000
weekly data, ARCBeta(1,1)
UK Beta to EU
UK Beta to Japan
UK Beta to US
200
400
600
800
1000
1200
1400

Figure 1:

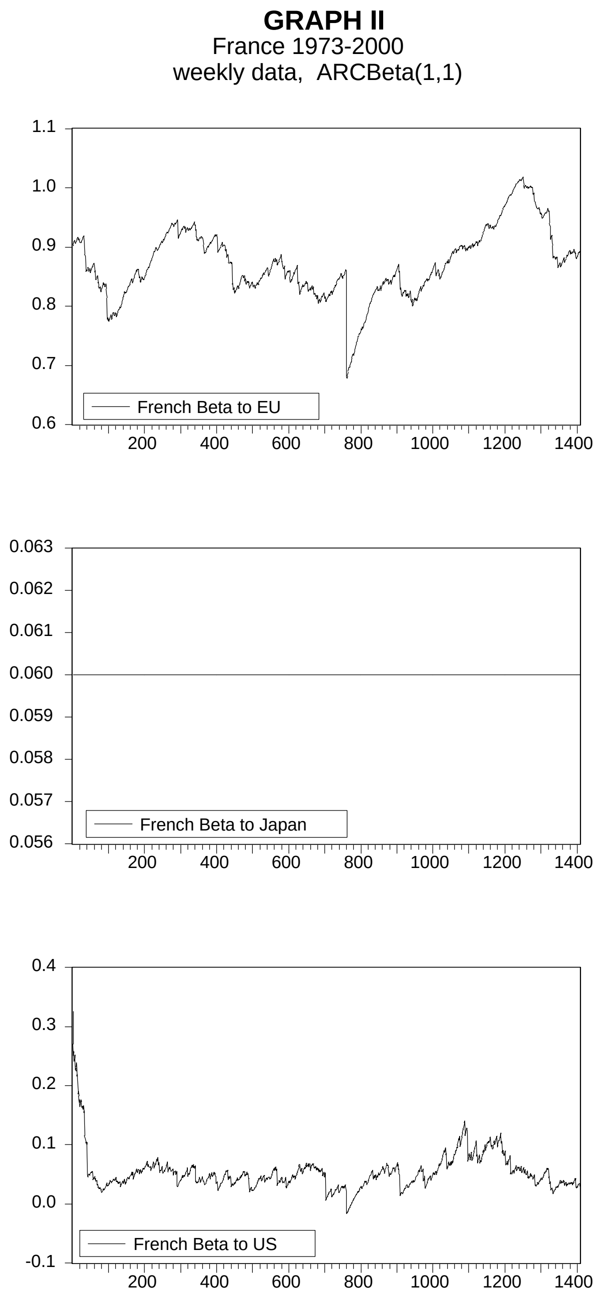

GRAPH II
France 1973-2000
weekly data, ARCBeta(1,1)
1.1
1.0
0.9
0.8
0.7
0.6
French Beta to EU
200
400
600
800
1000
1200
1400
0.063
0.062
0.061
0.060
0.059
0.058
0.057
0.056
French Beta to Japan
200
400
600
800
1000
1200
1400
0.4
0.3
0.2
0.1
0.0
-0.1
French Beta to US
200
400
600
800
1000
1200
1400

Figure 2:

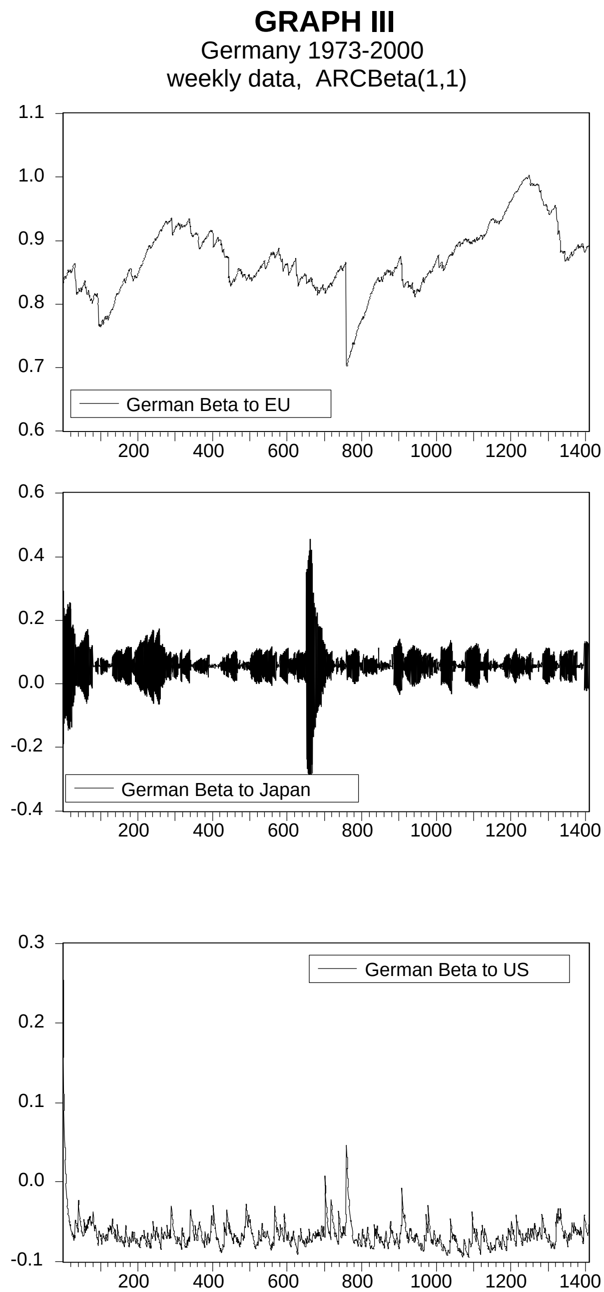
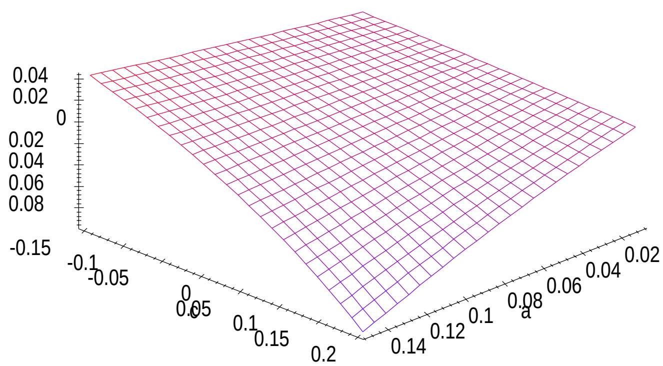
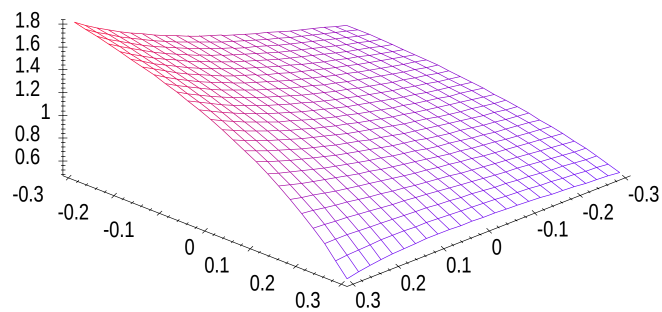

GRAPH III
Germany 1973-2000
weekly data, ARCBeta(1,1)
German Beta to EU
German Beta to Japan
German Beta to US
1.1
1.0
0.9
0.8
0.7
0.6
0.4
0.2
0.0
-0.2
-0.4
0.3
0.1
-0.1
200
400
600
800
1000
1200
1400

Figure 3:



Graph IV: plot of Cov$(\beta_{ij,t},\sigma^2_{j,t})$



Graph V: Cov $(\beta_{1j,t},\beta_{2j,t})$

References

[1] Bekaert G and C R Harvey (1997), "Emerging Equity Market Volatility", *Journal of Financial Economics*, 43, 29-77

[2] Bekaert G and C R Harvey (1995), "Time-Varying World Market Integration", Journal of Finance, L(2), 403-444

[3] Berndt E, B Hall, R Hall, J Hausmann (1974), "Estimation and Inference in Non-Linear Structural Models", *Annals of Economic and Social Measurement*, 3, 653-665

[4] Black F, M Jensen and M Scholes (1972), "The Capital Asset Pricing Model: some empirical tests", in M C Jensen, ed. *Studies in the Theory of Capital Markets*, NY, Praeger Pubblishing

[5] Blume M and I Friend (1973), "A New Look at the Capital Asset Pricing Model", *Journal of Finance*

[6] Blume M (1975), "Betas and their Regression Tendencies", Journal of Finance, XXX(3), 785-795

[7] Bollerslev T (1986a), "Generalized Autoregressive Conditional Heteroscedasticity", *Journal of Econometrics*, 31, 307-327

[8] Bollerslev T (1986b), "On the Correlation Structure for the Generalized Conditionally Heteroscedastic Process", *Journal of Time Series Analysis*, 9(2), 121-131

[9] Bollerslev T and J Wooldridge (1992), "Quasi-Maximum Likelihood Estimation and Inference in Dynamic Models with Time-Varying Covariances", *Econometric Reviews*, 11, 143-172

[10] Bond S (2000), "Modelling Static versus Dynamic Semi-variance - A Regime GARCH Approach", *manuscript*, Faculty of Economics, University of Cambridge

[11] Bos N and P Newbold (1984), "An Empirical Investigation of the Possibility of Stochastic Systematic Risk in the Market Model", *Journal of Business*, 57(1), 35-41

[12] Brandt A (1986), "The Stochastic Equation $Y_{n+1} = A_n Y_n + B_n$ with Stationary Coefficients", *Advances in Applied Probability*, 18, 211-220

[13] Braun P, D B Nelson and A Sunier (1995), "Good News, Bad News, Volatility and Betas", *Journal of Finance*, L(5), 1575-1603

[14] Brockwell P J and R A Davis (1987), "Time Series: Theory and Methods", New York, Springer Verlag

[15] Broyden C G (1967), "Quasi-Newton Methods and their Application to Function Minimization", *Mathematics of Computation*, 21, 368-381

[16] Bourgerol P and N Picard (1992a), "Strict Stationarity of Generalised Autoregressive Processes", *Annals of Probability*, 20, 1714-1730

[17] Bourgerol P and N Picard (1992b), "Stationarity of GARCH Processes and some Non-negative Time Series", *Journal of Econometrics*, 52, 115-27

[18] Brooks R D, R W Faff and J H H Lee (1992), "The Form of Time Variaton of Systematic Risk: some Austalian evidence", *Applied Financial Economics*, 2, 191-198

[19] Chen S-N (1982), “An Examination of Risk-Return Relationship in Bull and Bear Markets using Time-Varying Betas”, *Journal of Financial and Quantitative Analysis,* XVII(2), 265-286

[20] Chen S-N and A J Keown (1981), “Risk Decomposition and Portfolio Diversification when Beta is Non-Stationary”, *Journal of Finance*, XXXVI(4), 941-947

[21] Christopherson J, W Ferson and A Turner (1999), “Performance Evaluation Using Conditional Alphas and Betas”, *Journal of Portfolio Management*, Fall, 59-72

[22] Chung K L (1974), “A Course in Probability Theory”, 2nd ed. Academic Press, London

[23] Collins D, J Ledolter and J Rayburn (1987), “Some Further Evidence on the Stochastic Properties of Systematic Risk”, *Journal of Business*, 60(3), 425-448

[24] Connor G and O Linton (2000), “Semiparametric Estimation of a Characteristic-based Factor Model of Stock Returns”, *manuscript*, London School of Economics, UK

[25] DataStream (1990), DataStream Definitions Manual, Issue 2, page Equities-85

[26] Engle R (1982), “Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation”, *Econometrica*, 50(4), 987-1007

[27] Engle R, V Ng and M Rothchild (1990), "Asset Pricing with Factor-ARCH Covariance Structure: empirical estimates for treasury bills", *Journal of Econometrics*, 45, 213-237

[28] Episcopos A (1996), "Stock Return Volatility and Time Varying Betas in the Toronto Stock Exchange", *Quarterly Journal of Business and Economics*, 35(4), 28-38

[29] Fabozzi F and J C Francis (1978), "Beta as a Random Coefficient", *Journal of Financial and Quantitative Analysis*, 101-116

[30] Fama E and J D MacBeth (1973), "Risk, Return and Equilibrium: empirical tests", *Journal of Political Economy*

[31] Ferson W E and C R Harvey (1991), "The Variation of Economic Risk Premiums", *Journal of Political Economy*, 99, 285-315

[32] Ferson W E and C R Harvey (1993), "The Risk and Predictability in International Equity Returns", *Review of Financial Studies*, 6, 527-566

[33] Ferson W and R Korajczyk (1995), "Do Arbitrage Pricing Models Explain the Predictability of Stock Returns?", *Journal of Business*, 68(3), 309-349

[34] Furstenberg H and H Kesten (1960), "Products of Random Matrices", *Annals of Mathematical Statistics*, 31, 457-469

[35] Hall S, D Miles and M P Taylor (1989), "Modelling Asset Prices with Time-Varying Betas", *Manchester School*, LVII(4), 340-356

[36] Hall S, G Urga and A Zalewska-Mitura (1998), "Testing for Evolving Stock Market Efficiency", *manuscript*, London Business School, London

[37] Harvey C R (1995), "Predictable Risk and Returns in Emerging Markets", *Review of Financial Studies*, 8(3), 733-816

[38] He C and T Terasvirta (1999), "Properties of Moments of a Family of GARCH Processes", *Journal of Econometrics*, 92, 173-192

[39] Hildreth C and J P Houck (1968), "Some Estimators for a Linear Model with Random Coefficients", *Journal of the American Statistical Association*, 584-595

[40] Karanasos M (1999), "The Second Moment and the Autocovariance Function of the Squared Errors of the GARCH Model", *Journal of Econometrics*, 90, 63-76

[41] Klemkosky R and J Martin (1975), "The Adjustment of Beta Factors", *Journal of Finance*, 1123-1128

[42] Koutmos G, U Lee and P Theodossiou (1994), "Time-Varying Betas and Volatility Persistence in International Stock Markets", *Journal of Economics and Business*, 46, 101-112

[43] Kryzanowski L, S Lalancette and M C To (1997), "Performance Attribution using an APT with prespecified macrofactors and Time-Varying Risk Premia and Betas", *Journal of Financial and Quantitative Analysis*, 32(2), 205-224

[44] Lintner J (1965), "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets", *Review of Economics and Statistics*, 30-55

[45] Liu J and P J Brockwell (1988), "On the General Bilinear Time Series Model", *Journal of Applied Probability*, 25, 553-564

[46] Mark N (1988), "Time Varying Betas and Risk Premia in the Pricing of Foreward Foreign Exchange Contracts", *Journal of Financial Economics*, 22, 335-354

[47] Nelson D B and C Q Cao (1992), "Inequality Constraints in the Univariate GARCH Model", *Journal of Business and Economic Statistics*, 10, 229-235

[48] Nicholls D F and B G Quinn (1982), "Random Coefficient Autoregressive Models: An Introduction", Lecture Notes in Statistics, Springer-Verlag, New York

[49] Ng V, R Engle and M Rothchild (1992), "A Multi-Dynamic-Factor Model for Stock Returns", *Journal of Econometrics*, 52, 245-266

[50] Ohlson J and B Rosenberg (1982), "Systematic Risk for the CRSP equally-weighted Common Stock Index: A history estimated by stochastic parameter regression", *Journal of Business*, 55, 121-45

[51] Pham D T (1985), "Bilinear Markovian Representation and Bilinear Models", *Stochastic Processes and Their Appplications*, 20, 295-306

[52] Pourahmadi M (1986), "On Stationarity of the Solution of a Doubly Stochastic Model", *Journal of Time Series Analysis*, 7(2), 123-131

[53] Pourahmadi M (1988), "Stationarity of the Solution of $X_t = A_t X_{t-1} + \varepsilon_t$ and Analysis of Non-Gaussian Dependent Random Variables", *Journal of Time Series Analysis*, 9(3), 225-239

[54] Pourahmadi M (1999a), "Joint Mean-Covariance Models with Applications to Longitudinal Data: Unconstrained Parametrisation", *Biometrika*, 86(3), 677-690

[55] Pourahmadi M (1999b), "Joint Modelling of Mean, Dispersion and Correlation: Generalized Gaussian Estimating Equations", *manuscript*, Division of Statistics, Northern Illinois University, US

[56] Rao M Bhaskara, Rao T Subba and A M Walker (1983), "On the Existence of Some Bilinear Time Series Models", *Journal of Time Series Analysis*, 4(2), 95-110

[57] Rockinger M and G Urga (1999), "A Time Varying Parameter Model to Test for Predictability and Integration in Stock Markets of Transition Economies", *manuscript*, City University Business School, forthcoming, *Journal of Business and Economic Satistics* (2001)

[58] Ross S (1976), "The Arbitrage Theory of Capital Asset Pricing", *Journal of Economic Theory*, 341-360

[59] Schwert G W and P J Seguin (1990), "Heteroscedasticity in Stock Returns", *Journal of Finance*, XLV(4), 1129-1155

[60] Shanken J (1990), "International Asset Pricing: an Empirical Investigation", *Journal of Econometrics*, 45, 99-120

[61] Sharpe W (1964), "Capital Asset Prices: A theory of market equilibrium under conditions of risk", *Journal of Finance*, 425-442

[62] Vervaat W (1979), "On a Stochastic Difference Equation and a Representation of Non-negative Infinitely Divisible Random Variables", *Advances in Applied Probability*, 11, 750-783

[63] Wells C (1995), "The Kalman Filter in Finance", Dordrecht, Kluwer Academic Publishers

[64] White H (1980), "A Heteroscedasticity Consistent Covariance Matrix Estimator and a Direct Test for Heteroscedasticity", *Econometrica*, 48, 817-838

[65] Wooldridge J M (1990), "A Unified Approach to Robust, Regression-Based Specification Tests ", *Econometric Theory*, 6, 17-43