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A New Class of Commodity Hedging Strategies: A Passport Options Approach

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ABSTRACT

We provide a new way of hedging a commodity exposure which eliminates downside risk without sacrificing upside potential. The tool used is a variant on the equity passport option and can be used with both futures and forwards contracts as the underlying hedge instrument. Results are given for popular commodity price models such as Gibson-Schwartz and Black with convenience yield. Two different scenarios are considered, one where the producer places his usual hedge and undertakes additional trading, and the other where the usual hedge is not held. In addition, a comparison result is derived, showing that one scenario is always more expensive than the other. The cost of these methods are compared to buying a put option on the commodity.

Keywords: Passport Options, Futures, Forward Contract, Option Pricing, hedging, commodities

1. Introduction

A commodity producer typically hedges the commodity price risk at a specific future date by taking positions in a sold forward contract or in a futures contract. As commodity production is continuous or seasonal, hedging exposures at a series of dates can be achieved by strips of sold forward contracts which result in an equivalent swap rate. For commodities like gold and silver which are in contango, (that is the spot price is less than the futures price), this has the advantage of providing a higher forward price in the short term and a lower price in the future, presumably when it is needed less. For markets in backwardation, like oil, base metals and agricultural commodities, (where now the spot price is greater than the futures price), the selling of long-dated forwards need not be a sensible risk management practice, particularly when the futures prices are comparable to the costs of production. Typically, short term contracts are stacked and rolled to replicate a long-dated futures price which is generally associated with a high variance of profit and loss. The benefit of these simple hedging structures is that a commodity price is fixed at a series of dates, over a reasonable period of production, and results in a positive revenue stream with a variance which precludes losses with a high probability.

Whilst these so-called vanilla hedging products are sensible and popular, other derivative structures can be embedded in these forward or futures contracts which potentially result in a higher forward selling price. This is achieved essentially by the producer selling options to generate premium and thus the producer is taking views on the direction of the commodity price. In the gold market, for example, the selling of out-of-the-money gold call options is common and reflects a bearish view on high gold prices. Of course if higher prices eventuate the current prevailing forward prices may be superior to the result from the initial hedging structures, and this may result in the costly restructuring of the original hedge. Whilst it is possible to embed upside views on the price movements, these payoffs are typically European and cannot necessarily capture daily or weekly moves in the futures curve.

It seems reasonable to develop a hedging strategy whereby the potential profits of correctly revising a hedge are compared to the potential profits of a static futures hedge, with the hedger choosing the greater of the two. In the case when a producer must hedge to lock in a revenue stream, then this new structure will protect losses but keep the upside potential. This is useful when a producer has knowledge about the commodity price and wants to speculate on the upside whilst still retaining a downside hedge.

The producer enters into a contract with a bank which entitles him to receive the returns from the standard hedge, plus the gains from his additional trades provided these trades have made a profit. If the additional trades have made a loss, then the loss is borne by the financial institution. Thus the producer has a stop-loss, and receives at worst, the standard forward or futures hedge, in exchange for a premium. The only constraint on the additional trades is that there is a limit on the number of forwards that the producer may hold at any one time. In effect, this is simply a passport option on forward contracts (see Hyer [1], Henderson and Hobson [2], and also Andersen et al [3], Delbaen and Yor [4], Shreve and Večeř [5] and Ahn et al [6]). We refer to this as the 'downside hedged' strategy as the minimum payoff is the result of taking a simple forward hedge.

This type of structure could be sold by banks as an alternative to traditional hedging methods, giving the commodity producer flexibility to undertake an active strategy but with the knowledge that they will not be any worse off than if they had hedged passively. Thus, like the passport options in Henderson and Hobson [2], the downside risk of the holder of the option is limited but upside potential is not. This can be likened to portfolio insurance, where put options are bought (or constructed synthetically) to hedge downside risk of holding an index or stock, see Leland [7]. Portfolio insurance applies to portfolios which attempt to track an index and so is used for passive funds. In our context of hedging a commodity, the producer could buy a put option on the commodity. This would also provide a downside hedged position. In contrast, a passport option, and hence our commodity strategies, can be written on active portfolios. Thus the producer has the flexibility to change his position dynamically, and still be downside hedged. A rationale for using this type of product may be that producers who have access to the underlying commodity may be in a better, more informed position to ascertain whether movements in the curve have been influenced by fundamental supply and demand effects.

Compared to simply implementing a dynamic hedging strategy, the option allows the producer to benefit regardless of moves in prices. However, again, the disadvantage is the cost of undertaking such a flexible hedge. We compare the cost of the hedge to the cost of buying a put option, and find it is more expensive, as expected. However, when we adjust for the cost, we see the downside hedged strategy has a higher probability of being in the money, and thus is giving a better hedge against the worst outcomes.

We also consider a second scenario in which the commodity producer remains unhedged, but, subject to position limits trades actively in the hedge instrument. At expiry they are entitled to receive their gains from trade, if they have made a profit, but any losses are borne by the bank. Again in this second scenario where the commodity producer trades actively about an unhedged position we can find the price the bank should charge to sell the option, for a variety of models. Interestingly we can show that this second type of hedge is *more expensive* than the first, regardless of whether the hedge instrument is a forward contract or a future. This example is considered as is illustrates that there can be several hedging strategies based on the passport strategy that have quite different costs.

We derive the "fair" price for these options which is the price the bank should charge the commodity producer to enter the contract. In order to determine this price, we need to find the 'worst case' scenario from the point of view of the financial institution, or equivalently the optimal behaviour for the commodity producer. We allow the basic hedging instrument to either be a forward contract or a futures position, and use both the Black [8] and Gibson-Schwartz [9] models in each case. These models are well known commodity price models and are applied to commodities such as oil or copper. Our use of the Gibson-Schwartz model highlights that we can use models other than exponential Brownian motion to price passport options. For the equity passport option, it has been shown by Henderson and Hobson [10] that a range of stochastic volatility models may be used. When a forward contract is the hedging instrument, we can calculate the explicit cost for each of these models. When the hedging is done with futures, we show the option fits into the case of equity passport option with dividends in Henderson [11], Delbaen and Yor [4] and

Lipton [12]. For this we need to solve the pde numerically to obtain prices.

The comparison mentioned above between the two hedging scenarios collapses naturally into a comparison between the option prices under asymmetric and symmetric trading limits. See Henderson [11] and Shreve and Večeř [13] for a discussion of the passport option with asymmetric constraint. We present a formulation of the result for a wide class of models which includes Black (with convenience yield) and Gibson-Schwartz, for the case of forward prices. This is generalised in Henderson [11] to account for futures as the hedge instrument.

The remainder of the paper is structured as follows. First, modelling commodity futures (and forwards) prices is discussed and the models used in the paper given. Section 3 introduces the passport option and is closed with a discussion of the two hedging scenarios considered in the paper. Section 4 treats forward price models for the two hedging scenarios and gives explicit expressions for prices in each case, see Figure 1. The following section analyses the prices for the same scenarios using futures rather than forwards as the hedging instrument. A comparison between the downside hedged and unhedged scenarios is undertaken in Section 6, showing that one is always more expensive than the other. Section 7 compares the two scenarios to a portfolio insurance style hedge and the final section concludes.

2. Commodity Futures Price modelling

It is quite clear that the dynamics of the futures curve for the base metals, and storable commodities in general, exhibit quite complex behaviour. The original theory of storage by Kaldor [14], Working [15] and Brennan [16], has been developed to understand why backwardated markets arise. The basic idea of this model is that holders of the commodity receive a benefit, called the "convenience yield" because stocks-on-hand allow continuation of production and consumption during periods of a stock shortage. The marginal value of the convenience is dependent on the stock levels so that for large inventories the convenience yield declines. Since holders of the stock earn the convenience yield, arbitrage implies a relation between the spot and forward price because holders of the forward contract do not receive the convenience yield. The theory of storage has direct implications for the spread dynamics and volatility, see Ng and Pirrong [17]. The convenience yield has been modelled in a number of ways in the literature. Brennan and Schwartz [18] used a constant fraction of the spot process whilst Gibson and Schwartz [9] use a mean reverting Gaussian convenience yield. Also, stochastic interest rates and jumps are introduced by Schwartz [19] and Hilliard and Reis [20]. In each of these models, the futures price is determined by the spot price and the costs and benefits of storage.

We choose the the Gibson-Schwartz model for our calculations, as it has the important feature of mean-reversion for the asset price and reasonable probabilities for movements of futures curves from contango into backwardation and vice-versa. This follows because the convenience yield is modelled as a Vasicek style mean-reverting process. The intuition is that as inventories rise, the convenience yield falls as does the commodity price. Production would then decrease, which in turn raises prices and the convenience yield mean-reverts to a long-term average. This induces mean-reversion in the asset prices.

More recent models of commodity price dynamics, such as those of Routledge et al [21] and Pirrong [22], incorporate supply and demand shocks with explicit requirements on the relationship between the probability of a zero inventory and the current level of the inventory. Via a competitive inter-temporal arbitrage equation, which asserts that in equilibrium the expected appreciation of the commodity discounted at the risk free rate equals the marginal cost of storage, many salient features of the futures curve follow. For example, anticipated profits motivate storers to hold the commodity whenever expected price returns exceed the marginal cost of storage. Thus stock holding links supply and demand through time.

These rational expectations approaches are difficult to implement and calibrate. However these models have implications: in particular a nonlinearity is introduced in the price dynamics because of the constraint of non-negative inventories. This introduces an upper futures curve limited by full contango and an unbounded backwardated curve due to stock shortage. In addition, the spread has greater variance during times of backwardation and higher price variance with lower inventories, and, due to the probability of a zero inventory, the price distribution is positively skewed. There are other implications of this model, including the fact that the convenience yield is a function of the spot level, and the correlation between the futures contracts decrease during backwardation.

The remainder of this section introduces some notation and describes the models for the forward and futures prices. We follow Chapter 8 of Duffie [23] and Chapter 20 of Björk [24].

Let the price process for the commodity be given by $(P_t)_{t\geq 0}$. We assume there is a reference probability measure $\mathbb{P}^r$ and an equivalent risk-neutral measure $\mathbb{P}$ which is used for pricing. It is natural to want to consider production at a series of dates. However it is possible to concentrate on a single date T and to consider a strip of contracts.

2.1. *Forward Prices*

The forward price at time t for the commodity to be delivered at time T, denoted F_t^T, is defined by

$$\mathbb{E}_t[e^{\int_t^T -r_u du}(P_T - F_t^T)] = 0 \tag{1}$$

where $\mathbb{E}_t$ denotes expectation with respect to the risk-neutral probability measure conditional on information available at time t and r_u is the risk-free rate at time u.

Our aim is to illustrate the passport option in the context of forward markets rather than to discuss the implications of interest rate uncertainty on the forward price. Hence we will assume that interest rates are deterministic. For shorter maturity contracts, Schwartz [19] found that a model with stochastic interest rates did not outperform a deterministic rate model.

We will consider two particular models, each incorporating a dividend yield. For each of these models we will concentrate on deriving dynamics for the forward price process (see (2) and (5)). In each case it will turn out that F^T is a $\mathbb{P}$-martingale, a fact which will be necessary to use the passport options theory of Henderson and Hobson [2].

2.1.1. *The Black Model*

If the proportional price increments are Gaussian with constant variance, or equivalently if P is exponential Brownian motion, then, under the pricing measure the forward price F^T solves

$$\frac{dF_t^T}{F_t^T} = \sigma dB_t \tag{2}$$

which is a $\mathbb{P}$-martingale. In particular, if there is no dividend yield and r_u is constant, then $F_t^T = P_t e^{r(T-t)}$. This is the Black [8] model for forward prices.

If we introduce a (deterministic) convenience yield δ_t, then the risk-neutral dynamics of P become

$$\frac{dP_t}{P_t} = (r_t - \delta_t)dt + \sigma dB_t. \tag{3}$$

We find $F_t^T = e^{\int_t^T (r_u - \delta_u)du} P_t$ and we can use Itô's lemma to recover (2). This is the Black model with convenience yield. Further references include Brennan and Schwartz [18], Pindyck [25] and Paddock et al [26].

2.1.2. *The Gibson-Schwartz Model*

It is a well documented empirical fact that convenience yields exhibit mean reversion, see Gibson and Schwartz [9], Schwartz [19], Bjerksund [27] and Jamshidian and Fein [28]. In response to this Gibson and Schwartz [9] introduced a model which allows for a stochastic convenience yield. The model is two factor and the commodity price and the convenience yield are assumed to follow (under the risk neutral measure) equations of the form

$$\begin{aligned} dP &= (r - \delta_t)Pdt + \sigma_1 P dB_1, \\ d\delta &= \kappa\left(\left(\alpha - \frac{\sigma_2\lambda}{\kappa}\right) - \delta\right)dt + \sigma_2 dB_2. \end{aligned} \tag{4}$$

Here B_1, B_2 are correlated Brownian motions under the risk neutral measure with $dB_1 dB_2 = \rho dt$ and λ is the market price of convenience yield risk, assumed constant. In particular the convenience yield δ is of Vasicek form and the dynamics for $\log P$ are Gaussian.

Bjerksund [27] and Jamshidian and Fein [28] solve for the forward price to get

$$F_t^T = P_t e^{-\{(1-e^{-\kappa(T-t)})/\kappa\}\delta_t + A(T-t)}$$

where A is a complicated but deterministic function, as given in the appendix.

In the appendix we show further that F_t^T satisfies the SDE

$$\frac{dF_t^T}{F_t^T} = \gamma(T-t)d\beta \tag{5}$$

where β_t is a Brownian motion under the risk-neutral measure and γ is a function of time alone. It follows that the forward price has zero drift, and the log-forward price is Gaussian.

2.2. *Futures Prices*

We may also consider hedging with *futures* rather than forwards. Define ϕ_t^T to be the futures price at time t for the commodity with delivery at time T. Then ϕ_t^T is given by

$$\phi_t^T = \mathbb{E}_t P_T. \tag{6}$$

If we assume that interest rates are deterministic then $\phi_t^T = F_t^T$ since futures and forward prices coincide under deterministic interest rates, see Proposition 20.6 of Björk [24].

It follows that in both the Black and Gibson-Schwartz models, the futures price ϕ_t^T has the same dynamics as the forward price. For example in the Black model with deterministic convenience yield

$$\frac{d\phi_t^T}{\phi_t^T} = \sigma dB_t \tag{7}$$

and ϕ^T is a $\mathbb{P}$-martingale.

3. The Passport Option framework

This section introduces the passport option and outlines the hedging strategies we consider.

3.1. *The Gains from Trade and Hedging Scenarios*

Consider a copper producer who will have a fixed quantity of copper to sell at some future date T, and who is considering whether to lock in the price received for copper on that date by selling forward. This applies equally to other commodities such as oil.

Rather than simply selling a forward (or remaining unhedged), suppose that the copper producer makes additional trades in forwards (places a series of extra hedges), creating an extra trading profit or loss at time T of

$$\psi_T = \sum_{i=0}^{n-1} q^i (F_{t_{i+1}}^T - F_{t_i}^T) \tag{8}$$

where q^i is the number of forward contracts held between times t_i and t_{i+1} with $t_0 = 0, t_n = T$.

As we allow more trading dates we can denote the number of forwards held at time t by q_t and the gains process up to time t by

$$\psi_t = \int_0^t q_u dF_u^T. \tag{9}$$

We place no restrictions on the trading strategy q_t except that it should be adapted (i.e. q_t depends only on information available at time t) and bounded (in magnitude) by some constant L. In forward markets any gains (or losses) are realised at maturity so there is no need to account for interest earned.

To adapt the same framework to the futures case, recall that ψ is the undiscounted gains from trade, which will be different for futures because of the marking to market. As the producer will receive gains via this procedure before the end of the contract, this needs to be reflected in the dynamics. In particular gains (and losses) are realised as they occur. As part of the definition of the contract we will assume that these gains accrue interest at rate r.

Denote the gains from trade for the futures case by ψ^ϕ to distinguish it from ψ, the gains for forwards. We have

$$\psi_t^\phi = \int_0^t (q_u d\phi_u^T + r\psi_t^\phi dt). \tag{10}$$

where the final term arises from the interest paid on gains from trade to date. Again the set of admissible strategies q is the set of adapted strategies satisfying $|q_u| \leq L$.

3.2. *The Passport Option*

Consider a copper producer who wishes to trade on the futures market. This may be because they wish to speculate (in a controlled fashion) on the behaviour of the market or more generously that they may wish to take advantage of a perceived ability to utilise privileged information on demand and supply to predict futures price movements. Suppose the producer wishes to benefit from the upside from the gains from trade generated by his hedging strategy, but wishes to be protected from any losses. This can be achieved with a passport option.

A passport option (Hyer et al [1], Henderson and Hobson [2], Andersen et al [3], Delbaen and Yor [4], Shreve and Večeř [13], Henderson [11]) is a security sold by a financial institution to an investor (in this case the copper producer). The investor chooses any trading strategy (subject to the agreed position limit L) and the bank implements this strategy on the investor's behalf. At maturity the bank guarantees to pay the investor the positive part of any gains from trade from the strategy, whilst covering the losses itself. In return for this payoff the bank demands an initial premium. As mentioned earlier, this is similar to portfolio insurance but here, we may have an active futures portfolio.

Let P_T denote the spot price at maturity and F_0^T the forward price at $t = 0$. Consider a fixed cost of product K with total production A_T. The unhedged revenue is $A_T(P_T - K)$, but (fully hedged) with a sold forward the revenue becomes

$$A_T(P_T - K) - A_T(P_T - F_0^T) = A_T(F_0^T - K). \tag{11}$$

Thus when $F_0^T > K$ the producer may hedge and receive a fixed positive revenue. This simple forward selling strategy serves as a benchmark.

In the following we will assume $A_T = 1$. Since the problem has linear returns to scale this is of no loss of generality. Equivalently we are pricing per unit of production. There are (at least) two scenarios to consider.

3.2.1. *The Downside Hedged Scenario*

The producer hedges the copper with a forward at time 0, and undertakes any trading strategy subject to the position limit $q_t \leq L$. At maturity he has the option to either receive the terminal wealth which results from following his chosen strategy, or to forget the additional trades and to receive the monies due from undertaking a simple forward hedge. The payoff to the copper producer at time T is then

$$\max[\psi_T + (F_0^T - K), F_0^T - K] = \max[\psi_T, 0] + (F_0^T - K). \tag{12}$$

This payoff is in two parts: the payoff from the forward hedge plus an additional 'passport option' return.

3.2.2. *The Unhedged Scenario*

Alternatively, in this second scenario, the copper producer wishes to choose between the return from having remained unhedged $P_T - K$ and the final value of his trading strategy based on an initial fortune (denominated in units of cash at time T) of $(F_0^T - K)$. The quantity $(F_0^T - K)$ is the appropriate benchmark about which to measure trading gains since it is the expected worth of remaining unhedged, or equivalently the return available from undertaking a forward hedge. The payoff to the copper producer at time T is

$$\max[\psi_T + (F_0^T - K), P_T - K] = \max[\psi_T - (P_T - F_0^T), 0] + (P_T - K).$$

For both of these scenarios we can also imagine a producer who trades on futures markets instead of forward markets.

The vital observation is that these options can be considered as passport options. This allows us to make statements about the optimal strategy q for the copper producer, and hence to deduce the premium that the bank should charge for the option. If the investor trades on the forward market then the problem becomes a standard 'symmetric' passport option (Hyer et al [1]), although the underlying model (Black with convenience yield, or Gibson-Schwartz) is potentially different from the standard equity case. On the futures market, the marking to market means that the problem does not fit into a standard framework. Instead it is like an equity passport option with dividends, see Delbaen and Yor [4], Henderson [11] and Lipton [12].

4. Passport Option Prices under various Forward Price models

In this section we consider the problem facing the financial institution of pricing the options in both the downside hedged and unhedged scenarios using the forward market. We will assume that the maturity time T is fixed so that it can be dropped as a suffix for the forward price F_t^T. Further we will assume that the rate of interest is deterministic, and moreover that it is a fixed constant r.

The bank wishes to calculate a premium for the options. In the downside hedged case, the bank is selling a passport option with payoff ψ_T^+ and in the unhedged case, selling an option with payoff $(\psi_T - (P_T - F_0))^+$. In calculating a premium it must assume that the producer behaves optimally. Hence the bank computes the prices under a "worst case" assumption. Using the well known result that in a complete market options prices are the discounted payoff under the risk-neutral measure we have that the discounted prices in the downside hedged and unhedged scenarios respectively are

$$\sup_q e^{-rT}\mathbb{E}\psi_T^+ \tag{13}$$

and

$$\sup_q e^{-rT}\mathbb{E}[(\psi_T - (P_T - F_0))^+], \tag{14}$$

Here the suprema are taken over adapted strategies satisfying $|q_u| \leq L$. Our aim is to find closed form expressions for both (13) and (14) in both the Black and Gibson-Schwartz models.

4.1. *The Downside Hedged scenario*

The problem of finding the optimal q in (13) is a standard passport option problem. If the underlying follows an exponential Brownian motion, then several authors (Hyer [1], Andersen et al [3], Henderson and Hobson [2]) have shown that the optimal strategy is $q_t^* = -L\text{sgn}(\psi_t)$ where

$$\text{sgn}(x) = \begin{cases} 1; & x > 0 \\ -1; & x \leq 0. \end{cases}$$

Once the optimal strategy is known, it is a straightforward exercise to derive an expression for the price of the option.

In the Black model (including the case with deterministic dividend yield) the forward price as given by (2) is indeed exponential Brownian motion. Hence:

Lemma 1 *For the downside hedged option with payoff as in (13) and for the Black with convenience yield model, the price maximising strategy is to choose $q_t^* = -Lsgn(\psi_t)$ and the associated price is*

$$\sup_{|q|\leq L} e^{-rT}\mathbb{E}\left(\int_0^T q\,dF\right)^+ = \tfrac{1}{2}Le^{-rT}F_0\left\{(2N(d)-1) + \sigma\sqrt{T}(N'(d) + dN(d))\right\}$$

where $d = \frac{1}{2}\sigma\sqrt{T}$.

Remark 1 Note that the price of the option does not depend on the value of the convenience yield parameter δ. This is a consequence of the fact that although the risk-neutral dynamics of the underlying price process depend on the convenience yield δ, the dynamics of the forward price do not.

In the Gibson-Schwartz model the standard results for exponential Brownian motion cannot be applied directly. However from the work of Henderson and Hobson [2] which covers other models as well as exponential Brownian motion it follows that the form of the optimal strategy is unchanged. Then using (5) we have that $F_T = F_0 e^{\int_0^T \gamma_u dB_u - \frac{1}{2}\int_0^T \gamma_u^2 du}$ for a deterministic volatility function γ given in (A.3) in the appendix. If we set $\xi = \int_0^T \gamma_u^2 du$, (again an expression for ξ is given in the appendix) we deduce

Lemma 2 *For the downside hedged option with payoff as in (13) and for the Gibson-Schwartz model with stochastic convenience yield, then the price maximising strategy is to choose* $q_t^* = -L sgn(\psi_t)$ *and the associated price is*

$$\sup_{|q|\leq L} e^{-rT}\mathbb{E}\left(\int_0^T qdF\right)^+ = \tfrac{1}{2}Le^{-rT}F_0\left\{(2N(d)-1)+\sqrt{\xi}(N'(d)+dN(d))\right\} \tag{15}$$

where $d = \frac{1}{2}\sqrt{\xi}$.

Again the price of the option is independent of the initial value of the stochastic convenience yield δ. However the price does depend on the parameters which describe the dynamics of δ.

4.2. *The Unhedged scenario*

If we substitute $P_T \equiv F_T$ into (14) we find that the price of the passport option in the unhedged scenario becomes

$$\sup_{|q|\leq L} e^{-rT}\mathbb{E}\left(\int_0^T q_u dF_u - (F_T - F_0)\right)^+ = \sup_{|q|\leq L} e^{-rT}\mathbb{E}\left(\int_0^T (q_u - 1)dF_u\right)^+ \tag{16}$$

This corresponds to a passport option with asymmetric constraint. For both the Black and Gibson-Schwartz models we have from Henderson [11] that the supremum is attained when $q_t^* = -L\mathrm{sgn}(\psi_t^* + F_0)$, where $\psi_t^* = \int_0^t q^* dF$. In particular we have $q_t^* = L$ if $\psi_t^* \geq F_0$ and $q_t^* = -L$ if $\psi_t^* < F_0$.

From the results in Henderson [11] (and also from Shreve and Večeř [13]) we deduce:

Lemma 3 *For the unhedged option with payoff as in (16) and for the Black with convenience yield model, the price is*

$$\begin{aligned}\tfrac{1}{2}e^{-rT}LF_0\Big\{&\sigma\sqrt{T}N'(d_{+-}) + 2(1+\frac{1}{L})N(d_{-+}) - (\sigma\sqrt{T}d_{+-}+1)N(-d_{+-})\\ &-2(1+\frac{1}{L})N(d_{--}) + (1+\frac{1}{L})^2 N(-d_{++})\Big\}\end{aligned}$$

where $d_{\pm\pm} = \frac{1}{\sigma\sqrt{T}}\ln(1+\frac{1}{L}) \pm \frac{1}{\sigma\sqrt{T}}\ln(1+\frac{1}{L}) \pm \frac{\sigma\sqrt{T}}{2}$.

Lemma 4 *In the Gibson-Schwartz model, with asymmetric bounds on the class of admissible strategies, the price will be a modification of the expression in Lemma 3*

$$\tfrac{1}{2}e^{-rT}LF_0\left\{\sqrt{\xi}N'(d_{+-})+2(1+\frac{1}{L})N(d_{-+})-(\sqrt{\xi}d_{+-}+1)N(-d_{+-})\right.$$
$$\left.-2(1+\frac{1}{L})N(d_{--})+(1+\frac{1}{L})^2N(-d_{++})\right\}$$

where $d_{\pm\pm}=\frac{1}{\sqrt{\xi}}\ln(1+\frac{1}{L})\pm\frac{1}{\sqrt{\xi}}\ln(1+\frac{1}{L})\pm\frac{\sqrt{\xi}}{2}$.

Gibson and Schwartz empirically tested their model using weekly oil futures prices from 1984-88. We will use their estimated parameters for an example. We graph the prices to give a clearer idea of the relationships between the downside hedged and unhedged scenarios. We compare the Black prices in Lemmas 1 and 3 and the Gibson-Schwartz prices in Lemmas 2 and 4 on one graph, see Figure 1.

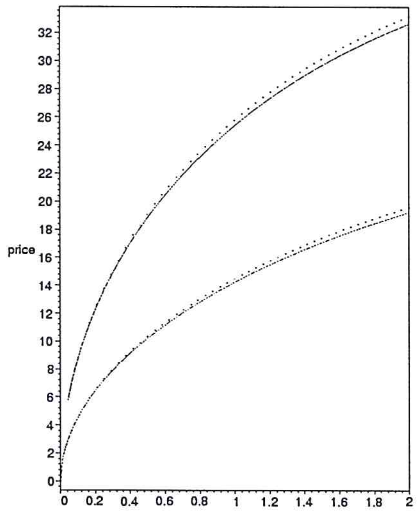

Fig. 1. *Prices of the Hedged and Unhedged scenarios using the Black and Gibson-Schwartz models for forwards, with* $F_0 = 100, K = 90$ *and* $L = 1$. *In both models* $r = 0.09$. *In the Black model we have* $\sigma = 0.35$ *and the parameter values for the Gibson-Schwartz model are* $\kappa = 16, \sigma_1 = 0.35, \sigma_2 = 0.4, \rho = 0.32$. *These parameters correspond to the values estimated from data by Gibson and Schwartz* [9]. *The Gibson-Schwartz model prices are represented by dotted lines and the solid lines are the Black model prices. The upper pair of lines are the unhedged scenario.*

It is immediately apparent from Figure 1 that the price in the unhedged scenario is greater than the downside hedged price for both the Black and Gibson-Schwartz models. In Section 6 we will show this conclusion holds quite generally.

Further from Figure 1 we see that the prices in the two models are very similar, but that in each case the price in the Gibson-Schwartz model is slightly higher than the Black price. This is down to parameter choice. In the Gibson-Schwartz model the mean reversion rate κ is relatively large (compared with the time scale $T = 1$). Hence the dominant term in the equation for the innovations for the forward price ((A.1) in the Appendix) is $\sigma_1 dB_1$ which is identical to the diffusion term for the Black model. Further, the observation that the prices are higher in the Gibson-Schwartz model is due to the presence of an additional diffusion term driven by the second Brownian motion B_2, although this effect is partially negated by the correlation between the two Brownian motions. For other choices of parameter values, the price relationship can be reversed and the Black prices will be higher.

5. Passport Option Prices under various Futures Price models

We now examine the prices of the downside hedged and unhedged strategies under the futures price models in Section 2.2. Even when interest rates are deterministic, so that $\phi_t \equiv F_t$, the marking to market affects the gains from trade process. In consequence, in futures markets there are no analytic solutions for the prices of passport options. Instead it is possible to derive a partial differential equation for the passport option price. This pde involves an optimisation over admissible strategies, and as a result it can be greatly simplified if the optimal strategy is known.

We will consider the Black model only, although the Gibson-Schwartz model would not be much more difficult. Recall from (13) and (16) that the prices in the downside hedged and unhedged scenarios respectively are

$$\sup_{|q|\leq L} e^{-rT}\mathbb{E}(\psi_T^\phi)^+ \tag{17}$$

and

$$\sup_{|q|\leq L} e^{-rT}\mathbb{E}(\psi_T^\phi - (\phi_T - \phi_0))^+ = \sup_{|q|\leq L} \mathbb{E}\left(\int_0^T (e^{-rt}q - e^{-rT})d\phi_t\right)^+. \tag{18}$$

5.1. *The Downside Hedged scenario*

We consider the downside hedged scenario (17) under the Black model with convenience yield. Define the time t value of (17) to be

$$V(t, \phi_t, \psi_t^\phi) = \sup_{|q|\leq L} e^{-r(T-t)}\mathbb{E}_t[(\psi_T^\phi)^+].$$

Applying Itô and (10) we obtain the Bellman equation

$$-\dot{V} = -rV + V_\psi r\psi^\phi + \sup_{|q|\leq L}\left[\frac{\sigma^2\phi^2}{2}(V_{\phi\phi} + 2V_{\phi\psi}q + V_{\psi\psi}q^2)\right] \tag{19}$$

subject to the terminal condition $V(T, \phi, \psi^\phi) = (\psi_T^\phi)^+$.

Before solving this equation it is convenient to use the scaling $V(t, \phi_t, \psi_t^\phi) = \phi\beta(t, \frac{\psi^\phi}{\phi})$ and if we set $z = \frac{\psi^\phi}{\phi}$ we obtain

$$\begin{aligned} -\dot{\beta} &= -r\beta + \beta_z rz + \sup_{|q|\leq L} [\frac{\sigma^2}{2}(z-q)^2 \beta_{zz}] \\ \beta(T,z) &= z^+ \end{aligned} \tag{20}$$

We know the supremum in (17) is obtained when $q^* = -L\mathrm{sgn}(z)$ from the dividend passport option work in Henderson [11]. This enables us to reduce (20) to

$$\begin{aligned} r\beta &= \dot{\beta} + \beta_z rz + \frac{\sigma^2}{2}(|z|+L)^2 \beta_{zz} \\ \beta(T,z) &= z^+ \end{aligned} \tag{21}$$

We will solve this pde numerically using a finite difference method, namely the Crank Nicholson scheme. Details of the numerical technique will not be given as the implementation is standard. See Andersen et al [3] for more details of a similar implementation for non-symmetric passport options.

In order to test the accuracy of the method, we first put $r = 0$ in (21) as this gives the traditional symmetric passport option. (Our other parameter values are $\phi_0 = 100, K = 90, T = 1, t = 0, \sigma = 0.3$ and $L = 1$.) Dividing time into 100 steps our numerical implementation gives $V(0, \phi_0 = 100, 0) = 13.1372$. This corresponds to a theoretical value from Lemma 1 of 13.1381 and is equivalent to an error in the value of σ of magnitude 0.0002. Thus we can have confidence that our numerical procedures give accuracy beyond two decimal places, and that this accuracy is beyond any reasonable error bounds on parameter values. For this first price calculation, we have chosen parameters to coincide with those in Table 2, Andersen et al [3] so we have an additional, external source of comparison. Indeed our calculated value corresponds exactly with the final value in Table 2 of Andersen et al, which gives us further confidence in our numerical implementation.

Now we consider our real problem, which is when the interest rate r is non-zero. We calculate the Black price in (17) for the futures model using the parameters from our forward price example of Section 3 ($\phi_0 = 100, K = 90, T = 1, t = 0, \sigma = 0.35, r = 0.09$ and $L = 1$). Using 100 timesteps we get a price of 14.1125. We will display prices on a graph in the next section, together with the prices from the unhedged scenario.

Although we cannot compare this to a theoretical price, we may make some simple comparisons with passport option prices to show that this price is plausible. From (17) we have that when $L = 1$

$$V(0, \phi_0, 0) = \sup_{|q|\leq 1} \mathbb{E}\left[\int_0^T e^{-rt} q_t d\phi_t\right]$$

and since $e^{-rT}q_t \leq e^{-rt}q_t \leq q_t$ we can conclude

$$\sup_{|q|\leq 1} e^{-rT}\mathbb{E}\left[\int_0^T q_t d\phi_t\right] \leq V(0,\phi_0,0) \leq \sup_{|q|\leq 1}\mathbb{E}\left[\int_0^T q_t d\phi_t\right].$$

Both these bounds can be calculated analytically using the analysis for passport options with the forward price as underlying. It follows that we have lower and upper bounds of 13.37 and 14.71 respectively for the passport option with the futures price as underlying.

5.2. *The Unhedged scenario*

We can also calculate the price of the unhedged scenario (18) in the Black model with convenience yield, and compare it to the hedged results in the previous section.

Let $H_t = e^{r(T-t)}\psi_t^\phi - (\phi_t - \phi_0)$ so that $dH_t = (e^{r(T-t)}q_t - 1)d\phi_t$. Define the time t value of (18) to be

$$\begin{aligned} V^a(t,\phi_t,H_t) &= \sup_{|q|\leq L} e^{-r(T-t)}\mathbb{E}_t[(\psi_T^\phi - (\phi_T-\phi_0))^+] \\ &= e^{-r(T-t)}\sup_{|q|\leq L}\mathbb{E}_t[H_T^+] \end{aligned} \tag{22}$$

Then V^a solves

$$-\dot{V}^a = -rV^a + \sup_{|q|\leq L}\left[\frac{\sigma^2\phi^2}{2}\left(V^a_{\phi\phi} + 2V^a_{\phi H}(e^{r(T-t)}q-1) + V^a_{HH}(e^{r(T-t)}q-1)^2\right)\right]$$

subject to $V(T,\phi_T,H_T) = H_T^+$.

Again it is convenient to use scaling properties of V^a: setting $V(t,\phi,H) = \phi\beta(t,z)$ where $z = \frac{H}{\phi}$ gives

$$\begin{aligned} -\dot{\beta} &= -r\beta + \sup_{|q|\leq L}\left[\frac{\sigma^2}{2}(z+1-qe^{r(T-t)})^2\beta_{zz}\right] \\ \beta(T,z) &= z^+ \end{aligned} \tag{23}$$

It can be shown the optimal strategy is $q^* = -L\text{sgn}(z+1)$. This enables us to reduce (23) to

$$\begin{aligned} r\beta &= \dot{\beta} + \frac{\sigma^2}{2}\left(|z+1| + Le^{r(T-t)}\right)^2\beta_{zz} \\ \beta(T,z) &= z^+ \end{aligned} \tag{24}$$

With $\phi_0 = 100, K = 90, T = 1, t = 0$ and parameter values $r = 0.09, \sigma = 0.35$ we again use the Crank Nicholson scheme to numerically solve the pde to get a price of 25.6035.

We can display the prices of both the hedged and unhedged scenarios as functions of time to maturity, on a graph, see Figure 2. Again note that the price in the

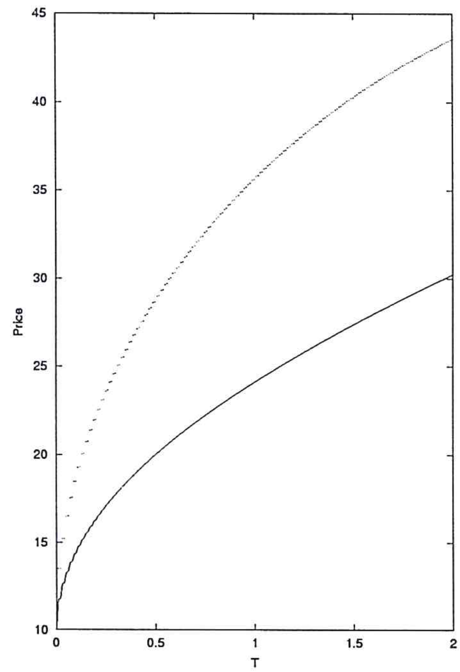

Fig. 2. *Expected payoff (price plus $e^{-rT}(\phi_0 - K)$) for the downside hedged and unhedged scenarios using the Black model for futures, with $\phi_0 = 100, K = 90$, $L = 1$, $r = 0.09$ and $\sigma = 0.35$. The upper dotted line is the unhedged scenario, and the lower solid line is the price in the downside hedged scenario.*

unhedged scenario exceeds the price in the hedged scenario. The graph shows the expected payoff which is the price of the passport option plus the discounted expected value of the futures hedge.

6. Price Comparisons between the Downside Hedged and Unhedged Scenarios

In Section 4 we observed that when the underlying is a forward price then the price of an option in the unhedged scenario is greater than the price in the downside hedged scenario. This relationship remained true when the underlying is a futures price, at least for the particular example we considered in Section 5. This is apparent for certain models and parameter values from the graphs in Figures 1 and 2.

The aim of this section is to prove a general result which says that for a wide class of models (including Black with convenience yield, and Gibson-Schwartz) the price of a passport option in the unhedged scenario is greater than the price in a downside hedged scenario.

Theorem 1 *If Q solves $dQ = \eta(t)QdB_t$ and if $L \geq L_0$ then for any $m \in \mathbb{R}$*

$$\sup_{\tilde{q}\in[m-L,m+L]} \mathbb{E}\left(\int_0^T \tilde{q}dQ\right)^+ \geq \sup_{|q|\leq L_0} \mathbb{E}\left(\int_0^T qdQ\right)^+ \tag{25}$$

Corollary 1 *When the underlying is a forward market it is sufficient to consider $Q_t = F_t$. Then in both the Black model with convenience yield ($\eta(t) \equiv \sigma$) and the Gibson-Schwartz model ($\eta(t) \equiv \gamma(T-t)$) we have*

$$\sup_{\tilde{q}\in[m-L,m+L]} e^{-rT}\mathbb{E}\left(\int_0^T \tilde{q}dF\right)^+ \geq \sup_{|q|\leq L_0} e^{-rT}\mathbb{E}\left(\int_0^T qdF\right)^+ . \tag{26}$$

Consequently, taking $m = -1$ and $L = L_0$ the price in the unhedged scenario is higher than the price in the downside hedged scenario.

Remark 2 This result is generalised in Henderson [11] in two important ways. Firstly the result is extended to cover a wider class of models for the forward price. These models are of the form $dQ = \eta(t, Q_t)dB_t$ (so that Q is a diffusion). The result is that, provided that η is increasing in the space variable for each t, then the inequality (25) holds. Secondly Theorem 1 is extended to cover the futures case. See Section 8.5 of Henderson [11].

Proof of Theorem 1:
If we can show that for some $\tilde{q} \in [m-L, m+L]$

$$\mathbb{E}\left(\int_0^T \tilde{q}dQ\right)^+ \geq \sup_{|q|\leq L_0} \mathbb{E}\left(\int_0^T qdQ\right)^+$$

then we are done.

For given q, let $G_t(q) = \int_0^t q_u dQ_u$ be the gains from trade from following the strategy q. From Chapter 5.3.3 of Henderson [11] and Delbaen and Yor [4] we know that the supremum on the right hand side is obtained when $q_t = q_t^* = -L_0 \text{sgn}(G_t(q^*))$, and that moreover, for this choice of strategy,

$$|G_T| = L_0 \left(\max_{0 \le u \le T} Q_u - Q_T \right), \tag{27}$$

and the sign of G_T is independent of Q_T.

Then if $\hat{q} = m + q^*$, we have

$$\int_0^T \hat{q} dQ = m(Q_T - Q_0) + G_T(q^*)$$

and

$$\begin{aligned}
\mathbb{E}\left(\int_0^T \hat{q} dQ \right)^+ &= \mathbb{E}(m(Q_T - Q_0) + G_T(q^*))^+ \\
&\geq \mathbb{E}\left([m(Q_T - Q_0) + G_T(q^*)] \, I_{(G_T > 0)} \right) \\
&= \mathbb{E}\left([m(Q_T - Q_0)] \, I_{(G_T > 0)} \right) + \mathbb{E}(G_T(q^*))^+
\end{aligned}$$

This first term of this last line is zero, by independence and the martingale property of Q so that

$$\sup_{\tilde{q} \in [m-L, m+L]} \mathbb{E}\left(\int_0^T \tilde{q} dQ \right)^+ \geq \mathbb{E}\left(\int_0^T \hat{q} dQ \right)^+ \geq \mathbb{E}(G_T(q^*))^+ = \sup_{|q| \le L} \mathbb{E}\left(\int_0^T q dQ \right)^+ .$$

□

7. A Comparison with Buying Put Options

The traditional method for providing protection for a passive (index linked) equities fund is to purchase portfolio insurance. The key phrase here is 'passive fund', since the strategy involves buying (or selling) put options on the index. This gives downside protection at the cost of the premium paid for the puts. Leland [7] is recognised as a main contributor to the development of portfolio insurance.

Our passport strategies of 'downside hedged' and 'unhedged' differ from this because they allow the producer to hold an active portfolio of forward/futures contracts and to dynamically alter this portfolio. Whilst retaining downside protection, the passport option provides a single option on an active portfolio.

To place our proposed hedging methods into context, we compare the cost of buying a put to that of the downside hedged scenario in Section 3.2.1. As in Section 2.4, P is the spot price of the commodity and F_t the forward price at time t. Suppose the copper producer buys a put option on copper with maturity T and with strike equal to the forward price of copper, F_0. Suppose the cost of this put, expressed in monetary units at time T is Put(T). The net position at time T of the copper producer with portfolio insurance is

$$\begin{aligned} Y_T &= F_T - K + (F_0 - F_T)^+ - \mathrm{Put(T)} \\ &= (F_0 - K) + (F_T - F_0)^+ - \mathrm{Put(T)}. \end{aligned} \tag{28}$$

Alternatively, suppose the copper producer uses the downside hedged strategy as given in Section 3.2.1, with position limit $L = 1$. The overall position of the producer is now

$$Z_T = (F_0 - K) + \psi_T^+ - \mathrm{DSH(T)} \tag{29}$$

where DSH(T) is the price of the downside hedged strategy.

Our goal in this section is to compare these two payoffs under the assumption that the underlying follows the Black model. The first step is to compare the cost of the portfolio insurance with the cost of the downside hedged strategy. The price of the put option is

$$e^{-rT}[F_0 N(\tfrac{1}{2}\sigma\sqrt{T}) - F_0 N(-\tfrac{1}{2}\sigma\sqrt{T})] \tag{30}$$

and the corresponding passport option costs

$$\tfrac{1}{2}e^{-rT}F_0 \left\{(2N(\tfrac{1}{2}\sigma\sqrt{T}) - 1) + \sigma\sqrt{T}(N'(\tfrac{1}{2}\sigma\sqrt{T}) + \tfrac{1}{2}\sigma\sqrt{T}N(\tfrac{1}{2}\sigma\sqrt{T}))\right\}. \tag{31}$$

These prices are plotted in Figure 3, with $F_0 = 100$ and $\sigma = 0.35$.

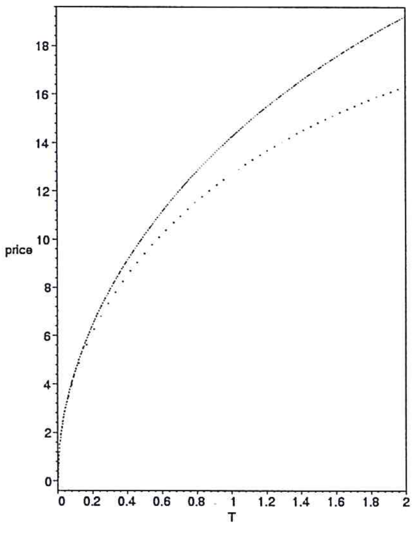

Fig. 3. *Prices of the downside hedged strategy (upper solid line) and portfolio insurance (lower dotted line) using Black model. Parameters are as before, with* $K = 80$.

It is clear from Figure 3 that portfolio insurance is slightly cheaper than the passport option. This is inevitable since one (suboptimal) strategy available to the passport option holder is to hold a single forward. However the price difference is surprisingly small given the great flexibility available to the passport option holder.

The second step is to compare the random payoffs from following the two strategies, under the assumption that the holder of the passport option behaves optimally. We do this by comparing the cumulative density function (cdf) of the undiscounted and random payoffs Y_T and Z_T. The cdf of Y_T can be obtained analytically using the fact that Y_T is a function of exponential Brownian motion. The cdf of Z_T is slightly harder to obtain. However, we have that under the optimal strategy

$$|\psi_T| = \sup_{0 \leq r \leq T} F_r - F_T \tag{32}$$

and $\mathrm{sgn}(\psi_T)$ and $|\psi_T|$ are independent. Hence by simulating $(\sup_{0 \leq r \leq T} F_r, F_T)$ and an independent $U\{+,-\}$ random variable we can simulate ψ_T^+. (We ensure that the simulation is sufficiently large by insisting that the simulated cdf of F_T agrees with the true value to two decimal places). The parameter values we choose are $F_0 = 100$, $L = 1$, $\sigma = 0.35$, $K = 80$.

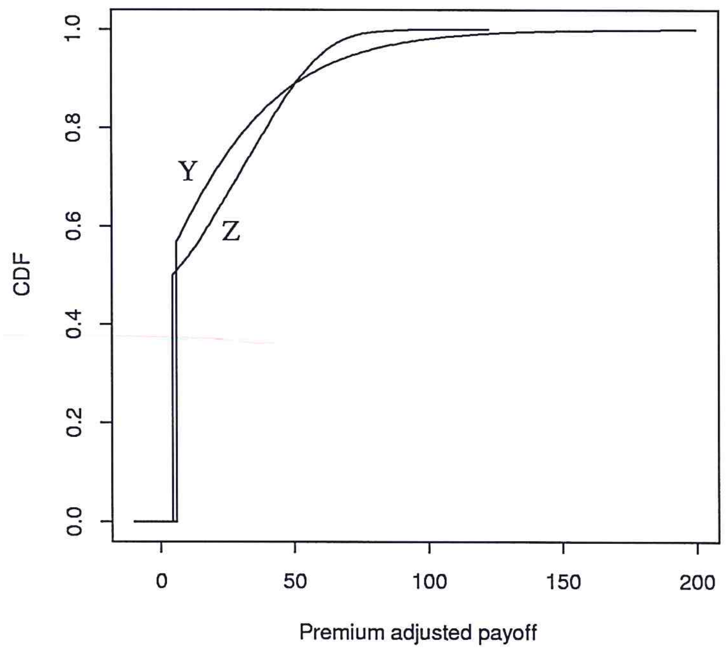

Fig. 4. *Cumulative distribution functions (CDF) for Y_T and Z_T under the Black model. Parameters as before, with $K = 80$.*

If the put option finishes out of the money, then $Y_T = (F_0 - K) - \mathrm{Put(T)}$.

Similarly, if $\psi_T \leq 0$ then $Z_T = (F_0 - K) - \text{DSH(T)}$. Since the price of the downside hedged strategy just exceeds that of the put option the worst case scenario for the downside hedged position is slightly worse than the worst case for portfolio insurance. This is reflected on the graph by the fact that the interval $\{x : (\text{CDF of } Z) > 0\}$ contains points which do not lie in $\{x : (\text{ CDF of } Y) > 0\}$.

However, one way in which the downside hedged position is preferable to the portfolio insurance is that $\mathbb{P}(\psi_T > 0) = 0.50 > \mathbb{P}(F_T - F_0 > 0) \approx 0.43$ so that the probability of this worst case scenario is smaller for the passport option holder. Moreover, for relatively small values of profit threshold H, (in our example this holds from profit levels lower than $H \approx 52$) the probability that the payoff from the downside hedged scenario is greater than H exceeds the probability that the payoff from the portfolio insurance exceeds H. For large profit thresholds, this relationship reverses. The holder of the portfolio insurance is more likely to experience large gains than the producer who purchases the passport option. This is because the large gains arise when the forward price rises sharply, the holder of the passport option only benefits if a large rise is followed by a subsequent fall.

8. Conclusion

We have given an alternative method for commodity hedging using either forwards or futures and passport options on these underlying hedge instruments. Rather than a simple forward contract or futures hedge, the producer may undertake extra trades in these instruments. This gives a position with a protected downside and unlimited upside.

Commodity markets often exhibit backwardation and contango and these phenomena are features of the models of Gibson-Schwartz [9] and Black with dividend yield [8]. Both these models can be used to price the options and hence the overall position of the producer, either using forwards or futures. With forwards the problem is similar to an equity passport option pricing problem for which a closed form solution is available. Using futures gives a more complex problem which can be solved numerically using pde methods. From Figures 1 and 2 it was natural to conjecture that the case where the additional trades replace the traditional hedge is more expensive than the case where they are in addition to the position. The conjecture was proved in Section 6.

One simple strategy available to commodity producers is to purchase a put option. This is a passive hedging strategy which gives the commodity producer the full benefit of any favourable price movements. In contrast, the downside hedged passport strategy and unhedged passport strategy allow the producer to hedge dynamically and potentially to take advantage of information they hold on likely future movements in supply and demand and hence price. The downside hedged strategy is only slightly more expensive than a simple put strategy, but provides an extremely flexible hedge. The unhedged passport strategy is still more expensive, but again provides a flexible hedge and gives unlimited upside potential. It fails to provide downside protection however.

The downside hedged passport option should be very attractive to a risk averse commodity producer who wishes to have downside protection, but does not necessarily insist on benefiting in full from favourable price moves. Instead, they seek to benefit from a dynamic trading strategy. Further, in this paper we have described the optimal trading behaviour of the producer. Knowledge of this strategy allows the financial institution who acts as counterparty to price the option in a direct manner. Thus it appears there are no barriers to the introduction of these contracts. Of course, it may be appropriate to adapt the precise specification of the option (for example, to allow for a continuous rate of production or to allow for strips of contracts) or to find prices and optimal strategies under alternative models, but the methods of this paper can potentially be extended to cover these cases as necessary.

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Appendix A

Forward prices in the Gibson Schwartz model

From Theorem 1 of Bjerksund [27] we get that the forward price in the Gibson-Schwartz model of (4) satisfies

$$F_t^T = P_t e^{-\{(1-e^{-\kappa(T-t)})/\kappa\}\delta_t + A(T-t)}$$

where

$$\begin{aligned} A(u) &= \left(r - \left(\alpha - \frac{\lambda\sigma_2}{\kappa}\right) + \tfrac{1}{2}\frac{\sigma_2^2}{\kappa^2} - \frac{\sigma_1\sigma_2\rho}{\kappa}\right) u + \frac{1}{4}\sigma_2^2 \left(\frac{1-e^{-2\kappa u}}{\kappa^3}\right) \\ &\quad + \left(\left(\alpha - \frac{\lambda\sigma_2}{\kappa}\right)\kappa + \sigma_1\sigma_2\rho - \frac{\sigma_2^2}{\kappa}\right)\left(\frac{1-e^{-\kappa u}}{\kappa^2}\right). \end{aligned}$$

By applying Itô's lemma, we find that F solves the SDE

$$\frac{dF_t^T}{F_t^T} = \sigma_1 dB_1 - (1 - e^{-\kappa(T-t)})\frac{\sigma_2}{\kappa} dB_2 \tag{A.1}$$

which can be transformed to an SDE involving only one Brownian motion β_t using Levy's theorem (see Chapter IV, Theorem 3.6 in Revuz and Yor [29]). In particular

$$\frac{dF}{F} = \gamma(T-t)d\beta \tag{A.2}$$

where β is a Brownian motion under $\mathbb{P}$ and γ is a time-dependent function which can be found by equating quadratic variation terms for F in the the two expressions (A.1) and (A.2). In particular

$$\{\gamma(u)\}^2 = \sigma_1^2 + \frac{\sigma_2^2}{\kappa^2}(1 - e^{-\kappa u})^2 - 2\frac{\rho\sigma_1\sigma_2}{\kappa}(1 - e^{-\kappa u}). \tag{A.3}$$

We also need to calculate $\int_0^T \gamma(u)^2 du$ for use in Section 3.1:

$$\int_0^T \gamma(u)^2 du = \sigma_1^2 T + \frac{\sigma_2^2}{3\kappa^3}\left[1 - (1 - e^{-\kappa T})^3\right] - \frac{\rho\sigma_1\sigma_2}{\kappa^2}\left[1 - (1 - e^{-\kappa T})^2\right].$$