

# Co-Volatility and Correlation Clustering : A Multivariate Correlated ARCH Framework

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May 2001

## Abstract

We present a new, full multivariate framework for modelling the evolution of conditional correlation between financial asset returns. Our approach assumes that a vector of asset returns is shocked by a vector innovation process the covariance matrix of which is time-dependent. We then employ an appropriate Cholesky decomposition of the asset covariance matrix which, when transformed using a Spherical decomposition allows for the modelling of conditional variances and correlations. The resulting asset covariance matrix is guaranteed to be positive definite at each point in time. We follow Christodoulakis and Satchell (2001) in designing conditionally autoregressive stochastic processes for the correlation coefficients and present analytical results for their distribution properties. Our approach allows for explicit out-of-sample forecasting of conditional correlations and generates a number of observed stylised facts such as time-varying correlations, persistence and correlation clustering, co-movement between correlation coefficients, correlation and volatility as well as between volatility processes (co-volatility). We also study analytically the co-movement between the elements of the asset covariance matrix which are shown to depend on their persistence parameters. We provide empirical evidence on a trivariate model using monthly data from Dow Jones Industrial, Nasdaq Composite and the 3-month US Treasury Bill yield which supports our theoretical arguments.

# 1 Introduction<sup>1</sup>

The evolution of correlation between asset returns is a well established stylized fact and of crucial importance for efficient financial decisions. It is well known that optimal financial variables such as asset weights in portfolio selection or prices of derivatives written on multiple assets e.g. basket options, are extremely sensitive to variations of correlation see Ingersoll (1987) and Rebbonato (1999). Thus, it is important to capture and predict the evolution of correlation between asset returns in a multivariate setting. Bollerslev (1990) proposed a full multivariate model in which conditional correlation between asset returns is assumed constant and variances follow generalized ARCH-type processes. As a result, conditional covariance varies over time as a constant correlation-adjusted product of conditional variances. Christodoulakis and Satchell (2001) build on Bollerslev's model and establish in a bivariate setting in which the correlation's *Fisher-z* transformation evolves as a conditionally autoregressive process.

In this paper we present a full multivariate  $N \times N$  generalization for modelling the evolution of conditional correlations between financial asset returns. Our approach assumes that a vector of asset returns is shocked by a vector innovation process the covariance matrix of which is time-dependent. We then employ an appropriate Cholesky decomposition of the asset covariance matrix which, when transformed using a Spherical decomposition allows for the modelling of conditional variances and correlations using conditionally autoregressive processes.

To the extent of our knowledge, this is the first model for evolving correlations, which guarantees a general  $N \times N$  covariance matrix to be positive definite at each point in time. As in Christodoulakis & Satchell (2001), we allow for explicit discrete-time stochastic processes for the correlation coef-

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<sup>1</sup>The author would like to thank Antonis Demos, Soosung Hwang, Steve Satchell, Elias Tzavalis and Mike Wickens for helpful comments, as well as participants of *The Econometrics of Financial Markets* international conference, Delphi, May 2001, the *Forecasting Financial Markets* conference, London, June 2001 and the *European Financial Management Association* conference, Lugano, June 2001.

ficients and present analytical results for their first and second steady-state moments. Also, our approach allows for explicit out-of-sample forecasting of conditional correlations. The assumption of non-zero covariances between the return innovation processes allows for the generation of a number of observed stylised facts: time-varying correlations, persistence and correlation clustering, co-movement between correlation coefficients, between correlation and volatility as well as between volatility processes (co-volatility), features that have understated in the literature so far. We study analytically the co-movement between volatilities as well as between volatility and correlation, which are shown to depend positively on their persistence parameters as well as the covariance of the innovation processes. We provide empirical evidence on a trivariate model using monthly data from Dow Jones Industrial, Nasdaq Composite and the 3-month US Treasury Bill yield which supports our theoretical arguments.

This section presents a number of stylized facts and modelling approaches for correlation and concludes with motivating arguments. In section two we present our model and in section three we study some of its properties. Sections four and five are devoted to estimation methodology and an empirical application respectively and section six concludes.

## 1.1 Stylized Facts

*Time Variation and Breakdown.* There is a number of studies examining the intertemporal stability and properties of the correlation structure between asset returns. Kaplanis(1988) uses the Jenrich(1970) statistic to test the equality of two sample correlation matrices of ten major stock markets. The null hypothesis that the correlation matrix was stable over any two adjacent subperiods could not be rejected at the fifteen percent level of significance. However, there was strong evidence at this significance level suggesting instability of the covariance matrix. Fustenberg and Jeon(1989) used daily data from the four major world stock markets to compare cross correlations before and after the crash of October 1987. They found that correlations among

the world's stock markets increased substantially after the crash. They report the robust finding that the pre-crash estimate of the average correlation was 0.2 which rose to 0.4 after the crash. Such periods of rapid correlation changes are known as *correlation breakdown*, see also Boyer et al (1997). The Brady Report(1988) supports the view that (rolling) correlations between stock markets change substantially over time, but do not exhibit any specific trend. Koch and Koch(1991) use simple Chow tests for eight national stock markets and uncover a 'growing market interdependence'. Finally, Erb et al (1994) present evidence for the G7 stock markets linking correlation instability to economic fundamentals.

*Persistence and Memory.* Luddin, Dacorogna et al (1999) examine the autocorrelation structure of cross correlations of a wide range of financial time series, with sampling frequencies varying from daily to tick-to-tick. They present evidence for approximately exponentially declining autocorrelation structure but with very different attenuation rates for each case. Similar evidence is also presented by Andersen et al (1999) for foreign exchange data. King, Sentana and Wadhwani(1994) estimate a multivariate Factor-GARCH model to account for time varying covariances between sixteen national stock markets. Their evidence implies strong variation of the correlation coefficients driven primarily by movements in latent factors.

*Volatility and Correlation Co-movement.* Bertero and Mayer(1990) confirmed the results of Fustenberg and Jeon but in addition they found that generally the correlations between stock markets tend to increase during periods of higher disturbances. King and Wadhwani (1990) construct a model in which contagion between markets is generated as a result of attempts by rational agents to infer information from price changes in other markets. They found significant evidence in favor of the hypothesis that correlation between markets rises following an increase in volatility. Further evidence for volatility and correlation co-movement is presented by Solnik et al (1996) as well as Andersen et al (1999).

*Co-movement across Volatilities (Covolatility).* Fleming et al (1998) construct a model of speculative trading that predicts strong volatility linkages in

the stock, bond and money markets due to common information that simultaneously affects expectations in these markets and information spillovers caused by cross market hedging. Their empirical results strongly support their model predictions. Also Andersen et al (1999) using a model-free framework for high-frequency data find high contemporaneous correlation across volatilities of FX returns.

*The Epps Effect.* The role of sampling frequency in correlation estimation has first been addressed by Epps (1979). Using stock return data, he presents evidence that indicates a substantial decrease in correlation estimates as sampling frequency increases beyond the hourly level. Evidence for foreign exchange returns has also been presented by Guillaume et al (1994), Low et al (1996) and further analyzed by Luddin, Dacorogna et al (1999).

## 1.2 Modelling Approaches

The interest of the literature on the dynamic properties of correlation has been strengthened in recent years. The availability of higher frequency data together with the development of multivariate volatility models has revealed a range of phenomena regarding correlation and motivated further research. It is possible to group the existing correlation modelling approaches in five major categories.

*Early Approaches.* Bergstrom and Henriksson (1981) examine models such as historical and mean and conclude that a common factor model with Bayesian adjusted estimates performs best. Eun and Resnick (1984) examined historical, mean and index models for the forecasting of the correlation structure. Their findings are in favor of the national mean model. Kaplanis (1988) also examined some historical and naive mean models as well as Bayesian and regression models. Her results uncover that the Bayesian model performs better.

*Regression.* Erb et al (1994) examined the stock correlation dynamics in the G7 in relation to the Business Cycle. In a regression framework they test the hypothesis that correlation can depend on such instruments as measures

of stock return and volatility persistence, the expected business cycle in two countries and the differential in expected returns in those two countries, see also Fustenberg and Jeon(1989). They present evidence that some of the variability in the correlations could be explained.

*GARCH-based Approaches.* Longin and Solnik(1995) and Cumby et al (1994) use a bivariate constant correlation GARCH of Bollerslev (1990) and make correlation time dependent. Conventional multivariate GARCH and Stochastic Volatility models have also been used to calculate correlation as the ratio of conditional covariance over the product of the individual standard deviations. This approach has been employed by Theodossiou and Lee (1993), Theodossiou et al (1997) as well as by Kahya (1997) and Burns, Engle and Mezrich (1998) using non-synchronicity return adjustments. An important problem of this approach is that correlation is not guaranteed to lie within  $(-1, 1)$ , with the exception of Kroner and Ng (1998) who provide a multivariate GARCH specification which nests many of the existing ones and implies correlations varying within the correct bounds. Also, Engle (2000) provides a multivariate GARCH approach to modelling the correlation dynamics which overcomes some of these difficulties. Finally, Ramchand and Susmel (1998) present a bivariate switching ARCH which allows for state varying covariances and correlations.

*RiskMetrics.* A popular approach among practitioners is the J P Morgan (1996) RiskMetrics volatility and correlation estimation system. Individual variances and covariances are assumed to follow exponentially weighted moving average processes, implying an IGARCH(1,1) structure at the limit. Correlations are then conventionally formed as ratios of the relevant covariance and standard deviation components. An additional to the GARCH criticism on RiskMetrics concerns its inflexible and often unrealistic model structure.

*Implied Correlations.* A recent approach to correlation forecasting exploits the availability of option prices for both individual underlying assets and portfolios formed by these assets. Inverting the option pricing formulas one obtains the implied volatility for each underlying asset as well as their

combinations. Then, correlation is immediately available by solving the formulae describing the portfolio variance with respect to correlation. This approach has been used by Campa and Chang (1997) as well as Martens and Poon (1999) for synchronous data. An evaluation of different procedures is given by Walter and Lopez (1997) and Gibson and Boyer (1997).

*Model-free Volatility Distribution.* In a recent study Andersen et al (1999) estimate the daily exchange rate volatility and correlation from within-the-day, ultra high frequency data. Under general conditions, volatility and correlation are treated as observed rather than latent, allowing to characterize their joint distributional properties.

### 1.3 Motivation for a New Model

All the above models contribute to our existing knowledge about the stability and the estimation of correlations in financial asset returns but also carry various disadvantages and restrictions. Bayesian approaches require the econometrician to have prior beliefs about the true parameters of the model and the sample information. The regression methods appear to be capturing some systematic variability but their explanatory power is low. GARCH-based models and RiskMetrics provide richer dynamics but violate a number of restrictions with the notable exception of Kroner and Ng (1998). Implied correlation methods are subject to all the imperfections of option pricing models as well as their use is restricted by the availability and reliability of option prices. Finally, the method of Andersen et al (1999) provides an analysis of the distributional characteristics of correlation but it's subject to intraday aggregation bias and its use is restricted to foreign exchange data only and in-sample applications.

The existing literature provides implicit rather than explicit methods for the evolution of correlation, with the exception of Cumby et al (1994) which can be seen as a special case of our approach. Such methods, although very useful and illuminating for in-sample analyses, suffer from at least three points of view: first, they are subject to disadvantages and restrictions of

intermediate models, second they fail to achieve sufficient generality. Finally, none of the existing approaches presents an explicit out-of-sample correlation forecasting framework.

In this paper, we shall attempt to present a framework which overcomes the existing difficulties. We build a framework for the *joint generating process* of the evolution of conditional correlations and variances over time. Our method achieves sufficient generality in the sense that the full conditional covariance matrix is guaranteed positive definite, correlations always lie within  $(-1, 1)$ , and is consistent with all the reported empirical stylized facts. The proposed model also presents an explicit approach to out-of-sample forecasting. There is a number of key issues which motivate our model building strategy. First, King, Sentana and Wadhvani (1994) attribute the dramatic changes in correlations of high frequency returns to latent rather than observed factors. This implies the existence of unobserved components driving the evolution of the conditional correlations. Second, Lundin et al (1999) as well as Andersen et al (1999) detect persistence and long memory in correlation variations. Finally and perhaps most importantly, the work of Fleming, Kirby and Ostdiek (1998) which based on an economic model, provides justification and strong evidence for the existence of common information shocks across markets.

## 2 Multivariate Correlated ARCH

Let  $\mathbf{y}_t$  be a  $N \times 1$  vector of asset returns with conditional mean equation

$$\mathbf{y}_t = \boldsymbol{\mu}_t + \boldsymbol{\varepsilon}_t \tag{1}$$

where  $\boldsymbol{\mu}_t$  is a  $N \times 1$  vector with general structure representing conditionally known information and  $\boldsymbol{\varepsilon}_t$  is a vector of error terms such that

$$\boldsymbol{\varepsilon}_t = \boldsymbol{\sigma}_t \odot \mathbf{v}_t \tag{2}$$

where  $\odot$  is the Hadamard element-by-element product,  $\boldsymbol{\sigma}_t$  a  $N \times 1$  vector of conditionally known and time dependent standard deviations and,  $\mathbf{v}_t$  is a

$N \times 1$  vector of serially independent heterogeneously distributed innovation processes, independent of  $\boldsymbol{\sigma}_t$ , such that

$$\mathbf{v}_t \mid I_{t-1} \sim D(0, R_t) \quad (3)$$

The  $N \times N$  innovation covariance matrix,  $R_t$ , has units on the main diagonal and the off-diagonal covariance elements take values within the  $(-1, 1)$  region, plus other restrictions consistent with Hawkins-Simon conditions<sup>2</sup>. The asset returns will share innovation shocks at the extent to which the covariance terms in  $R_t$  allow. By independence, we immediately see that  $E(\boldsymbol{\varepsilon}_t \mid I_{t-1}) = 0$  and the conditional covariance matrix of asset returns

$$H_t = E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t' \mid I_{t-1}) = \boldsymbol{\sigma}_t \boldsymbol{\sigma}_t' \odot R_t \quad (4)$$

The elements of  $\boldsymbol{\sigma}_t$  are assumed to be conditionally known and can follow any process from the ARCH class of models. Our purpose in this paper is to model the elements of the correlation matrix,  $R_t$ , as a function of conditionally known information and ensure that the full covariance matrix of asset returns remains positive definite at each point in time. For  $N = 2$  it is well known that positive definiteness is ensured if and only if each element of  $\boldsymbol{\sigma}_t$  is non-negative and the correlation coefficient  $\rho$  is restricted within the  $(-1, 1)$  interval. Christodoulakis and Satchell (2001), henceforth CS (2001), construct a bivariate framework and ensure the positive definiteness using an appropriately restricted form of the hypergeometric function<sup>3</sup>, known as the

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<sup>2</sup>If  $N = 3$ ,  $R_t = \begin{bmatrix} 1 & \rho_{12,t} & \rho_{13,t} \\ & 1 & \rho_{23,t} \\ & & 1 \end{bmatrix}$ , in addition to  $|\rho_{ij,t}| < 1$ , Hawkins-Simon conditions imply that all principal minors are positive, namely  $1 > 0$ ,  $1 - \rho_{12,t} > 0$ , and  $\det(R_t) > 0$ .

<sup>3</sup>An entire class of alternatives, inducing different non-linearities and speeds of transformation, can be obtained from properly restricted forms of the Hypergeometric function. For example, using Abramowitz and Stegun (1972, 15.1.4, p 556) the *Fisher's-z* transformation of  $\rho$  can be seen as

$$z = \rho F\left(\frac{1}{2}, 1, \frac{3}{2}, \rho^2\right) = \frac{1}{2} \ln\left(\frac{1+\rho}{1-\rho}\right)$$

where  $F(a, b, c, \rho)$  is the Hypergeometric function of  $\rho$  with parameters  $a, b, c$ . Further,

Fisher's-z transformation of the correlation coefficient. However, this result does not generalize easily to the full multivariate case, as cross correlation restrictions should also hold, consistent with the Hawkins-Simon conditions mentioned above. In this paper we shall model the time evolution of the full correlation matrix using a Cholesky decomposition of  $H_t$ .

From Lutkepohl (1996) we consider the Cholesky decomposition of the  $N \times N$  positive definite covariance matrix

$$H = \Delta' \Delta \tag{5}$$

where  $\Delta$  is a unique lower (upper) triangular  $N \times N$  matrix of full rank and have dropped the subscripts for convenience. Pinheiro and Bates (1996) propose a spherical parametrization of  $H$  in which, the elements of  $\Delta$  are expressed in terms of sines and cosines of their spherical coordinates. This approach allows to express the variances and correlations of  $H$  in terms of the parametrized elements of  $\Delta$ . Let  $\boldsymbol{\delta}_i$  be a vector containing the spherical coordinates of the elements of the  $i$ -th column of  $\Delta$ , the latter denoted as  $\Delta_i$ . Also let  $[\Delta]_{ij}$ ,  $[\Delta_i]_j$  and  $[\boldsymbol{\delta}_i]_j$  be the  $ij$ -th element of matrix  $\Delta$ , the  $j$ -th element of vector  $\Delta_i$  and the  $j$ -th element of vector  $\boldsymbol{\delta}_i$  respectively. Then from Pinheiro and Bates (p. 291) we have

$$\begin{aligned} [\Delta_i]_1 &= [\boldsymbol{\delta}_i]_1 \cos [\boldsymbol{\delta}_i]_2 \\ [\Delta_i]_2 &= [\boldsymbol{\delta}_i]_1 \sin [\boldsymbol{\delta}_i]_2 \cos [\boldsymbol{\delta}_i]_3 \\ &\vdots \\ [\Delta_i]_{i-1} &= [\boldsymbol{\delta}_i]_1 \sin [\boldsymbol{\delta}_i]_2 \dots \sin [\boldsymbol{\delta}_i]_{i-1} \cos [\boldsymbol{\delta}_i]_i \\ [\Delta_i]_i &= [\boldsymbol{\delta}_i]_1 \sin [\boldsymbol{\delta}_i]_2 \dots \sin [\boldsymbol{\delta}_i]_{i-1} \sin [\boldsymbol{\delta}_i]_i \end{aligned} \tag{6}$$

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using  $F(a, b, c, \rho)$  as a basis, any cumulative probability density function  $D(\rho)$  defined on the real line, can also serve as an appropriate transformation since  $1 - 2D(\rho)$  takes values on the desired interval. Additional possibilities would be the logistic transformation used by Hansen (1994) and the Spherical transformation which we use in a later section of the chapter.

For example let  $N = 3$ , then

$$\Delta = \begin{pmatrix} [\boldsymbol{\delta}_1]_1 & [\boldsymbol{\delta}_2]_1 \cos [\boldsymbol{\delta}_2]_2 & [\boldsymbol{\delta}_3]_1 \cos [\boldsymbol{\delta}_3]_2 \\ 0 & [\boldsymbol{\delta}_2]_1 \sin [\boldsymbol{\delta}_2]_2 & [\boldsymbol{\delta}_3]_1 \sin [\boldsymbol{\delta}_3]_2 \cos [\boldsymbol{\delta}_3]_3 \\ 0 & 0 & [\boldsymbol{\delta}_3]_1 \sin [\boldsymbol{\delta}_3]_2 \sin [\boldsymbol{\delta}_3]_3 \end{pmatrix}$$

which implies

$$\begin{aligned} \sigma_{ii}^2 &= [\boldsymbol{\delta}_i]_1^2, \quad i = 1, 2, 3 \\ \sigma_{1j}^2 &= [\boldsymbol{\delta}_1]_1 [\boldsymbol{\delta}_j]_1 \cos [\boldsymbol{\delta}_j]_2, \quad j = 2, 3 \\ \sigma_{23}^2 &= [\boldsymbol{\delta}_2]_1 [\boldsymbol{\delta}_3]_1 (\cos [\boldsymbol{\delta}_2]_2 \cos [\boldsymbol{\delta}_3]_2 + \sin [\boldsymbol{\delta}_2]_2 \sin [\boldsymbol{\delta}_3]_2 \cos [\boldsymbol{\delta}_3]_3) \end{aligned}$$

and thus

$$\begin{aligned} \rho_{1,j} &= \cos ([\boldsymbol{\delta}_j]_2), \quad j = 2, 3 \\ \rho_{2,3} &= \rho_{1,2}\rho_{1,3} + \sin [\boldsymbol{\delta}_2]_2 \sin [\boldsymbol{\delta}_3]_2 \cos [\boldsymbol{\delta}_3]_3 \end{aligned}$$

In general,  $\sigma_{ii}^2 = [\boldsymbol{\delta}_i]_1^2$ ,  $i = 1, \dots, N$ ,  $\rho_{1,j} = \cos ([\boldsymbol{\delta}_j]_2)$ ,  $j = 2, \dots, N$ , and the correlations between the rest of the variables will be formed as linear combinations of products of sines and cosines of the elements of  $\boldsymbol{\delta}_j$ ,  $j = 2, \dots, N$ .

We are interested in making the elements of the spherical parametrization measurable with respect to the available information set, while maintaining its uniqueness. First, we need to ensure that  $\sigma_{ii}^2 = [\boldsymbol{\delta}_i]_1^2 > 0$  for all  $i$ . We can parametrize  $[\boldsymbol{\delta}_i]_1^2 = \sigma_{ii}^2$ ,  $i = 1, \dots, N$ , as a univariate processes from the ARCH family by making the first element of  $\boldsymbol{\delta}_i$  measurable with respect to past values of  $\left(\frac{v_{i,t-k}}{\sigma_{i,t-k}}\right)^2$ , subject to the parameter restrictions of the chosen ARCH process. Second, we need to ensure that  $[\boldsymbol{\delta}_i]_j \in (0, \pi)$  for  $i = 2, \dots, N$  and  $j = 2, \dots, i$  so that  $\cos ([\boldsymbol{\delta}_j]_2) = \rho_{1,j} \in (-1, 1)$ . One possibility is to define a parameter  $\dot{z}_{j-1,i} = \dot{z}([\boldsymbol{\delta}_i]_j)$  such that

$$\dot{z}_{j-1,i} = \ln \left( \frac{[\boldsymbol{\delta}_i]_j}{\pi - [\boldsymbol{\delta}_i]_j} \right)$$

which implies

$$[\boldsymbol{\delta}_i]_j = \pi \frac{e^{\dot{z}_{j-1,i}}}{e^{\dot{z}_{j-1,i}} + 1} \quad (7)$$

The choice of the subscript of  $\dot{z}$  will be justified later in the text. As  $\dot{z}_{j-1i}$  varies on the real line,  $[\boldsymbol{\delta}_i]_j$  will be taking values on  $(0, \pi)$  and  $\cos([\boldsymbol{\delta}_j]_2) = \rho_{1,j} \in (-1, 1)$ . To model the evolution of correlations we follow the approach of CS (2001) which make  $\dot{z}$  a linear function of the available information set. Our assumptions in equations (1) to (4) imply that

$$E(v_{j-1,t}v_{i,t} \mid I_{t-1}) = E\left(\frac{\varepsilon_{j-1,t}\varepsilon_{i,t}}{\sigma_{j-1,t}\sigma_{i,t}} \mid I_{t-1}\right) = \rho_{j-1i,t}$$

thus the sequence of conditionally known errors

$$\frac{\varepsilon_{j-1,t-1}\varepsilon_{i,t-1}}{\sigma_{j-1,t-1}\sigma_{i,t-1}}, \frac{\varepsilon_{j-1,t-2}\varepsilon_{i,t-2}}{\sigma_{j-1,t-2}\sigma_{i,t-2}}, \dots$$

provide a natural minimal information set for the evolution of  $\dot{z}_{ij,t}$ . Let  $\Phi(L)$  be a lag operator polynomial of order  $p$ , we define

$$\dot{z}_{j-1,i,t} = a_0 + \Phi(L) \left[ \frac{\varepsilon_{j-1,t}\varepsilon_{i,t}}{\sigma_{j-1,t}\sigma_{i,t}} - \rho_{j-1,i} \right]$$

as a Correlated ARCH (CorrARCH) process of order  $p$ . The stability and autocorrelation properties of the process are discussed in CS (2001). Under standard results in time series analysis a high lag order CorrARCH can be generalized<sup>4</sup> to a Correlated GARCH (CorrGARCH) process of order  $(p, q)$  of the form

$$(1 - Z(L)) \dot{z}_{j-1,i,t} = (1 - Z(1)) a_0 + \Phi(L) \left[ \frac{\varepsilon_{j-1,t}\varepsilon_{i,t}}{\sigma_{j-1,t}\sigma_{i,t}} - \rho_{j-1,i} \right]$$

where  $Z(L)$  is a polynomial of order  $q$ .

To justify the choice of the subscript of  $\dot{z}$ , for  $[\boldsymbol{\delta}_i(\dot{z}_{1i,t})]_2$ ,  $i = 2, \dots, N$ , which determines the individual correlations between the first asset and the remaining  $N - 1$ , the parameter  $\dot{z}_{1i,t}$  will follow a CorGARCH process, as a linear function of appropriate past cross products of standardized residuals  $\frac{v_{1,t-k}v_{i,t-k}}{\sigma_{1,t-k}\sigma_{i,t-k}}$ ,  $k \geq 1$ . In general, we define the elements of  $[\boldsymbol{\delta}_i]_j$  as functions

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<sup>4</sup>The model of Christodoulakis and Satchell (2001) includes as a special case the model proposed by Cumby et al (1994).

$\delta(\cdot)$  of the past history of the  $i$ -th innovation if  $j = 1$  and the cross product between the  $j - 1$  and  $i$ -th innovations if  $j \geq 2$ , that is

$$\begin{aligned} [\boldsymbol{\delta}_i]_1 &= \delta \left( \left( \frac{\varepsilon_{i,t-k}}{\sigma_{i,t-k}} \right)^2, k \geq 1 \right), i = 1, \dots, N \\ [\boldsymbol{\delta}_i]_2 &= \delta \left( \frac{\varepsilon_{1,t-k} \varepsilon_{i,t-k}}{\sigma_{1,t-k} \sigma_{i,t-k}}, k \geq 1 \right), i = 2, \dots, N \\ [\boldsymbol{\delta}_i]_3 &= \delta \left( \frac{\varepsilon_{2,t-k} \varepsilon_{i,t-k}}{\sigma_{2,t-k} \sigma_{i,t-k}}, k \geq 1 \right), i = 3, \dots, N \\ &\dots \\ [\boldsymbol{\delta}_i]_N &= \delta \left( \frac{\varepsilon_{N-1,t-k} \varepsilon_{i,t-k}}{\sigma_{N-1,t-k} \sigma_{i,t-k}}, k \geq 1 \right), i = N \end{aligned}$$

Thus, the remaining  $N \left( \frac{N-3}{2} \right) + 1$  correlations will be formed as combinations of the first  $N - 1$  correlations and sines and cosines of the remaining parametrized  $[\delta_i(\dot{z}_{j-1,i,t})]_j$ ,  $i = 3, \dots, N$ ,  $j = 3, \dots, N$ . Our notational convention for  $\dot{z}_{j-1,i,t}$  signifies that  $\dot{z}$  is parametrized as a function of cross products between the  $j - 1$  and  $i$ -th innovations. In our example, for  $N = 3$  we have

$$\begin{aligned} \rho_{12,t} &= \cos [\boldsymbol{\delta}_2(\dot{z}_{12,t})]_2 \\ \rho_{13,t} &= \cos [\boldsymbol{\delta}_3(\dot{z}_{13,t})]_2 \\ \rho_{23,t} &= \rho_{12,t} \rho_{13,t} + \sin [\boldsymbol{\delta}_2(\dot{z}_{12,t})]_2 \sin [\boldsymbol{\delta}_3(\dot{z}_{13,t})]_2 \cos [\boldsymbol{\delta}_3(\dot{z}_{23,t})]_3 \end{aligned}$$

where  $\dot{z}_{j-1,i,t}$ ,  $i, j = 2, 3$  follow CorGARCH processes. Since we employ a Cholesky decomposition, an important question concerns the effects of the asset return ordering in  $\mathbf{y}_t$  on the evolution of the individual correlation coefficients. In the above example for  $N = 3$ , rotation of  $\mathbf{y}_t$  by one position results in rotation of the correlation triangle by one position as well. Thus, correlations would now be given by

$$\begin{aligned} \rho_{23,t} &= \cos [\boldsymbol{\delta}_2(\dot{z}_{23,t})]_2 \\ \rho_{12,t} &= \cos [\boldsymbol{\delta}_3(\dot{z}_{12,t})]_2 \\ \rho_{13,t} &= \rho_{23,t} \rho_{12,t} + \sin [\boldsymbol{\delta}_2(\dot{z}_{23,t})]_2 \sin [\boldsymbol{\delta}_3(\dot{z}_{12,t})]_2 \cos [\boldsymbol{\delta}_3(\dot{z}_{13,t})]_3 \end{aligned}$$

We shall see in section 5 that the impact of rotation on correlation coefficients is very small.

We summarize the stationarity properties of the CorGARCH process in the following theorem. We use a Markovian representation for the CorGARCH(p,q) process in which  $Z_t$ ,  $\Lambda$ ,  $\bar{V}$  are properly defined vectors and  $A$  is the parameter matrix, see proof in the appendix.

**Theorem 1** *A CorGARCH(p,q) process is stable if and only if the associated characteristic polynomial has all its roots within the unit circle. The steady-state of  $z_t$  is given by the first element of*

$$E(Z_t) = (I - A)^{-1} \Lambda$$

and its steady-state variance is given by the first element of

$$\begin{aligned} \text{Var}(Z_t) = & (I - (A \otimes A))^{-1} \left( \text{vec}(\Lambda \Lambda') + \bar{V} \right) \\ & + (I - (A \otimes A))^{-1} [(A \otimes \Lambda) + (\Lambda \otimes A)] (I - A)^{-1} \Lambda \\ & - [(I - A) \otimes (I - A)]^{-1} \text{vec}(\Lambda \Lambda') \end{aligned}$$

**Proof:** see Appendix

As an example, let  $z_t$  follow a gaussian CorGARCH(1,1) process of the form

$$z_t = \lambda + \zeta_1 z_{t-1} + \phi_1 \tilde{v}_{t-1}$$

and  $\tilde{v}_t = \frac{\varepsilon_{j-1,t} \varepsilon_{1,t}}{\sigma_{j-1,t} \sigma_{1,t}}$ . Then by Theorem 1 we have  $E(z_t) = \frac{\lambda}{1-\zeta_1}$  and  $\text{Var}(z_t) = \frac{\phi_1^2(1+2\rho^2)}{1-\zeta_1^2}$  where  $\lambda = (1 - Z(1)) a_0$  and  $(1 + 2\rho^2)$  is the second joint moment of two standard gaussian innovations, see Johnson and Kotz (1972) (9.1) and (9.2).

As we shall see explicitly later in the paper, our assumption on correlated return innovation processes allows for the flexibility of correlated volatilities. This is in line with recently reported empirical facts, although an earlier literature<sup>5</sup> has addressed a similar issue in the form of volatility spillover effects.

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<sup>5</sup>see for example Hamao et al (1990).

The latter refer to the phenomenon that the lagged volatility or squared error of one asset may have a significant impact on the current volatility of another. Spillover effects can be accommodated in our framework by allowing the asset volatility be generated by any GARCH-X type of processes.

### 3 Model Properties and Stylized Facts

The corner stone of our proposed model is the assumption that the innovation processes of asset returns exhibit covariances. This assumption distinguishes our approach from the existing multivariate GARCH models which either assume independent innovation processes (see Diebold and Nerlove (1989) and Engle et al (1990)) or ignore this aspect (see Engle and Kroner (1995)). This is an important flexibility of our model which first of all allows to model conditional correlation between assets as a conditionally autoregressive process, exhibiting time variation and persistence. Further, it enables the model to generate a number of reported stylized facts such as the co-movement between volatilities (see for example Hamao, Masulis and Ng (1990)) as well as between volatilities and correlation (see King and Wadhvani(1990), Bertero and Mayer(1990), Solnik et al (1996), Ramchand and Susmel (1998)). The purpose of this section is to characterize the existence and the steady-state of the covariances  $\text{Cov}(\sigma_{i,t}^2, \sigma_{j,t}^2)$  and  $\text{Cov}(z_t, \sigma_{j,t}^2)$ . As we shall see, these can be stated as functions of the corresponding processes parameters and joint moments of the innovation processes. We summarize the relevant results in the following propositions. We denote  $\Sigma_{1,t}$ ,  $\Sigma_{2,t}$  and  $Z_t$  the state-space form of two correlated GARCH volatility processes and the CorGARCH process respectively,  $\Gamma_{ij}$  are parameter matrices which are properly defined in the proof, see appendix.

**Proposition 2** *Let  $\Sigma_{1,t}$  and  $\Sigma_{2,t}$  follow stationary GARCH processes of order  $(p_j, q_j)$ ,  $j = 1, 2$  respectively, in state-space form, with correlated innovation processes. If  $E\left(\dot{\Gamma}_{t-j} \otimes \dot{\Gamma}_{t-j}\right)$  has all its eigen values within the unit circle,  $E\left(\sigma_{1,t}^2 \sigma_{2,t}^2\right)$  has a strictly stationary solution and  $\text{Cov}(\sigma_{1,t}^2, \sigma_{2,t}^2)$*

is given by the first element of  $\overline{\text{Cov}}(\Sigma_{1,t}, \Sigma_{2,t}) = \text{vec}(\text{Cov}(\Sigma_{1,t}, \Sigma_{2,t}))$

$$\begin{aligned} & \overline{\text{Cov}}(\Sigma_{1,t}, \Sigma_{2,t}) \\ = & \text{vec}(\Gamma_{10}\Gamma'_{20}) + \Gamma_{2\Gamma_{10}}(I - \Gamma_2)^{-1}\Gamma_{20} + \Gamma_{20\Gamma_1}(I - \Gamma_1)^{-1}\Gamma_{10} \\ & + ((I - \Gamma_{2\Gamma_1})^{-1}\Gamma_{2\Gamma_1})\text{vec}(\Gamma_{10}\Gamma'_{20}) \\ & + ((I - \Gamma_{2\Gamma_1})^{-1}\Gamma_{2\Gamma_1})(\Gamma_{2\Gamma_{10}}(I - \Gamma_2)^{-1}\Gamma_{20} + \Gamma_{20\Gamma_1}(I - \Gamma_1)^{-1}\Gamma_{10}) \\ & - ((I - \Gamma_2) \otimes (I - \Gamma_1))^{-1}\text{vec}(\Gamma_{10}\Gamma'_{20}) \end{aligned}$$

**Proof:** *see appendix*

As an example, for two correlated gaussian GARCH(1,1) processes, where the vector innovation process is drawn from a standard bivariate gaussian distribution with correlation  $\rho$ , the above proposition states that

$$\text{Cov}(\sigma_{1,t}^2, \sigma_{2,t}^2) = \frac{2\gamma_{20}\gamma_{10}\gamma_1\gamma_2\rho^2}{(1 - \gamma_1\gamma_2(1 + 2\rho^2) - \delta_2\gamma_1 - \gamma_2\delta_1 - \delta_2\delta_1)(1 - \gamma_1 - \delta_1)(1 - \gamma_2 - \delta_2)}$$

where  $\rho \in (-1, 1)$  and  $(1 + 2\rho^2)$  is the second joint moment of two standard gaussian innovation processes (see Johnson and Kotz (1972) (9.1) and (9.2)) and  $\gamma_j, \delta_j$  are ARCH and GARCH parameters respectively. Provided that stability conditions hold, covariance between volatilities will be positive irrespective of the sign of correlation between innovations. This is a GARCH-specific result due to the fact that the fundamental volatility shock is the squared return innovation process and may change for alternative volatility processes. We set  $\gamma_{10} = \gamma_{20} = 1$  and  $\gamma_1 = \gamma_2 = 0.01$  and plot the above relationship as a function of the GARCH persistence parameters,  $\delta_1$  and  $\delta_2$ , and for various values of  $\rho$ , see appendix Graph I. It is clear that the covariance of volatilities is a positive function of  $\rho$ , as well as of common persistence in volatility.

Our results on the relationship between volatility and correlation are summarized in the following proposition.

**Proposition 3** *Let  $\Sigma_t$  and  $Z_t$  follow stationary GARCH and CorGARCH processes in state-space form, of order  $(p_j, q_j)$ ,  $j = 1, 2$  respectively, with*

correlated innovation processes. If  $(\Gamma \otimes A)$  has all its eigen values within the unit circle and  $E(\overline{V_t' V_t}) < \infty$ ,  $E(V_t' V_t) < \infty$ , then  $E(\sigma_t^2 z_t)$  has a strictly stationary solution and  $\text{Cov}(\sigma_t^2, z_t)$  is given by the first element of

$$\begin{aligned} \overline{\text{Cov}}(Z_t, \Sigma_t) &= \kappa + \Lambda_1 (I - \Gamma)^{-1} \Gamma_0 + \Lambda_2 (I - A)^{-1} A_0 \\ &\quad + [I - (\Gamma \otimes A)]^{-1} (\Gamma \otimes A) \kappa \\ &\quad + [I - (\Gamma \otimes A)]^{-1} (\Gamma \otimes A) \Lambda_1 (I - \Gamma)^{-1} \Gamma_0 \\ &\quad + [I - (\Gamma \otimes A)]^{-1} (\Gamma \otimes A) \Lambda_2 (I - A)^{-1} A_0 \\ &\quad - [(I - \Gamma) \otimes (I - A)]^{-1} \text{vec} \left( A_0 \Gamma_0' \right) \end{aligned}$$

**Proof:** *see appendix.*

For the special case of a gaussian GARCH(1,1) and a CorGARCH(1,1), using Proposition 3 and the joint moments of two standard gaussian processes  $\mu_{2r+1, 2s+1}$  for  $r = 0, s = 1$  or  $r = 1, s = 0$ , see Johnson and Kotz (1972) (9.1) and (9.2), we obtain

$$\overline{\text{Cov}}(\dot{z}_t, \varepsilon_t^2) = \frac{2\rho\gamma_0\gamma_1\phi_1}{(1 - \gamma_1 - \delta_1)(1 - \gamma_1\zeta_1 - \delta_1\zeta_1)}$$

where  $\gamma_0, \gamma_1, \delta_1$  are the GARCH parameters and  $\phi_1, \zeta_1$  CorGARCH parameters. Thus, covariance may take both positive and negative values if we consider that  $\phi_1$  and  $\zeta_1$  do not carry any sign restrictions. We set  $\gamma_0 = 1$  and  $\gamma_1 = \phi_1 = 0.01$  and plot the above relationship as a function of the GARCH ( $\delta_1$ ) and CorGARCH ( $\zeta_1$ ) persistence parameters, for various values of  $\rho$ , see appendix Graph II. It is clear that the covariance between conditional volatility and the driving force of conditional correlation  $\dot{z}_t$  is a positive function of common volatility as well as  $\rho$ , and its sign depends on the sign of  $\rho$ .

We shall discuss in more detail the empirical relevance of the above propositions in a later section, where we illustrate the estimation of the multivariate correlated ARCH model using data.

## 4 Likelihood Function and Estimation

The Maximum Likelihood estimation of the multivariate Correlated ARCH model follows the standard practice for multivariate ARCH models. Since volatilities and correlations are driven by conditionally autoregressive processes, thus  $\sigma_{i,t}^2, \sigma_{j-1i,t}^2, \rho(z_{ij-1,t}) \in I_{t-1}$ , the unknown parameters can be estimated using a (Quasi) Maximum Likelihood (ML) approach. Equations (1) to (4) state that

$$y_t = \boldsymbol{\mu}_t + \boldsymbol{\varepsilon}_t$$

where

$$\boldsymbol{\varepsilon}_t \mid I_{t-1} \sim (0, \boldsymbol{\sigma}_t \boldsymbol{\sigma}_t' \odot R_t)$$

$\sigma_{i,t}^2, \sigma_{j-1i,t}^2, \rho(z_{ij-1,t})$  are assumed to follow univariate GARCH-type and CorGARCH processes.

A standard approach is to estimate the unknown parameters by maximizing a conditional multivariate normal, which for a sample size  $T$ , yields a log-likelihood function of the form

$$\begin{aligned} \ln L &= -\frac{TN}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^T \ln |H_t| - \frac{1}{2} \sum_{t=1}^T \boldsymbol{\varepsilon}_t' H_t^{-1} \boldsymbol{\varepsilon}_t \\ &= -\frac{TN}{2} \ln(2\pi) - \sum_{t=1}^T \ln |C_t| - \frac{1}{2} \sum_{t=1}^T \ln |R_t| - \frac{1}{2} \sum_{t=1}^T \boldsymbol{\varepsilon}_t' H_t^{-1} \boldsymbol{\varepsilon}_t \end{aligned}$$

where  $C_t$  is a diagonal matrix of conditional standard deviations such that  $H_t = C_t R_t C_t$ . The positive definiteness of the full covariance matrix,  $H_t$ , is ensured by imposing the transformations (5) to (7). Now our log-likelihood function is expressed in terms of  $\boldsymbol{\varepsilon}_t$ 's which are functions of the mean parameters contained in  $\boldsymbol{\mu}_t$ , of  $\sigma_{i,t}^2$ 's which evolve as univariate GARCH-type processes and of  $z_{ij-1,t}$ 's which evolve as CorrARCH or CorGARCH process of any order as presented in section 2. Our purpose is to estimate the values of the unknown parameters involved in the conditional mean, conditional variance and the conditional correlation equations that will give the globally maximum value of the likelihood function. For example, in the

very simple case of a trivariate system with no predetermined variables or ARMA terms in the conditional means and where the conditional variances are assumed to follow GARCH(1,1) processes and the conditional correlations CorGARCH(1,1) process, we shall need to estimate a total of twenty one parameters, three coming from the conditional means, nine from the univariate conditional volatility processes and nine from the three conditional correlation processes.

To maximize the likelihood function, we need to evaluate its first and second order derivatives with respect to the vector of the unknown parameters. It is clear that the log-likelihood function is highly non-linear and therefore we need an algorithm for non linear, iterative maximization since we cannot have a closed form expression for the maximizing vector. In addition, the complexity of analytical derivatives suggests that utilization of numerical derivatives would facilitate the estimation procedure. There is a tradition in the GARCH literature to employ the Berndt, Hall, Hall, Hausman(1974) (BHHH) algorithm, but others like the Newton-Raphson and Quasi-Newton algorithms may be as appropriate. Our experience in estimating CorGARCH-type models with daily stock market returns, suggests that Newton-Raphson will be more stable, though less fast. Finally, based on the arguments of Nelson and Cao(1992) the estimation is performed without imposing non-negativity constraints on the GARCH processes.

Excess kurtosis and conditional non-normality of the data do not affect the asymptotic properties of the quasi ML estimator. In particular, Bollerslev and Wooldridge(1992), (henceforth BW), establish the consistency and asymptotic normality of the Quasi MLE for general dynamic multivariate models that jointly parametrize the conditional mean and the conditional covariance matrix. Assuming that some regularity conditions hold and that the first two conditional moments are correctly specified, by theorem 2.1 of BW(1992) even when the normality assumption is violated, maximization of a normal likelihood function will yield consistent estimates with normal asymptotic distribution. However, these results are based on regularity conditions (see BW(1992) appendix A, A.1) that are very abstract and their

verification for any model cannot be guaranteed.

As it is well known, ignorance of excess kurtosis will cause underestimation of the standard errors of the estimated coefficients. The literature has addressed this issue either by maximizing a likelihood function based on a conditionally non-normal density, see Nelson(1991), or by employing bootstrap methods to estimate heteroscedasticity robust coefficient standard errors or by using covariance matrix adjustments such as the White's and that proposed by BW(1992).

## 5 An Empirical Application

We illustrate the empirical relevance of our theoretical model using a trivariate specification. We shall study the evolution of monthly correlation between the Dow Jones Industrial (DJ) index, the Nasdaq Composite (NC) and the three-month US Treasury Bill (TB) yield. We use a decade of monthly returns  $y_{j,t} = (\ln P_{j,t} - \ln P_{j,t-1}) \times 100$ , where  $P_{j,t}$  denotes the  $j$ -th price at time  $t$ , for the period of August 05, 1989 to July 05, 2001 resulting in 144 data points for each series.

We first perform a preliminary univariate statistical analysis of the data. All three series were found to exhibit excess kurtosis and a slight negative skewness for the cases of Nasdaq and Treasury Bill returns. We also find no serial correlation for the Nasdaq returns and first order serial correlation for DJ and TB. Further, all series exhibit strong serial correlation for the squared returns and we strongly rejected the null of no GARCH effects. A univariate model selection procedure for each series resulted in an MA(1), a constant and an MA(2) conditional mean, along with GARCH(1,1) variances for DJ, NC and TB respectively.

Our model building strategy proceeds from specific to general. We first jointly estimate the first and second conditional moments of all three series assuming zero correlation, see Table I. Then, we relax the zero correlation assumption and re-estimate the model assuming constant correlations, Table II. Further, we sequentially relax the assumption of constant correlation

between two portfolios and re-estimate the model allowing correlation to be generated by a CorGARCH(1,1) process. In each case, we record the coefficient estimates together with heteroscedasticity robust standard errors, and also calculate Likelihood Ratio (LR) tests and model selection criteria such as AIC and SIC. To save space we report the full joint estimation results for conditional means, variances and correlations in Table III. Additional tables with only one or two varying correlations are available upon request. Note that in table III the  $ij$ -th correlation equation parameters are in fact those of the  $\dot{z}_{ij,t}$  process.

An inspection of the tables I and II shows that we reject the null of no correlation over a constant correlation specification. This is supported by the values of the t-statistic that the corresponding parameters take as well as the LR tests and model selection criteria. Further, Table III presents evidence for strongly time-varying and persistent  $\dot{z}_{ij,t}$  again on the basis of t-statistics and the likelihood ratio test and Information Criteria.

Using the estimated model of Table III we graph the empirical estimates of the three correlation processes in Figure 1. It is clear that correlation in all cases is strongly time varying and persistent. Correlation between DJ and NC fluctuates from 0.13 to 0.60 with a steady-state around 0.35. The correlations of DJ and NC to TB take both positive and negative values ranging from  $-0.12$  to  $0.33$  and both have a steady-state around zero. We can gain some further intuition about the degree of persistence by looking the on average ‘half-life’ of a shock associated to the  $\dot{z}_{ij,t}$  process, that is the number of periods  $s$  such that  $\zeta_1^s = \frac{1}{2}$ . Using the estimated values of  $\zeta_1$  from table III we see that shocks on correlation of DJ to NC and TB have half-lives of about three and five months respectively, while the half-live of shocks to correlation between NC and TB is exhausted in one and a half month.

In Figures 2 to 4 we graph each individual correlation together with the conditional variances of the corresponding assets. An inspection uncovers that variances tend to move together to a large extent and that in period of common high volatility, correlation is also high. These empirical observations are justified by our theoretical results in section 3 and are consistent with

the existing literature.

Had we used daily data from geographically sequential markets, our results on correlation would be subject non-synchronous trading biases. Kahya (1997) shows that “observed” correlation in one period equals  $\frac{n-l}{n} \rho$ , where  $n$  is the total number of hours in one period,  $l$  the number of hours by which the closing times in sequential markets differ, ( $l < n$ ), and  $\rho$  is the “true” correlation. In our context, we can use Theorem 1 and a CorGARCH(1,1) process and see that  $E(\dot{z})$  is free of  $\rho$  and thus remains unaffected, while its variance becomes  $Var(\dot{z}) = \frac{\phi_1^2 \left(1 + 2\left(\frac{n-l}{n}\rho\right)^2\right)}{1 - \zeta_1^2}$ . Clearly,  $Var(\dot{z})$  is an increasing function of  $\rho$  irrespective of its sign, and decreasing function of the distance between closing times,  $l$ , thus introducing a downward bias. Also, as the number of hours in one period,  $n$ , increases<sup>6</sup> the term  $\frac{n-l}{n}$  approaches unity thus eliminating the non-synchronicity bias.

Graphs III to V present the News Impact Surface on  $\rho_{ij,t}$  following the methodology established by Engle and Ng (1993) for volatility processes. In particular, we fix the information set to  $I_{t-2}$ , evaluate lagged  $\dot{z}_{t-l}$  at its steady state, and then plot  $\rho_{ij,t}$  as a function of  $v_{i,t-1}$  and  $v_{j,t-1}$ . We can observe in Graph III, that correlation  $\rho_{ij,t}$  tends to unit when both innovations take large values of the same sign and to minus one when both innovations take large values of opposite sign. The impact of news on correlation is a non linear one, but symmetric in that positive and negative shocks generate equal impact on correlation. One could of course design such an error structure for the CorGARCH process that would generate various sorts of asymmetries, analogous to EGARCH and GJR-GARCH processes. In Graphs IV and V we plot the effects of  $v_{1,t-1}v_{2,t-1}$  and  $v_{1,t-1}v_{3,t-1}$  (which shock  $\rho_{12,t}$  and  $\rho_{13,t}$  respectively) on  $\rho_{23,t}$ . We observe that the relative impact is very small which implies that asset rotation in  $\mathbf{y}_t$  would substantially leave the correlation structure unaffected.

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<sup>6</sup>This is equivalent to moving from a higher to a lower frequency of data.

## 6 Conclusions

This paper presents a new, full multivariate framework for modelling the evolution of conditional correlation between financial asset returns. Our approach assumes that a vector of asset returns is shocked by a vector innovation process the covariance matrix of which is time-dependent. We guarantee the positive definiteness of the covariance matrix using an appropriate Cholesky decomposition which, when expressed in terms of its spherical coordinates allows for the modelling of conditional variances and correlations using conditionally autoregressive processes. The resulting asset covariance matrix is guaranteed to be positive definite at each point in time. We follow Christodoulakis and Satchell (2001) in designing explicit discrete-time stochastic processes for the correlation coefficients and present analytical results for their distribution properties. Our approach allows for explicit out-of-sample forecasting of conditional correlations and generates a number of observed stylised facts. These concern the time dependence of conditional correlations, persistence and correlation clustering, co-movement between the correlation coefficients, between correlation and volatility as well as between volatility processes (co-volatility). We also study analytically the co-movement between the elements of the asset covariance matrix which are shown to depend on their persistence parameters as well as the covariance of the asset innovation processes. We provide empirical evidence using monthly returns on Dow Jones Industrial average, Nasdaq Composite and the three-month US Treasury Bill, which supports our theoretical arguments.

## 7 Appendix

### 7.1 Proof of Theorem 1

Let a CorGARCH(p,q) be

$$z_t = \lambda + \zeta_1 z_{t-1} + \cdots + \zeta_p z_{t-p} + \phi_1 \tilde{v}_{t-1} + \cdots + \phi_q \tilde{v}_{t-q}$$

Its state-space form can be written

$$\begin{bmatrix} z_t \\ z_{t-1} \\ \vdots \\ z_{t-p+1} \\ \tilde{v}_t \\ \vdots \\ \tilde{v}_{t-q+1} \end{bmatrix} = \begin{bmatrix} \lambda \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \zeta_1 & \zeta_2 & \cdots & \zeta_p & \phi_1 & \cdots & \phi_q \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} z_{t-1} \\ z_{t-2} \\ \vdots \\ z_{t-p} \\ \tilde{v}_{t-1} \\ \vdots \\ \tilde{v}_{t-q} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \tilde{v}_t \\ 0 \\ 0 \end{bmatrix}$$

and more compactly

$$Z_t = \Lambda + AZ_{t-1} + \tilde{V}_t \quad (\text{Th1.1})$$

Upon recursive substitution  $k$  times

$$Z_t = \sum_{j=0}^{k-1} A^j \Lambda + A^k Z_{t-k} + \sum_{j=0}^{k-1} A^j \tilde{V}_{t-j} \quad (\text{Th1.1a})$$

As  $k \rightarrow \infty$  the first term of the r-h-s in the above equation will be absolutely summable if all the eigen values of  $A$  lie within the unit circle. Under the same condition the last term of the r-h-s will be convergent in mean square (see Lutkepohl (1991) proposition C.7, p. 490) provided that  $E(\tilde{V}'_{t-j} \tilde{V}_{t-j})$  is finite. Then

$$E(Z_t) = (I - A)^{-1} \Lambda$$

the first element of which is  $E(z_t) = \frac{\lambda}{1 - \zeta_1 - \dots - \zeta_p}$ .

Also, upon recursive substitution on  $\text{vec}(Z_t Z'_t)$  we obtain

$$\begin{aligned} \text{vec}(Z_t Z'_t) &= \sum_{j=0}^k (A \otimes A)^j \text{vec}(\Lambda \Lambda') + \sum_{j=0}^k (A \otimes A)^j K_{t-j} Z'_{t-j-1} \\ &\quad + \sum_{j=0}^{k-1} (A \otimes A)^j \tilde{V}_{t-j} + (A \otimes A)^k \text{vec}(Z_{t-k} Z'_{t-k}) \end{aligned} \quad (\text{Th1.2})$$

where

$$K_t = (A \otimes \Lambda) + (\Lambda \otimes A) + (\tilde{V}_t \otimes A) + (A \otimes \tilde{V}_t)$$

and

$$\bar{V}_t = \text{vec} \left( \Lambda \tilde{V}_t' \right) + \text{vec} \left( \tilde{V}_t \Lambda' \right) + \text{vec} \left( \tilde{V}_t \tilde{V}_t' \right)$$

$(p+q)^2 \times 1$

For  $k \rightarrow \infty$  in Th1.2, the term  $\sum_{j=0}^k (A \otimes A)^j \text{vec}(\Lambda \Lambda')$  will be absolutely summable provided that  $(A \otimes A)$  has all its eigen values within the unit circle. Under the same condition  $\sum_{j=0}^{k-1} (A \otimes A)^j \bar{V}_{t-j}$  will be convergent in mean square provided that in addition  $E(\bar{V}_{t-j}' \bar{V}_{t-j}) = E(2\lambda^2 v_t^2 + v_t^4) < \infty$ . Last, taking into account Th1.1a, the second term on the r-h-s of Th1.2 is written as

$$\sum_{j=0}^k (A \otimes A)^j K_{t-j} \sum_{i=j+2}^k A^i \Lambda + \sum_{j=0}^k (A \otimes A)^j K_{t-j} \sum_{i=j+2}^k A^i \tilde{V}_{t-i}$$

which converges in mean square provided that  $(A \otimes A)$  has all its eigen values within the unit circle and  $\sum_{i=j+2}^k A^i \Lambda$  and  $\sum_{i=j+2}^k A^i \tilde{V}_{t-i}$  are absolutely summable and convergent in mean square respectively.

For  $k \rightarrow \infty$  in Th1.2 and using independence,  $\text{Var}(z_t) = E(z_t^2) - E(z_t)^2$  is given by the first element of

$$\begin{aligned} \text{Var}(Z_t) &= (I - (A \otimes A))^{-1} \left( \text{vec} \left( \Lambda \Lambda' \right) + \bar{V} \right) \\ &\quad + (I - (A \otimes A))^{-1} [(A \otimes \Lambda) + (\Lambda \otimes A)] (I - A)^{-1} \Lambda \\ &\quad - [(I - A) \otimes (I - A)]^{-1} \text{vec} \left( \Lambda \Lambda' \right) \end{aligned}$$

where  $\bar{V} = E(\bar{V}_{t-j})$  with  $E(\tilde{v}_{t-j}^2)$  as its  $(p+q)p + p + 1$  element and zero elsewhere.  $\yenumber$

## 7.2 Proof of Proposition 2

We adopt the state-space representation of a GARCH(p,q) process of Bourgerol and Picard (1992) where for  $p \geq 2$

$$\Sigma_{i,t} = \Gamma_{i0} + \Gamma_{i,t-1} \Sigma_{i,t-1}$$

where  $\Sigma'_{i,t} = (\sigma_{i,t}^2 \dots \sigma_{i,t-q+1}^2, \varepsilon_{t-1}^2 \dots \varepsilon_{t-p+1}^2)$  and  $\Gamma'_{i0} = (\gamma_0, 0, \dots, 0)$  are  $(p+q-1) \times 1$  vectors,  $\Gamma_{i,t-1}$  is a  $(p+q-1) \times (p+q-1)$  matrix with first row

$$(\gamma_1 v_{t-1}^2 + \delta_1, \delta_2, \dots, \delta_q, \gamma_2, \dots, \gamma_p)$$

and  $q+1$  row  $(v_{t-1}^2, 0, \dots, 0)$  and zeros elsewhere. Upon recursive substitution on  $\text{vec}(\Sigma_{1,t}\Sigma'_{2,t})$  we obtain

$$\begin{aligned}
\text{vec}(\Sigma_{1,t}\Sigma'_{2,t}) &= \text{vec}(\Gamma_{10}\Gamma'_{20}) + (\Gamma_{2,t-1} \otimes \Gamma_{10}) \Sigma_{2,t-1} + (\Gamma_{20} \otimes \Gamma_{1,t-1}) \Sigma_{1,t-1} \\
&+ \sum_{r=1}^{k-1} \prod_{j=1}^r (\Gamma_{2,t-j} \otimes \Gamma_{1,t-j}) \text{vec}(\Gamma_{10}\Gamma'_{20}) \\
&+ \sum_{r=2}^k \prod_{j=1}^{r-1} (\Gamma_{2,t-j} \otimes \Gamma_{1,t-j}) (\Gamma_{2,t-r} \otimes \Gamma_{10}) \Sigma_{2,t-r} \\
&+ \sum_{r=2}^k \prod_{j=1}^{r-1} (\Gamma_{2,t-j} \otimes \Gamma_{1,t-j}) (\Gamma_{20} \otimes \Gamma_{1,t-r}) \Sigma_{1,t-r} \\
&+ \prod_{j=1}^k (\Gamma_{2,t-j} \otimes \Gamma_{1,t-j}) \text{vec}(\Sigma_{1,t-k}\Sigma'_{2,t-k}) \quad (\text{Pr2.1})
\end{aligned}$$

Given stationarity of  $\Sigma_{2,t}$  and  $\Sigma_{1,t}$ , we wish to establish conditions for the convergence of the vector process  $\sum_{r=1}^{k-1} \prod_{j=1}^r \dot{\Gamma}_{t-j} \dot{\Gamma}$  where  $\dot{\Gamma}_{t-j} = (\Gamma_{2,t-j} \otimes \Gamma_{1,t-j})$  and  $\dot{\Gamma}$  is a conformable vector. Denote  $(\cdot)_l$  the  $l$ -th element of a vector and  $(\cdot)_{l,n}$  the  $(l, n)$ -th element of a matrix. By Chung (1974, xi, p 42)

$$\sum_{r \geq 1} E \left| \prod_{j=1}^r \dot{\Gamma}_{t-j} \dot{\Gamma} \right|_l < \infty$$

for all  $l$ , implies that  $\sum_{r \geq 1} \prod_{j=1}^r \dot{\Gamma}_{t-j} \dot{\Gamma}$  is absolutely convergent almost surely.

Step 1. We show this for  $l = 1$  and  $\dot{\Gamma} = \text{vec}(\Gamma_{10}\Gamma'_{20})$ :

$$E \left| \prod_{j=1}^r \dot{\Gamma}_{t-j} \dot{\Gamma} \right|_1 = E \left| \sum_{s=1}^{(p+q-1)^2} \left( \prod_{j=1}^r \dot{\Gamma}_{t-j} \right)_{1,s} (\dot{\Gamma})_s \right| \leq \sum_{s=1}^{(p+q-1)^2} E \left| \left( \prod_{j=1}^r \dot{\Gamma}_{t-j} \right)_{1,s} (\dot{\Gamma})_s \right|$$

by triangle inequality. Also by Cauchy-Schwartz inequality

$$\sum_{s=1}^{(p+q-1)^2} E \left| \left( \prod_{j=1}^r \dot{\Gamma}_{t-j} \right)_{1,s} (\dot{\Gamma})_s \right| \leq \sum_{s=1}^{(p+q-1)^2} \left[ E \left( \left( \prod_{j=1}^r \dot{\Gamma}_{t-j} \right)_{1,s} (\dot{\Gamma})_s \right)^2 \right]^{\frac{1}{2}}$$

Now denote  $(\cdot \otimes \cdot)_{k,l;m,n}$  the product of the  $(\cdot)_{k,l}$   $(\cdot)_{m,n}$  elements of two matrices and evaluate for  $s = 1, \dots, (p+q-1)^2$  using independence

$$\begin{aligned} E \left( \prod_{j=1}^r \dot{\Gamma}_{t-j} \right)_{1,s}^2 &= E \left( \prod_{j=1}^r \dot{\Gamma}_{t-j} \otimes \prod_{j=1}^r \dot{\Gamma}_{t-j} \right)_{1,s;1,s} \\ &= \prod_{j=1}^r E \left( \dot{\Gamma}_{t-j} \otimes \dot{\Gamma}_{t-j} \right)_{1,s;1,s} = \left( E \left( \dot{\Gamma}_{t-j} \otimes \dot{\Gamma}_{t-j} \right) \right)_{1,s;1,s}^r \end{aligned}$$

which will be finite if  $E \left( \dot{\Gamma}_{t-j} \otimes \dot{\Gamma}_{t-j} \right)$  has all its eigen values within the unit circle.

Step 2. Let  $r \rightarrow \infty$ . Provided that  $\Sigma_{1,t}$  and  $\Sigma_{2,t}$  are stationary processes and by independence between  $\Gamma_{i,t-r}$ ,  $i = 1, 2$  and  $(\Gamma_{2,t-j} \otimes \Gamma_{1,t-j})$   $j < r$ , using the same arguments as in step 1 we conclude that if  $E \left( \dot{\Gamma}_{t-j} \otimes \dot{\Gamma}_{t-j} \right)$  has all its eigen values within the unit circle the third and fourth lines of Pr2.1 will be absolutely convergent almost surely. The term on the fifth line will vanish under the same condition.

For  $r \rightarrow \infty$ , taking expectations in Pr2.1 and provided the above results hold, we obtain  $\overline{\text{Cov}}(\Sigma_{1,t}, \Sigma_{2,t}) = \text{vec}(\text{Cov}(\Sigma_{1,t}, \Sigma_{2,t}))$

$$\begin{aligned} \overline{\text{Cov}}(\Sigma_{1,t}, \Sigma_{2,t}) &= \text{vec} \left( \Gamma_{10} \Gamma'_{20} \right) + \Gamma_{2\Gamma_{10}} (I - \Gamma_2)^{-1} \Gamma_{20} + \Gamma_{20\Gamma_1} (I - \Gamma_1)^{-1} \Gamma_{10} \\ &\quad + ((I - \Gamma_{2\Gamma_1})^{-1} \Gamma_{2\Gamma_1}) \text{vec} \left( \Gamma_{10} \Gamma'_{20} \right) \\ &\quad + ((I - \Gamma_{2\Gamma_1})^{-1} \Gamma_{2\Gamma_1}) \left( \Gamma_{2\Gamma_{10}} (I - \Gamma_2)^{-1} \Gamma_{20} + \Gamma_{20\Gamma_1} (I - \Gamma_1)^{-1} \Gamma_{10} \right) \\ &\quad - ((I - \Gamma_2) \otimes (I - \Gamma_1))^{-1} \text{vec} \left( \Gamma_{10} \Gamma'_{20} \right) \end{aligned}$$

where  $\Gamma_{2\Gamma_{10}} = E(\Gamma_{2,t-j} \otimes \Gamma_{10})$ ,  $\Gamma_{20\Gamma_1} = E(\Gamma_{20} \otimes \Gamma_{1,t-j})$ ,  $\Gamma_{2\Gamma_1} = E(\Gamma_{2,t-j} \otimes \Gamma_{1,t-j})$ .  $\yen$

### 7.3 Proof of Proposition 3

Let  $Z_t$  be represented as in Theorem 1. Also let a GARCH(p,q)  $\Sigma_t$  take the state space form of its ARMA representation (Baillie and Bollerslev (1992)) for the squared errors  $\varepsilon_t^2$  with, zero-mean and serially uncorrelated innovations  $v_t = \varepsilon_t^2 - \sigma_t^2$  :

$$\begin{bmatrix} \varepsilon_t^2 \\ \varepsilon_{t-1}^2 \\ \vdots \\ \varepsilon_{t-m+1}^2 \\ v_t \\ \vdots \\ v_{t-p+1} \end{bmatrix} = \begin{bmatrix} \gamma_0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} \gamma_1 + \delta_1 & \cdots & \gamma_m + \delta_m & -\delta_1 & \cdots & -\delta_p \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{t-1}^2 \\ \varepsilon_{t-2}^2 \\ \vdots \\ \varepsilon_{t-m}^2 \\ v_{t-1} \\ \vdots \\ v_{t-p} \end{bmatrix} + \begin{bmatrix} v_t \\ 0 \\ \vdots \\ 0 \\ v_t \\ 0 \\ 0 \end{bmatrix}$$

and more compactly  $\Sigma_t = \Gamma_0 + \Gamma \Sigma_{t-1} + V_t$  where  $\Sigma_t, \Gamma_0, V_t$  are  $(m+p) \times 1$  vectors and  $m = \max(p, q)$ .

Upon recursive substitution on  $\text{vec}(Z_t \Sigma_t')$  we obtain

$$\begin{aligned} \text{vec}(Z_t \Sigma_t') &= \text{vec}\left((A_0 + \bar{V}_t) (\Gamma'_0 + V'_t)\right) \\ &+ [(\Gamma \otimes A_0) + (\Gamma \otimes \bar{V}_t)] \Sigma_{t-1} + [(\Gamma_0 \otimes A) + (V_t \otimes A)] Z_{t-1} \\ &+ \sum_{j=1}^{k-1} (\Gamma \otimes A)^j \text{vec}\left((A_0 + \bar{V}_{t-j}) (\Gamma'_0 + V'_{t-j})\right) \\ &+ \sum_{j=1}^{k-1} (\Gamma \otimes A)^j [(\Gamma \otimes A_0) + (\Gamma \otimes \bar{V}_{t-j})] \Sigma_{t-j-1} \\ &+ \sum_{j=1}^{k-1} (\Gamma \otimes A)^j [(\Gamma_0 \otimes A) + (V_{t-j} \otimes A)] Z_{t-j-1} \\ &+ (\Gamma \otimes A)^k \text{vec}\left(Z_{t-k} \Sigma'_{t-k}\right) \end{aligned} \quad (\text{Pr3.1})$$

Both components of the second line Pr3.1 are stationary by assumption. For  $k \rightarrow \infty$  in Pr3.1, the term  $\sum_{j>1} (\Gamma \otimes A)^j \text{vec}\left((A_0 + \bar{V}_{t-j}) (\Gamma'_0 + V'_{t-j})\right)$  will be absolutely summable provided that  $(\Gamma \otimes A)$  has all its eigen values within the unit circle and

$$E \left( \text{vec}\left((A_0 + \bar{V}_{t-j}) (\Gamma'_0 + V'_{t-j})\right)' \text{vec}\left((A_0 + \bar{V}_{t-j}) (\Gamma'_0 + V'_{t-j})\right) \right) < \infty$$

(see Lutkepohl (1991) proposition C.7, p. 490). Similarly, provided stationarity of  $\Sigma_t$  and  $Z_t$ , the fourth and the fifth lines will be absolutely summable

if  $(\Gamma \otimes A)$  has all its eigen values within the unit circle and  $E(\bar{V}_t' \bar{V}_t) < \infty$ ,  
 $E(V_t' V_t) < \infty$ .

For  $k \rightarrow \infty$  in Pr3.1 and taking expectations we obtain  $\overline{\text{Cov}}(Z_t, \Sigma_t) = \text{vec}(\text{Cov}(Z_t, \Sigma_t))$

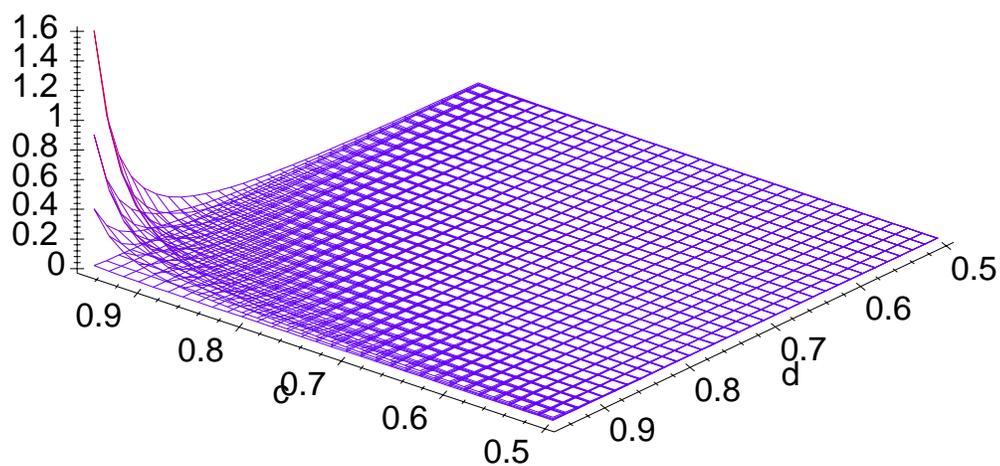
$$\begin{aligned} \overline{\text{Cov}}(Z_t, \Sigma_t) &= \kappa + \Lambda_1 (I - \Gamma)^{-1} \Gamma_0 + \Lambda_2 (I - A)^{-1} A_0 \\ &\quad + [I - (\Gamma \otimes A)]^{-1} (\Gamma \otimes A) \kappa \\ &\quad + [I - (\Gamma \otimes A)]^{-1} (\Gamma \otimes A) \Lambda_1 (I - \Gamma)^{-1} \Gamma_0 \\ &\quad + [I - (\Gamma \otimes A)]^{-1} (\Gamma \otimes A) \Lambda_2 (I - A)^{-1} A_0 \\ &\quad - [(I - \Gamma) \otimes (I - A)]^{-1} \text{vec}(A_0 \Gamma_0') \end{aligned}$$

where  $\kappa = E[\text{vec}((A_0 + \bar{V}_t)(\Gamma_0' + V_t'))]$  and

$$\begin{aligned} \Lambda_1 &= E[(\Gamma \otimes A) + (\Gamma \otimes \bar{V}_t)] \\ \Lambda_2 &= E[(\Gamma_0 \otimes A) + (V_t \otimes A)] \end{aligned}$$

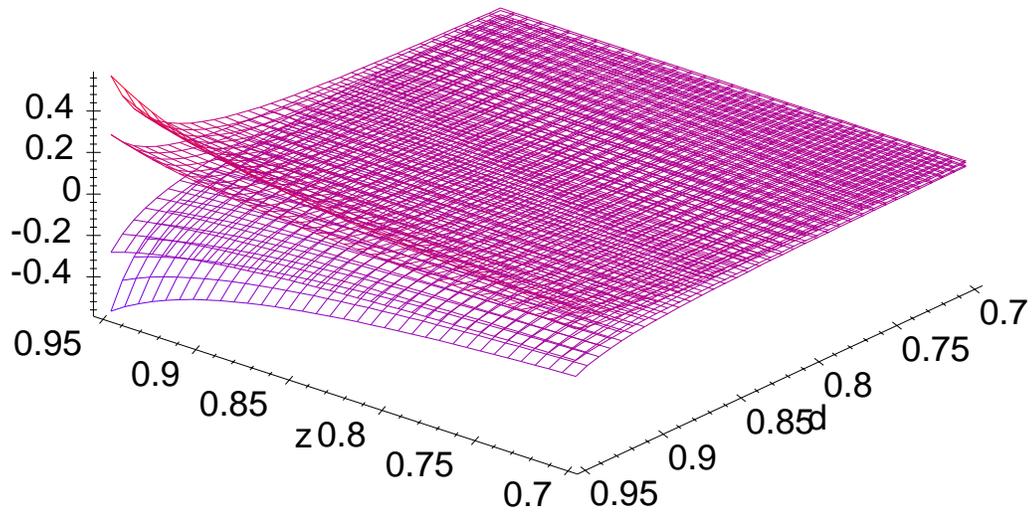
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## 7.4 Theoretical Graphs



Graph 1: Steady State Covariance of two GARCH(1,1) Variances

Note:  $c, d$  are GARCH(1) parameters,  $\rho = .1, .5, .75, 1$ , intercepts and ARCH(1) parameters normalised to 1, .01 respectively



Graph II: Steady State Covariance of CorGARCH(1,1) and GARCH(1,1)

Note:  $z, d$  are CorGARCH(1) and GARCH(1) parameters,  $\rho = -1, -.5, .5, 1$ , intercepts,

ARCH(1) and CorrARCH(1) parameters normalised to 1, .01, .01 respectively

## 7.5 Tables

Table I: DJ - NC - 3m TB Zero Correlations

Parameters (17)	D-J Industrial	Nasdaq	Treasury Bill
<b>MEAN</b>			
intercept	0.9644 (4.100)	1.2951 (11.16)	0.0060 (0.063)
MA(1)	-.1265 (1.674)		0.2368 (3.037)
<b>VARIANCE</b>			
intercept	0.4741 (1.075)	0.7668 (0.920)	7.4351 (220.7)
ARCH(1)	0.0943 (2.027)	0.1863 (2.643)	0.2489 (2.339)
GARCH(1)	0.8808 (15.32)	0.8212 (11.22)	0.6728 (3.513)
Log Likelihood	-1,269.01		

Notes: Trivariate estimation with zero correlations. Robust t-statistics in brackets.

Table II: Dow Jones - Nasdaq - 3m TB Constant Correlations

Parameters (17)	D-J Industrial	Nasdaq	Treasury Bill
<b>MEAN</b>			
intercept	0.9646 (4.325)	1.2242 (3.557)	0.0066 (0.063)
MA(1)	-.1324 (1.783)		0.2375 (3.154)
<b>VARIANCE</b>			
intercept	0.4804 (1.075)	0.7882 (17.19)	7.4400 (4.050)
ARCH(1)	0.0908 (2.141)	0.1863 (3.152)	0.2506 (2.167)
GARCH(1)	0.8838 (16.23)	0.8193 (18.27)	0.6717 (3.241)
<b>CORRELATION</b>			
	DJ - NC	DJ - TB	NC - TB
intercept	0.3437 (4.268)	0.0428 (0.502)	0.0505 (0.654)
Log Likelihood	-1,260.68		
LR $H_0$ : Zero Corr.	16.66		
AIC = $-2 \left( \frac{\ln L}{N} - \frac{k}{N} \right)$			
SIC = $-2 \left( \frac{\ln L}{N} - \frac{k \ln N}{2N} \right)$			

Notes: Trivariate estimation with constant correlations. Robust t-statistics in brackets.  $X^2(3)$  critical value 7.81.

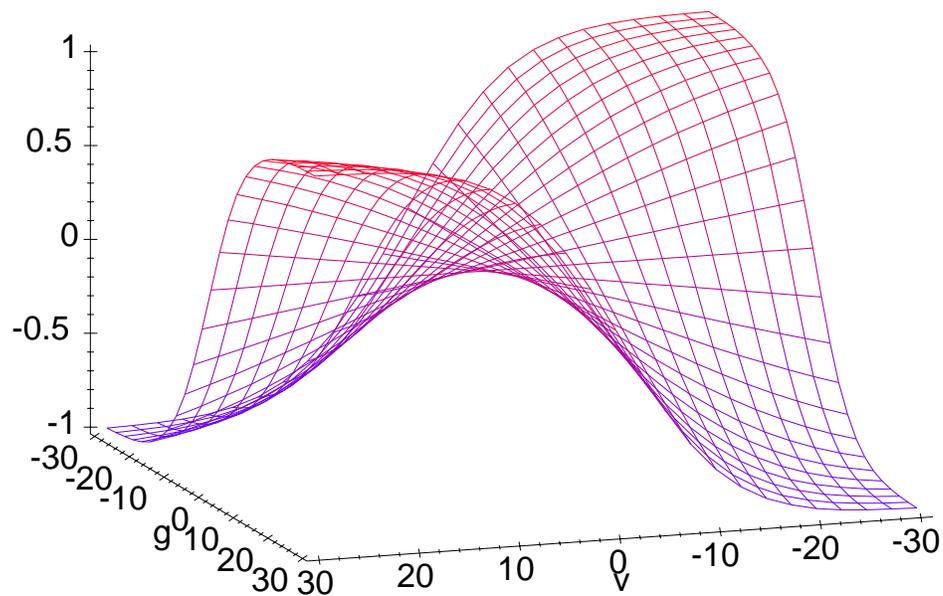
Table III: Dow Jones - Nasdaq - 3m TB Varying Correlations

Parameters (23)	D-J Industrial	Nasdaq	Treasury Bill
<b>MEAN</b>			
intercept	0.9945 (6.563)	1.2795 (20.20)	0.0197 (0.063)
MA(1)	-.1359 (1.821)		0.2374 (3.248)
<b>VARIANCE</b>			
intercept	0.4343 (1.075)	0.7834 (0.972)	7.5700 (266.0)
ARCH(1)	0.1005 (2.135)	0.1900 (2.730)	0.2368 (2.591)
GARCH(1)	0.8798 (16.69)	0.8137 (11.19)	0.6769 (8.204)
<b>CORRELATION</b>			
	DJ - NC	DJ - TB	NC - TB
intercept	0.0455 (1.630)	0.0025 (0.197)	0.0016 (0.043)
CorrARCH(1)	0.0345 (1.963)	0.0306 (0.724)	0.0394 (0.825)
CorGARCH(1)	0.8031 (3.359)	0.8735 (4.867)	0.6129 (16.14)
Log Likelihood	-1,249.59		
LR $H_0$ : Const. Corr.	22.18		
AIC	$-2 \left( \frac{\ln L}{N} - \frac{k}{N} \right)$		
SIC	$-2 \left( \frac{\ln L}{N} - \frac{k \ln N}{2N} \right)$		

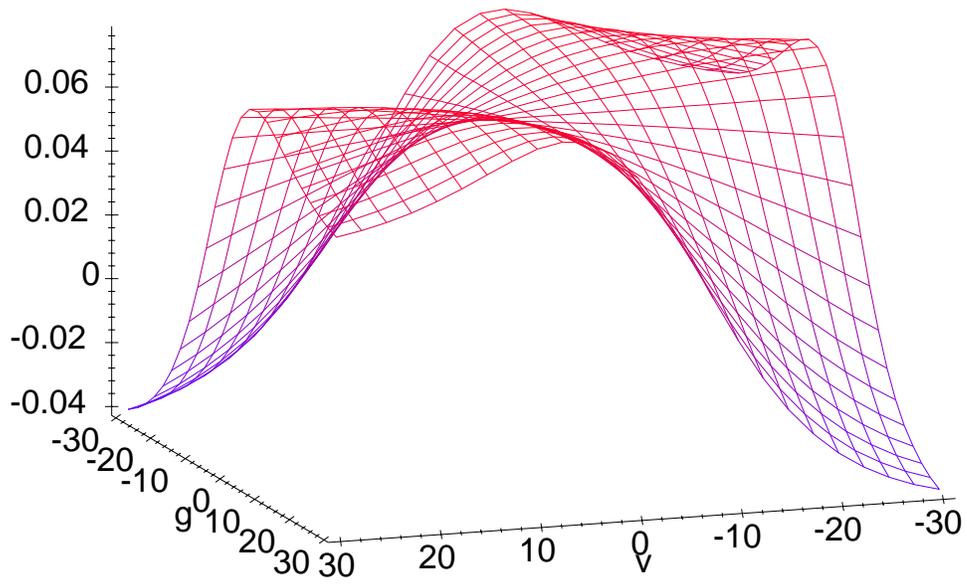
Notes: Trivariate estimation with CorGARCH(1,1) correlation processes.

Robust t-statistics in brackets.  $X^2(6)$  critical value: 12.59.

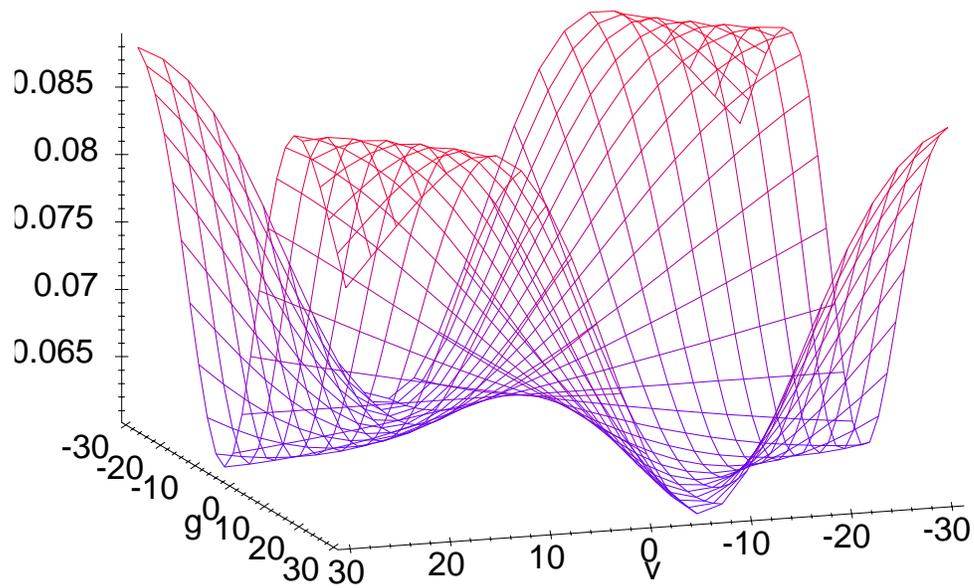
## 7.6 Empirical Graphs



Graph III: News Impact Surface:  $v_{1,t-1}, v_{2,t-1}$  on  $\rho_{12,t}$ ,  
Note:  $v, g$  are the Nasdaq and Easdaq standardized residuals  
in a CorGARCH(1,1)



Graph IV: News Impact Surface:  $v_{1,t-1}, v_{2,t-1}$  on  $\rho_{23,t}$ ,



Graph V: News Impact Surface:  $v_{1,t-1}, v_{3,t-1}$  on  $\rho_{23,t}$ ,

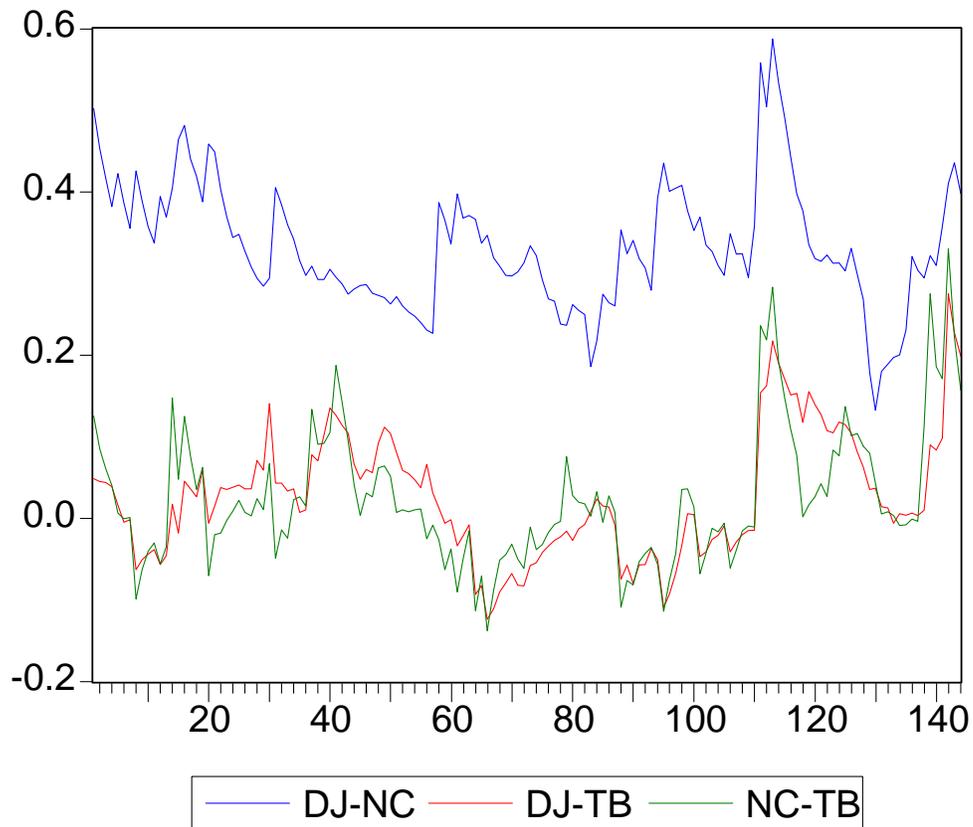


Figure 1: CorGARCH(1,1) dynamic correlations between Dow Jones Industrial (DJ), Nasdaq Composite (NC) and 3m US T Bill (TB), 1989-2001, monthly data

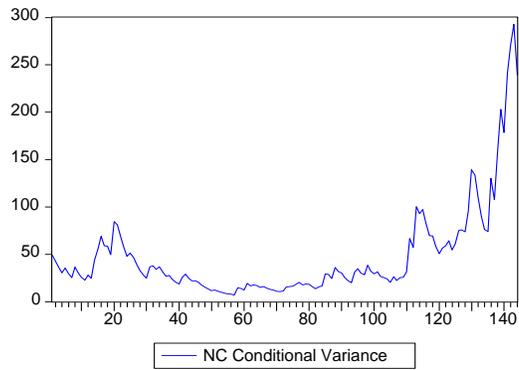
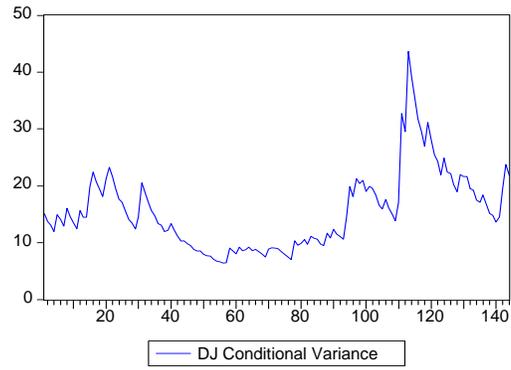
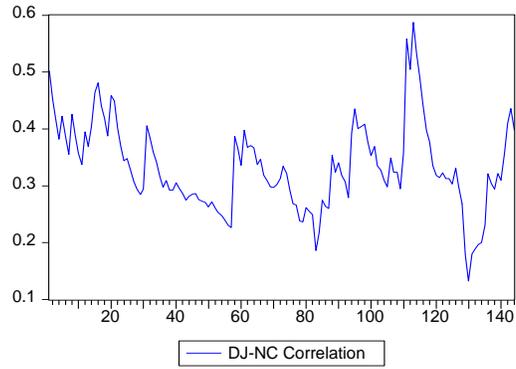


Figure 2: Dow Jones - Nasdaq Composite Correlation and Conditional Variances, 1989 - 2001, monthly data

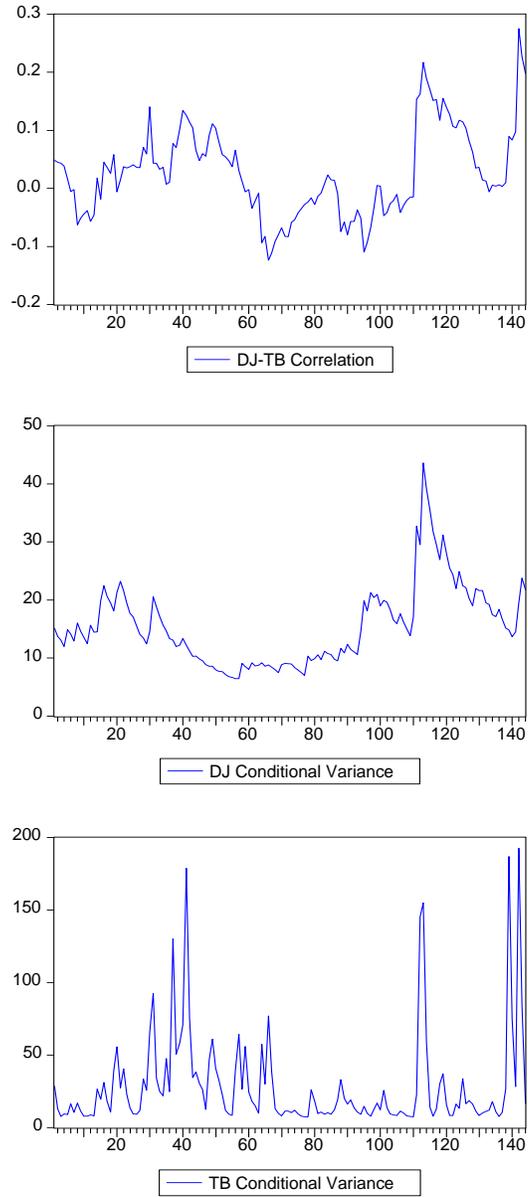


Figure 3: Dow Jones - Treasury Bill Correlation and Conditional Variances, 1989 - 2001, monthly data

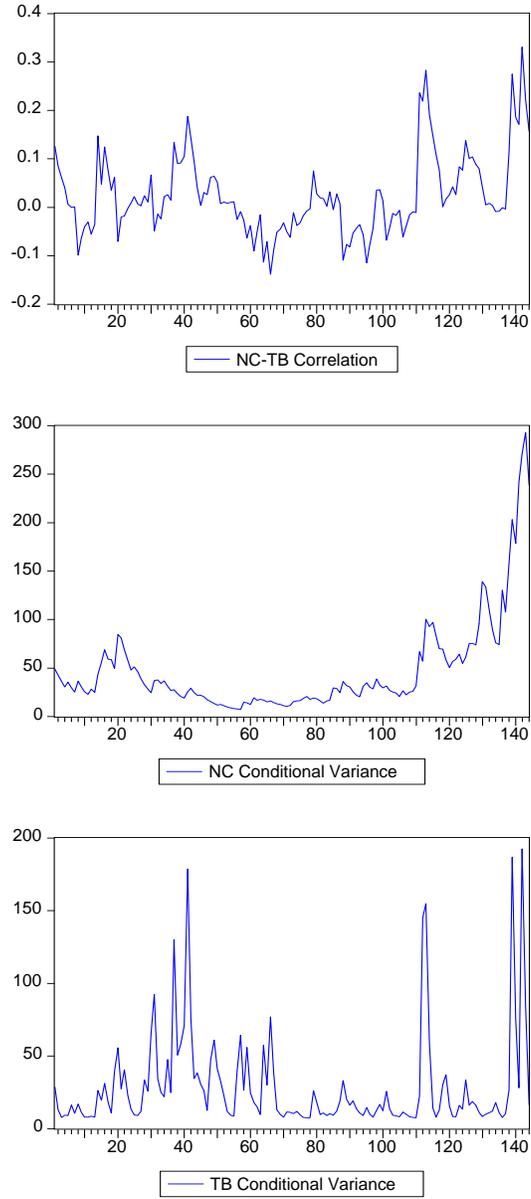


Figure 4: Nasdaq Composite - Treasury Bill Correlation and Conditional Variances, 1989 - 2001, monthly data

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