Equilibrium Price Processes, Mean Reversion, and Consumption Smoothing

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ABSTRACT

Motivated by the empirical observation that there exists some degree of mean reversion in asset prices, this paper investigates the time-varying behaviour of the price of risk in a partial equilibrium framework. The paper characterizes the equilibrium conditions under which the asset price processes must satisfy and obtains closed-form solutions in a time homogeneous economy. We construct a model where the consumption is relative smooth and risk premium falls as wealth rises. The representative agent also becomes less risk averse as wealth rises.

Much of recent empirical research provides evidence that stock returns are serially correlated (see Poterba and Summers (1988) and Fama and French (1988a), Fama and French (1988b)) and often points to the rejection of the random walk hypothesis. It has also been found that returns over long horizons are likely to be negatively autocorrelated. This property is usually termed as mean reversion, possibly one of the most contentious phenomena in the stock market. The economic intuition of mean reversion means that the expected returns declines as the wealth (or the value of the market portfolio) increases. However, since expected returns are not observable, it remains debatable whether mean reversion is attributed to market inefficiency, or it should be consistent with an equilibrium asset pricing model.

From a theoretical viewpoint, mean reversion may help explain two long standing puzzles in finance: the “consumption smoothing puzzle” and the “equity premium puzzle”. The two puzzles are related to one another. The “consumption smoothing puzzle” refers to the finding that consumption is much smoother relative to income and prices (see, for example, Hansen and Singleton (1983)). The “equity premium puzzle”, first branded by Mehra and Prescott (1985), refers to the high risk premium in the stock market which, together with smooth consumption and low real interest rates, is unable to be explained by a conventional model with additive separable utility.
Many researchers have strived to resolve these puzzles by introducing some form of “imperfections” such as habit formation, non-time-additive utility, liquidity constraints, information asymmetry and so forth (see, for example, Sundaresan (1989), Campbell and Cochrane (1999), Constantinides (1990), Constantinides, Donaldson, and Mehra (2002), Zeldes (1989) and Zhou (1999)). Although these models do seem to offer promising results, one is still left to query whether mean reversion can possibly be explained in a relatively simple equilibrium asset pricing framework. To put it more explicitly, can we explain the puzzles in a conventional model without turning to “imperfections”. Cecchetti, Lam, and Mark (1990) show that when economic agents are concerned about smoothing their consumption, mean reverting stock prices could be consistent with an equilibrium model calibrated by Porterba and Summers’ variance ratio statistics and Fama and French’s long-horizon return regression coefficients.

In a conventional continuous time setting, the representative agent’s total wealth is the value of the market portfolio and the risk premium of the market portfolio is expressed as a product of squared volatility and instantaneous relative risk aversion. Therefore, a time-varying risk premium can be produced by allowing a time-varying volatility and/or a time-varying risk aversion. Traditionally, it is more common to assume constant relative risk aversion in financial modeling. Black (1990) engineers a simple equilibrium model that allows mean reversion by separating the measures of direct and derived relative risk aversion. Both risk aversion are constant and the volatility varies depend on wealth and time.

In this paper, we also demonstrate that mean reversion can be consistent with a simple equilibrium asset pricing model without features of “imperfections”. In contrast to the usual assumption of constant relative risk aversion, the main idea of this investigation is to allow risk aversion to vary over time with the state of the economy as time-varying risk aversion has strong implications in stock return predictability. Time-varying risk aversion is also an important feature produced by some “imperfection” models.
Our modeling approach is to first characterize equilibrium asset price processes by following the framework employed by Bick (1990), He and Leland (1993), and Hodges and Carverhill (1993), and then seek for analytic solutions in which the asset price processes satisfy the equilibrium conditions and display the mean reversion property. This approach is closely related to those of Hodges and Carverhill (1993) and Hodges and Selby (1997) but differs in that we allow the presence of intermediate consumption. As we will show later in the paper, this feature adds significant flexibility and enables us to obtain the mean reversion result that is not possible when economic agents consume only at the horizon date.

The remainder of the paper is organized as follows. We set up our basic model in section I, and give complete characterization of equilibrium conditions for the asset price dynamics in section II. In section III, the special case of a time-homogeneous economy is studied and explicit solutions are obtained given the assumption of constant real interest rate and volatility of market portfolio. Realistic parameterization is then used to demonstrate the main results of the model:

1. Risk premium falls as wealth rises
2. Investors become less risk averse as wealth rises
3. Consumption varies less than wealth

Section IV finally concludes the paper.

I. The Formulation

We consider a continuous-time, finite horizon, pure exchange economy of Lucas (1978). The financial markets are assumed to be complete and we assume that the economy can be described by a single representative agent. The agent trades and acts as an expected utility maximizer and in equilibrium will optimally hold the market portfolio (representing the aggregate wealth of the whole economy) through time until the horizon date \( T < \text{inf} \). There are
two long-lived financial securities available for trading: a risky asset (the *stock*), and a locally riskless asset (the *bond*). At time \( t \), the *trading price* of the stock is denoted by \( S_t \) and the holder of the stock is entitled to its dividends, if the stock is dividend-paying. The bond price is denoted by \( R_t \), and increases at the instantaneous riskless rate of interest \( r_t \). In equilibrium, there is one share of the stock outstanding and the bond is in zero net supply.

In the rest of this section, we shall describe two distinct economic settings. First, we consider an economy where the representative agent is concerned with her terminal wealth only. Second, we consider an economy where the representative agent is concerned not only with the terminal wealth but also with intermediate consumption. We will refer to the former case as the *one-consumption* economy, and the latter one as the *continuous-consumption* economy. Finally, we discuss two important properties that arise from the first-order conditions and explain the significance of the path-independence result which leads us to the fundamental PDEs as the necessary conditions for equilibrium.

**A. Without Intermediate Consumption**

Consider a simple economy in which the only consumption good is produced exogenously and can only be consumed by individuals at time \( T \), when all economic activity will end. There are two infinitely divisible securities that are continuously and frictionlessly traded in the market: a risky asset (the stock) that entitles the holder to the ownership of the consumption good at time \( T \), and a riskless asset (the bond) that pays one unit of consumption at time \( T \). The stock (the market portfolio) pays no dividends and its price \( S \) follows a diffusion process

\[
\frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma(S_t, t)dz_t, \tag{1}
\]

where \( \mu \) and \( \sigma \) are functions of the stock price and time.

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where $z$ is the standard Brownian motion under the objective probability measure $\mathbb{P}$ and the drift $\mu$ and the diffusion $\sigma$ are deterministic functions of $S$ and $t$. Denote by $\alpha$ the price of risk that represents the instantaneous reward per unit of risk, i.e.

$$
\alpha(S_t, t) = \frac{\mu(S_t, t) - r(S_t, t)}{\sigma(S_t, t)},
$$

(2)

where $r$ is the instantaneous riskless interest rate and is assumed to be a deterministic function of $S_t$ and $t$ as well. Hence, we can rewrite (1) as

$$
\frac{dS_t}{S_t} = \left[ r(S_t, t) + \sigma(S_t, t) \cdot \alpha(S_t, t) \right] dt + \sigma(S_t, t) dz_t.
$$

(3)

Moreover, the bond price $B$ accumulates at the riskless rate $r$:

$$
\frac{dB_t}{B_t} = r_t dt.
$$

(4)

Assume that the agent is endowed with a positive initial amount but receives no intermediate income. In addition, consumption only occurs at the agent’s investment horizon date $T$, the same as that of the economy. She then aims to allocate her wealth optimally in the stock and bond in order to maximize her expected utility of the time-$T$ wealth, $W_T$:

$$
\max \ E[U(W_T)],
$$

(5)

where $U$ is a strictly increasing, state-independent, and continuously differentiable von Neumann-Morgenstern utility function.

Denote by $\Phi$ the amount of money invested in the stock. Then the wealth function $W$ follows the process:

$$
\frac{dW_t}{W_t} = \left[ \Phi_t (\mu_t - r_t) \right] dt + \Phi_t \sigma_t dz_t,
$$

(6)
with \( W_0 = w > 0 \) (positive initial wealth) and \( W_t \geq 0 \) (nonnegative wealth constraint), for \( 0 < t \leq T \).

From Harrison and Kreps (1979), it is well known that in any arbitrage-free pricing system there exists a risk-neutral probability measure under which the drift of the stock returns is the riskless rate \( r \). Hence we can let \( Q \) denote the risk-neutral probability measure and \( \mathcal{M} = dQ/dP \) denote the change of measure (i.e. the Radon-Nykodym derivative) from \( P \) to \( Q \). By the Girsanov’s Theorem, the state-price density (SPD) can be defined as

\[
\xi_t = \exp \left( -\int_0^t r_s \, ds \right) \cdot \mathcal{M}_t,
\]

or more specifically,

\[
\xi_t = \exp \left( -\int_0^t r_s \, ds - \int_0^t \alpha_s \, dz_s - \frac{1}{2} \int_0^t \alpha_s^2 \, ds \right), \quad \text{(7)}
\]

where the price of risk \( \alpha = (\mu - r)/\sigma \) satisfies the usual Novikov’s condition.

Since in equilibrium the representative agent will be not to trade at all and should hold the market portfolio, the agent’s marginal utility can be related to the SPD through the first order condition

\[
\frac{\partial U(S_T)}{\partial S_T} = \lambda \cdot \xi_T, \quad \text{(8)}
\]

where \( \lambda \) is the Lagrange multiplier.

**B. With Intermediate Consumption**

If alternatively we allow the presence of intermediate consumption and assume the stock pays dividends, the trading price of the stock \( S \) can be formulated as:

\[
dS_t = [\mu(S_t,t)S_t - D(S_t,t)] \, dt + \sigma(S_t,t)S_t \, dz_t, \quad \text{(9)}
\]
where \( z \) is the standard Brownian motion under the \( \mathbb{P} \)-measure and \( D \) is the cash dividend paid by the stock and is assumed to be a deterministic function of \( S \) and \( t \). Similarly, (9) can be rewritten as

\[
dS_t = \left[ (r(S_t, t) + \sigma(S_t, t) \cdot \alpha(S_t, t)) S_t - D(S_t, t) \right] dt + \sigma(S_t, t) S_t dz_t, \tag{10}
\]

where \( r \), \( \sigma \), and \( \alpha \) are as defined before.

The agent’s maximization problem is now formulated as follows:

\[
\max \, E_0 \left[ \int_0^T e^{-\rho t} U_1(C_t) dt + U_2(W_T) \right], \tag{11}
\]

where \( \rho \) is the rate of time preference and \( U_1 \) and \( U_2 \) are strictly increasing, time additive, and state-independent von Neumann-Morgenstern utility functions, and are continuously differentiable where applicable.

Again let \( \Phi_t \) denote the amount of money invested in the stock at the beginning of time-\( t \) period and \( C_t \) the amount being consumed during time \( t \)-period. Thus the investor’s indirect utility (or value function) can be defined as:

\[
J(W_t, S_t, t) = \max_{\Phi_t, C_t} E_t \left[ \int_t^T e^{-\rho(s-t)} U_1(C_s) ds + U_2(W_T) \right], \tag{12}
\]

where \( J(W_T, S_T, T) = U_2(W_T) \) is the boundary condition. The wealth process follows:

\[
dW_t = \left[ W_t r_t + \Phi_t (\mu_t - r_t) - \left( \frac{\Phi_t}{S_t} \right) C_t \right] dt + \Phi_t \sigma_t dz_t, \tag{13}
\]

where \( W_0 = w > 0 \) and \( W_t \geq 0 \) for \( 0 < t \leq T \). As is standard in the literature, the first order condition of optimality for the Hamilton-Jacobi-Bellman (HJB) equation yields:

\[
e^{-\rho t} \frac{\partial U_1(C_t)}{\partial C_t} = \frac{\partial J(W_t, S_t, t)}{\partial W_t} = \gamma \cdot \xi_t, \tag{14}
\]
where $\gamma$ is the Lagrange multiplier.

Recall that in equilibrium the representative agent’s optimal strategy is to hold the market portfolio and consume all the dividends received from the stock investment. Thus, in equilibrium $W_t = S_t$ and $C_t = D_t$ for $t < T$.

C. Monotonicity Property and the Path Independence Result

It is now well known through the work of Cox and Leland (2000) that an efficient trading strategy must be path-independent and generate a payoff that is monotonically increasing in the market level, at least in our construction of economy. These properties must also apply to the market portfolio itself, if we assume that the market is always efficient.

Let us briefly recall these results. By inspection of the first-order conditions (8) and (14), we first note that with the assumption of increasing utility functions, wealth should be monotonically and inversely related to the marginal utility (or the SPD). This property applies to both cases as it can be seen that from (8), $W_T = V(\lambda \cdot \xi_T)$, where $V$ is the inverse function of the marginal utility $U'$, and from (14), $W_t = I(\gamma \cdot \xi_t)$, where $I$ is the inverse function of the marginal indirect utility $J'$. Thus a portfolio strategy which creates state-dependent wealth will be called an efficient strategy only if the monotonicity property is satisfied (see Dybvig (1988a), Dybvig (1988b)).

In the context of equilibrium, since the stock price, $S_t$, represents the agent’s wealth and the monotonicity property should apply, the process of the SPD, $\xi_t$, must be path-independent at each point in time $t \in [0, T]$, regardless the stock price history. It is this very result that paves the way for us to analyze the equilibrium asset price dynamics in the economy.\(^1\)
II. Equilibrium Conditions of the Asset Prices Dynamics

In this section, we provide the characterization of equilibrium price processes for both the one-consumption economy and the continuous-consumption economy. In each case, we first derive a general partial differential equation for the intertemporal relative risk aversion, \( f \), with respect to \( S \). For convenience, we then choose to work on a change of variable \( x \) which is a martingale under the risk neutral probability measure. The assumption of the constancy of \( r \) and \( \sigma \) further enables us to simplify the PDEs in terms of the price of risk \( \alpha \) with respect to the new state variable \( x \). For ease of exposition, we will omit the time reference for the variables and the subscripts refer to the derivatives unless otherwise state.

A. Without Intermediate Consumption

Theorem 1 (Equilibrium conditions: without consumption) Assume that in the economy, there exists one non-dividend-paying risky asset (the stock) and one riskless asset (the money account or the bond). The representative agent continuously allocates her wealth among these two assets according to her objective function (5) subject to the wealth process (6) and then consumes her terminal wealth at time \( T \). The necessary condition for the asset price dynamics (1) to be an equilibrium process when \( r, \mu \) and \( \sigma \) are deterministic functions of \( S \) and \( t \) is that the coefficients must satisfy the following PDE:

\[
\mathcal{L}f + f_t + rS(f - 1) + \sigma \sigma_S S (f_S + f^2 - f) = 0, \tag{15}
\]

where

\[
f(S,t) = \frac{\mu(S,t) - r(S,t)}{(\sigma(S,t))^2},
\]

\[
\mathcal{L}f = \frac{1}{2} \sigma^2 S^2 f_{SS} + \mu S f_S,
\]
and the boundary condition is
\[ f(S, T) = -S \frac{U''(S)}{U'(S)}. \]

**Proof:** The main idea is to exploit the path-independence property on the state-price density. Given the process of \( \xi \) from (7), we now define a new variable \( Z(S,t) = \ln \xi(S,t) \). We then apply It\( \hat{o} \)'s Lemma to derive \( dZ \) and equate it with \( d(\ln \xi) \). Collecting \( dt \) and \( dz \) terms respectively yields the following equations:

\[
\begin{align*}
Z_t + \mu SZ_S + \frac{1}{2} \sigma^2 S^2 Z_{SS} & = -r - \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 = -r - \frac{1}{2} \sigma^2 f^2, \quad (16) \\
\sigma S Z_S & = -\left( \frac{\mu - r}{\sigma} \right) = -\sigma f. \quad (17)
\end{align*}
\]

Note that for notational simplicity, we have suppressed the time index so that it will not be confused with the partial derivatives.

From equation (17), we can derive \( Z_{St} = -\frac{f_t}{S} \), \( Z_S = -\frac{f}{S} \) and \( Z_{SS} = -\frac{f_t S}{S^2} \). Substitute \( Z_S \) and \( Z_{SS} \) into the first equation and use \( \mu = r + \sigma^2 f \) to obtain

\[
Z_t = -r + rf + \frac{1}{2} \sigma^2 f^2 + \frac{1}{2} \sigma^2 S f_S - \frac{1}{2} \sigma^2 f.
\]

Differentiate the above equation with respect to \( S \) and equate it with \( Z_{St} \) to obtain

\[
Z_{tS} = r_S (f - 1) + (r + \sigma^2 f) f_S + \frac{1}{2} \sigma^2 S f_{SS} + \sigma \sigma_S (S f_S + f^2 - f) = Z_{St} = -\frac{f_t}{S}.
\]

The resulting PDE (15) then immediately follows. Q.E.D.

Theorem 1 states the equilibrium conditions which the intertemporal relative risk aversion \( f \) must satisfy. We now discuss some special cases. The first case is a Black-Scholes economy
where both the interest rate $r$ and the volatility of the stock returns $\sigma$ are constant. By letting $\sigma_S = 0$ and $r_S = 0$, (15) can be simplified as

\[ \mathcal{L}f + f_t = 0. \]  

(18)

The strong assumption of constancy of $r$ and $\sigma$ enables us to obtain a nice result known as the Burgers’ equation. It seems that the finance application of this equation first appeared in Hodges and Carverhill (1993) and in an independent work of He and Leland (1993). For completeness, we recall the result in the following theorem but give the proof in a slightly different way.

**Theorem 2 (Burgers’ equation)** Assume constant $r$ and $\sigma$ and define the transformed state variable $x$ as $x = \ln S - (r - \sigma^2/2)t$. The price of risk $\alpha$ in the wealth-only economy must evolve over time according to the PDE:

\[ \alpha_t = \frac{1}{2} \sigma^2 \alpha_{xx} + \sigma \alpha \alpha_x \]  

(19)

where $\tau = T - t$.

**Proof:** By definition, we have $\mu - r = \sigma^2 f = \sigma \alpha$ and $x = \ln S - \left(r - \frac{\sigma^2}{2}\right)t$. Thus, we can write

\[ \alpha(S, t) \equiv \alpha(x, t) = \sigma \cdot f \left( e^{x+\left(r - \frac{\sigma^2}{2}\right)t}, t \right) \]

and its partial derivatives

\[ \alpha_x = \sigma f_S, \]
\[ \alpha_{xx} = \sigma (f_S + S f_{SS}) S, \]
\[ \alpha_t = \sigma f_S \left( r - \frac{\sigma^2}{2} \right) S + \sigma f. \]
Rearranging the above equations to obtain \( f_S, f_{SS} \) and \( f_t \) and substituting them into (18) yields

\[
\alpha_t + \sigma \alpha x + \frac{1}{2} \sigma^2 \alpha x = 0.
\]

This immediately gives (19). Q.E.D.

Another interesting case is to assume that \( f \) is time-invariant in the sense that \( \mu, r, \) and \( \sigma \) are functions of \( S \) only. We then obtain the following proposition:

**Proposition 1** When \( \mu, r, \) and \( \sigma \) are functions of \( S \) only, the equilibrium condition for the economy stated in Theorem 1 is that the following equation must be satisfied:

\[
\frac{\partial}{\partial S} \left[ \sigma^2 (S f_S + f^2 - f) + 2r(f - 1) \right] = 0.
\]

(20)

Or equivalently, there exists a constant \( K \) such that

\[
\sigma^2 (S f_S + f^2 - f) + 2r(f - 1) = K.
\]

**Proof:** By assumption, \( \mu = r + \sigma^2 f \) and \( f_t = 0 \). After some simple manipulation, (20) can then easily be derived from (15):

\[
0 = \mathcal{L} f + f_t + r_S (f - 1) + \sigma \sigma_S S (S f_S + f^2 - f) + \frac{1}{2} \sigma^2 S^2 f_{SS} + (r + \sigma^2 f) S f_S + S \frac{\partial}{\partial S} [r(f - 1)] - r S f_S
\]

\[
+ \frac{1}{2} S (S f_S + f^2 - f) \frac{\partial^2 \sigma^2}{\partial S} + S \frac{\partial}{\partial S} (S f_S + f^2 - f) \frac{\partial \sigma}{\partial S}
\]

\[
= \frac{1}{2} S \sigma^2 \frac{\partial}{\partial S} (S f_S + f^2 - f) + S \frac{\partial}{\partial S} [r(f - 1)] + \frac{1}{2} S (S f_S + f^2 - f) \frac{\partial^2 \sigma^2}{\partial S}
\]

\[
= \frac{1}{2} S \frac{\partial}{\partial S} \left[ \sigma^2 (S f_S + f^2 - f) + 2r(f - 1) \right]. \quad \text{Q.E.D.}
\]
In a same setting as ours, except assuming constant interest rate, He and Leland (1993) also derive a similar necessary condition for the time-invariant case, namely, \( \sigma^2(f^2 + Sf_S - f) = K \). Therefore, it should be pointed out that their condition holds only when \( r = 0 \).

Without further assumptions, the PDE (15) is in general difficult to solve. To this day, several functional form solutions to the time-invariant case with constant \( \sigma \) and \( r = 0 \) can be found as examples in Bick (1990) and in He and Leland (1993). For a more general definition on time independence of the diffusion processes, Hodges and Selby (1997) carried out a time-homogeneous analysis for the case with constant volatility and constant interest rate. They seek to find steady-state solutions to the Burgers’ equation (19) by constraining the risk premium to vary depending on the level of the market in such a way that the functional form does not depend on time. Interestingly, they conclude that there are only two possible viable solutions and one non-viable one for the steady state: the price of risk can be constant or increasing in aggregate wealth, but the only steady state solution with decreasing price of risk admits arbitrage (and is not viable).

The finding of an increasing price of risk is somewhat disappointing as it would be nice if we can produce a mean-reverting behaviour in such a simple model. Nevertheless, it is conceivable that the introduction of intermediate consumption might be sufficient to modify this behaviour. As we shall illustrate in the next section, it is indeed the case: with large enough intermediate consumption, a decreasing price of risk can be obtained in the steady state from a decreasing relative risk aversion of the representative agent.

**B. With Intermediate Consumption**

In light of the limitations described above, we now extend our investigation to characterize the equilibrium conditions for the economy with intermediate consumption. Since it is more convenient to work with the dividend yield rather than the cash dividend, we denote by \( \delta \) the dividend yield to mean \( \delta(S,t) = D(S,t)/S \). The next two theorems generalize on equa-
tions (15) and (19) to include dividends. The approach is similar to that used in the single consumption case.

**Theorem 3 (Equilibrium conditions: with consumption)** Assume that in the economy, there exists one dividend-paying stock and one riskless bond. The representative agent continuously allocates her wealth among the two assets according to her objective function (11) subject to the wealth process (13) and consumes the dividends paid by the stock investment. The necessary condition for the asset price dynamics (9) to be an equilibrium process when \( r, \mu, \delta \) and \( \sigma \) are deterministic functions of \( S \) and \( t \) is that the coefficients must satisfy the following PDE:

\[
\mathcal{L} f + f_t - \delta S f + r S (f - 1) + \sigma \sigma S (S f_S + f^2 - f) = 0,
\]

where

\[
f(S,t) = \frac{\mu(S,t) - r(S,t)}{(\sigma(S,t))^2} = -\frac{U''(D(S,t))SD_S(S,t)}{U'(D(S,t))},
\]

\[
\mathcal{L} f = \frac{1}{2} \sigma^2 (S,t) S^2 f_{SS}(S,t) + (\mu(S,t) - \delta(S,t)) S f_S(S,t),
\]

and the boundary condition is given by

\[
f(S,T) = -S \frac{U''(S)}{U'_2(S)}.
\]

**Proof:** This is simply a rederivation of Theorem 1 with the presence of intermediate consumption (dividends). Again define a new variable \( Z(S,t) = \ln \xi(S,t) \). We apply Ito’s Lemma to derive \( dZ(S,t) \) and equate it with \( d(\ln \xi) \). Collecting \( dt \) and \( dz \) terms respectively yields

\[
Z_t + (\mu S - D) Z_S + \frac{1}{2} \sigma^2 S^2 Z_{SS} = -r - \frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 = -r - \frac{1}{2} \sigma^2 f^2,
\]

\[
\sigma S Z_S = -\left( \frac{\mu - r}{\sigma} \right) = -\sigma f.
\]
Following the same technique in the proof of Theorem 1 and applying $\delta = D/S$, (21) can then easily be obtained. Q.E.D.

Theorem 3 provides the equilibrium conditions for the asset price processes to satisfy in an economy with intermediate consumption. A casual inspection of the PDE (21) shows that it is not an easy task to obtain an explicit solution. However, if we assume the constancy of $r$ and $\sigma$, the problem can greatly be simplified. The following theorem provides an equation which is analogous to the Burgers’ equation in Theorem 2 but with some additional terms.

**Theorem 4** Assume $r$ and $\sigma$ are constant and define the transformed state variable $x$ as $x = \ln S - (r - \sigma^2 t)/2$. Then the price of risk $\alpha$ in the consumption economy must evolve over time according to the PDE:

$$\alpha_t = \frac{1}{2} \sigma^2 \alpha_{xx} + \sigma \alpha_x - \delta \alpha_x - \delta_x \alpha,$$

where $\tau = T - t$.

**Proof:** Apply $rS = 0$ and $\sigma S = 0$ to (21) and rearrange to obtain

$$f_t + \frac{1}{2} \sigma^2 S^2 f_{SS} + \mu S f_S - \delta S f_S - \delta_S S f = 0.$$

By definition, $\sigma f = \alpha$ and $x = \ln S - (r - \sigma^2 t)/2$. Thus, (22) immediately follows by substituting $\sigma S f_S = \alpha_x$, $\delta_S S = \delta_x$ and

$$f_t + \frac{1}{2} \sigma^2 S^2 f_{SS} + \mu S f_S = \frac{1}{\sigma} \left( \alpha_t + \sigma \alpha_x + \frac{1}{2} \sigma^2 \alpha_{xx} \right)$$

into (23). Q.E.D.
III. The Time-Homogeneous Case in a Black-Scholes Economy

To shed more light on how the price of risk must behave given the constraints on the equilibrium price processes, we study an important special case of the time homogeneous economy. Our objective is to find the steady-state solutions to the equilibrium conditions given in the previous section. It is more sensible that the price of risk and the dividend (consumption) should vary with the market level in such a way that the functional forms are independent of time. In what follows, we will first solve the PDE (22), and then we will give a simple numerical example to illustrate our results.

Mathematically, by homogeneity in time we mean that the price of risk, $\alpha$, and the dividend yield, $\delta$, can be specified as functions of the change of variable, $u$, where $u$ is a function of both $x$ and $\tau$. That is, the two functions are represented as:

$$\alpha(x, \tau) \equiv y(u)$$  \hspace{1cm} (24)

and

$$\delta(x, \tau) x \equiv g(u),$$  \hspace{1cm} (25)

where $u = x + \theta \tau$, for some functions $y$ and $g$ and some constant $\theta$. For our modeling purpose, both $y$ and $g$ are assumed to be positive.

Applying the above specification, the partial derivatives in (22) can be expressed by $\alpha_x = \theta y'$, $\alpha_x = y'$, $\alpha_{xx} = y''$ and $\delta_x = g'$. It follows that the PDE (22) can be reduced to an ODE:

$$-\theta y' + \sigma y' + \frac{1}{2} \sigma^2 y'' - g' y - g' y = 0.$$  \hspace{1cm} (26)

A casual inspection tells that the above equation involves the joint behaviour of $y$ and $g$. Therefore, it can not be solved without further knowledge of the relationship between the two
variables. We can, of course, view (26) as an ODE in $y$ given a $g$ function, but then the question will be how we should choose $g$. Since $g$ could be a function of a rather arbitrary form, a more plausible question might be to ask whether there are cases which permit closed-form solutions.

One way to achieve this is to restrict the dividend yield to be affine with respect to the risk premium. Since $y$ is the price of risk, this assumption amounts to

$$g(u) = p_0 + p_1 \sigma y(u),$$

(27)

for some constants $p_0$ and $p_1$.

Substituting (27) for $g$ into (26), (26) can be reduced to

$$-(\theta + p_0)y' + (1 - 2p_1)\sigma y'y' + \frac{1}{2} \sigma^2 y'' = 0.$$

(28)

At this point, it is worth noting that the linear form assumed in (27) is in the spirit of a first order Taylor Series approximation. Given that both variables $g$ and $y$ are likely to have limited ranges, and a monotonic relationship with each other (through their relation to the market level), we expect this to be a somewhat satisfactory representation. Nevertheless, in principle we have selected a small subclass of the infinitely many possible dividend yield functions in order to obtain an analytic solution.

To solve $y$, we first integrate (28) once to obtain

$$\frac{1}{2} \sigma^2 y' = (\theta + p_0)y - \frac{1}{2}(1 - 2p_1)\sigma y^2 + \text{constant}.$$  

(29)

The solution to (29) depends on conditions imposed on the free parameters, $\theta$, $p_0$, and $p_1$. For ease of exposition, it is natural to discuss three cases as follows:
Case 1 \((p_0 = -\theta\text{ and } p_1 = 1/2)\) In this case, (29) reduces to

\[
\frac{1}{2} \sigma^2 y' = \text{constant.} \tag{30}
\]

Rearrange to yield

\[
\frac{1}{2} \sigma^2 \int \frac{dy}{\text{constant}} = \int du, \tag{31}
\]

and then integrate it to obtain the linear form solution:

\[
y(u) = k_1 u + k_2, \tag{32}
\]

where \(k_1\) and \(k_2\) are constants. Provided \(k_2 > 0\), \(y\) can be constant if \(k_1 = 0\), or decreasing (increasing) if \(k_1 < 0\) (\(k_1 > 0\)).

A limitation of this linear solution is that except for the constant case (i.e. \(k_1 = 0\)), \(y\) is unbounded above and below. Thus, unless we impose additional constraint for \(x\) to stay within a certain range, the risk premium will go negative when the market level is large or small (depending on the sign of \(k_1\)).

Case 2 \((p_0 \neq -\theta\text{ and } p_1 = 1/2)\) In this case, (29) reduces to

\[
\frac{1}{2} \sigma^2 y' = (\theta + p_0)y + \text{constant.} \tag{33}
\]

Rearrange to yield

\[
\frac{\sigma^2}{2(\theta + p_0)} \int \frac{dy}{y + \text{constant}} = \int du, \tag{34}
\]

and then integrate it to obtain

\[
\frac{\sigma^2}{2(\theta + p_0)} \ln(y + c_1) = u + c_2, \quad \text{for } y \in (-c_1, +\infty), \tag{35}
\]

where \(c_1\) and \(c_2\) are constants.
It follows that $y$ has an exponential affine solution:

$$y(u) = e^{a + bu} + c,$$  \hfill (36)

where

$$b = \frac{2(\theta + p_0)}{\sigma^2},$$  \hfill (37)

and $a$ and $c$ are constants. Our assumption of a nonnegative price of risk requires $c_1 \leq 0$ and $c \geq 0$. Finally, by differentiating $y$ with respect to $x$, it shows that $y$ is decreasing (increasing) in $x$ if $p_0 < -\theta$ ($p_0 > -\theta$).

The exponential affine solution is slightly better than the linear one in that it ensures the risk premium will stay positive. However, the fact that it is unbounded above is still undesirable.

**Case 3** ($p_1 \neq 1/2$) In this case, we first rearrange (29) to obtain

$$\frac{\sigma}{1 - 2p_1} \int \frac{dy}{d + \frac{2(\theta + p_0)}{\sigma(1 - 2p_1)}y - y^2} = \int du,$$  \hfill (38)

where $d$ is a constant. We then integrate it to yield

$$\frac{\sigma}{2(1 - 2p_1)c_2} \ln \left| \frac{y + c_2 - c_1}{y - c_2 - c_1} \right| = u + \text{constant},$$  \hfill (39)

where

$$c_1 = \frac{\theta + p_0}{\sigma(1 - 2p_1)}, \quad \text{and} \quad c_2 = \pm \sqrt{d + c_1^2} \neq 0.$$  \hfill (40)

Thus, provided $c_2$ is a nonzero real number, we obtain two alternative solutions for $y$. In the region of $y \in (c_1 - c_2, c_1 + c_2)$,

$$y(u) = \frac{(c_2 + c_1)e^{a(u)} + c_1 - c_2}{e^{a(u)} + 1};$$  \hfill (41)
in the region of \( y \notin (c_1 - c_2, c_1 + c_2) \),

\[
y(u) = \frac{(c_2 + c_1)e^{\mu(u)} + c_2 - c_1}{e^{\mu(u)} - 1},
\]

(42)

where \( s(u) = k + \frac{2(1-2p_1)c_2}{\sigma}u \) and \( k \) is a constant.

The first solution (41) provides a stable travelling wave solution with \( y \) ranging from \( c_1 - c_2 \) to \( c_1 + c_2 \). A positive risk premium will require \( c_1 > 0 \). That is, we have ruled out the possibility of \( p_0 = -\theta \). From (40) for \( c_1 \), we have

\[
\text{sign}(\theta + p_0) = \text{sign}(1 - 2p_1).
\]

(43)

Provided that \( c_2 > 0 \), two scenarios can then be obtained as:

1. \( (p_0 > -\theta \text{ and } p_1 < 1/2) \) The solution \( y \) is increasing and is bounded below and above

\[
\begin{align*}
& \begin{cases}
  y \to c_1 - c_2, & \text{when } x \to -\infty \\
  y \to c_1 + c_2, & \text{when } x \to +\infty
\end{cases} \\
& \quad \text{(44)}
\end{align*}
\]

2. \( (p_0 < -\theta \text{ and } p_1 > 1/2) \) The solution \( y \) is decreasing and is bounded below and above

\[
\begin{align*}
& \begin{cases}
  y \to c_1 + c_2, & \text{when } x \to -\infty \\
  y \to c_1 - c_2, & \text{when } x \to +\infty
\end{cases} \\
& \quad \text{(45)}
\end{align*}
\]

It is worth noting that the increasing travelling wave solution Hodges and Selby (1997) obtained is the steady-state solutions to the Burgers’ equation (2) and is a special case of (44) with \( p_0 = p_1 = 0 \) (i.e., without intermediate consumption). On the other hand, the decreasing travelling wave (45) demonstrates the kind of behaviour that is more desirable since it implies investors will demand a higher risk premium when the market level is low. More importantly, this result cannot be obtained without introducing intermediate consumption in the economy.
While the solution from equation (41) is a hyperbolic tangent function consistent with equilibrium, the other equation (42), in general, entails a trigonometric function which could not possibly be supported by any reasonable utility function of an economic agent. To see this, we first realize that the price of risk $y$ has an unacceptable singularity at $s(u) = 0$ for the second equation. Namely, the state variable $x$ is prevented from reaching the point

$$x = -\frac{k\sigma}{2(1 - 2p_1)c_2} - \theta\tau.$$  

The singularity in the solution implies that arbitrage is permitted in the economy. Thereby, it is not a viable solution and must be ruled out.

Another special case is to let $k = 0$ and take the limit as $c_2$ tends to zero. The solution will then be

$$y = c_1 + \frac{\sigma}{(1 - 2p_1)u},$$

which implies

$$\alpha(x, \tau) = c_1 + \frac{\sigma}{(1 - 2p_1)(x + \theta\tau)}, \quad (46)$$

Unfortunately, however, (46) has a singularity at $p_1 = 1/2$ and $x = -\theta\tau$. Again, this model permits arbitrage and must be ruled out.

To summarize the results of our investigation, we first show, in Figure 1, the possible shapes of the price of risk function in the kind of stationary economy described before. Panels A, B and C depict the representative plots of the linear, exponential, and travelling wave solutions, respectively. Panel D shows the hyperbolic function with singularity (the nonviable solution). The solid lines represent the patterns which can be obtained only when nonzero intermediate consumption is allowed, whereas the dashed lines represent the patterns which can also be obtained in the one-consumption economy. As we can see, including intermediate consumption has added flexibility in the economic behaviour of the price of risk.
Before closing this section, we provide a further analysis on the decreasing travelling wave solution (41) followed by a numerical example. Since the price of risk \( y \) (or equivalently the risk premium) and the dividend yield \( g \) are likely to have limited ranges, we first denote by \( y_{\text{min}} = c_1 - c_2 \) and \( y_{\text{max}} = c_1 + c_2 \) the minimum and maximum values of \( y \), respectively, and denote by \( g_{\text{min}} \) and \( g_{\text{max}} \) the minimum and maximum values of \( g \), respectively. Therefore, we can easily obtain \( c_1 = (y_{\text{max}} + y_{\text{min}})/2 \) and \( c_2 = (y_{\text{max}} - y_{\text{min}})/2 \). After some simple manipulation of (27), we can also obtain \( p_1 \) by

\[
p_1 = \frac{g_{\text{max}} - g_{\text{min}}}{\sigma(y_{\text{max}} - y_{\text{min}})}.
\] (47)

Recall that \( p_1 \) must be greater than \( 1/2 \) to produce the decreasing form. Hence, we can impose this constraint on (47) and rearrange to obtain \( \sigma(y_{\text{max}} - y_{\text{min}}) < 2(g_{\text{max}} - g_{\text{min}}) \). This means that the range of the risk premium ought to be less than twice of the range of the dividend yield, or equivalently, the range of the dividend yield should be greater than half of the range of the risk premium. The coefficient \( p_0 \) can then be obtained by substituting \( p_1 \) in (47) to (27):

\[
p_0 = \frac{g_{\text{min}}y_{\text{max}} - g_{\text{max}}y_{\text{min}}}{y_{\text{max}} - y_{\text{min}}}.
\] (48)

Subsequently, the parameter \( \theta \) can be determined from (40). The only free parameter left undetermined is \( k \) in the function \( s(u) \).

Since the market portfolio represents the representative agent’s total wealth, the volatility and risk premium on the market portfolio should be substantially less than the volatility and risk premium on equity. Hence, if we assume the estimates for the wealth volatility \( \sigma = 0.1 \) and the risk premium \( \mu - r = 0.05 \), we will get a price of risk \( \alpha = 1/2 \). To apply these estimates to our model, we set the price of risk to be in the region of 0.1 and 1, i.e., \( y_{\text{min}} = 0.1 \) and \( y_{\text{max}} = 1 \). This implies that the risk premium on the market portfolio is in the region of 1% and 10%. Without loss of generality, we assume the interest rate \( r = 0 \) and the initial value
of the market portfolio $S_0 = 1$. Hence, the initial value of $x$ is $x_0 = 0$. Table 1 summarizes the parameter values employed in the model.

Figure 2 demonstrates how the price of risk and the dividend vary over time for an investment horizon of 100 years. Panel A shows the positive affine relationship between the price of risk and the dividend yield. Panels B illustrates the mean reversion feature, that is, the price of risk decreases as the (transformed) market level increases. Finally, Panel C shows that the change of consumption is less sensitive than the change of wealth. This also implies that consumption is rather smooth.

IV. Conclusion

There seems to be a consensus among financial economists that there is some predictability in stock index returns. It remains, however, something of a puzzle as to whether this has something to do with pricing anomalies or whether it reflects the nature of the risk premia within the underlying economy.

In this paper, we explored the possibility of a decreasing price of risk in the state of economy. We employed the framework established in the literature to characterize the equilibrium conditions of the asset price dynamics and obtained an explicit solution for the stationary economy case. It was shown that the model is capable of capturing the time variation in the expected returns and most interestingly the decreasing relative risk aversion behaviour of the representative agent. Given a postulated risk premium ranging from 1% to 10% and a dividend yield ranging from 1% to 6%, the model indicates the potential that the resulting time-varying price of risk and time-varying risk aversion can better explain the time-varying equity risk premium and leads to resolving the “equity premium puzzle” and the “consumption smoothing puzzle.”
Our assumptions of constancy of interest rate and volatility, on which our time-homogeneous solutions were based, are rather strong. It would seem natural to relax these assumptions and extend to models which can handle stochastic volatility and/or stochastic interest rate. We also excluded some real-world features such as imperfections and rule-of-thumb consumption. It remains to be investigated whether adding these more realistic features will significantly increase our understandings towards both these puzzles.

Our objective of this paper was to analyse the nature of the behaviour which is possible within the simpler framework and a representative agent equilibrium. What we do not know is whether it is consistent with general equilibrium. To this end, our model can be viewed as a reduced-form model derived from a partial equilibrium framework. The important issue of examining the consistency with the price processes and the dividend process in a general equilibrium framework is one of the ongoing investigations of the authors.

Finally, although the empirical issues are beyond the scope of our analysis, it would be nice to see how we can devise some kind of procedures so that the model can be estimated/tested empirically. These areas are challenges for future research.
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The table summarizes the parameter values that are employed in the numerical example. The parameters are used to illustrate the behaviour of the price of risk in a stationary economy. The dividend yield is approximated as a linear function of the risk premium, and the price of risk function is a decreasing traveling wave (decreasing with respect to the state of the economy). The investment horizon is assumed to be 100 years.

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Figure 1. Typical plots of the price of risk function. The figures are plotted based on the analytical solutions of the price of risk in the time-homogeneous case. From A. to D., they show linear, exponential, hyperbolic tangent, and hyperbolic forms, respectively. The solid lines represent the patterns which can be obtained only from the economy with intermediate consumption, while the dashed lines represent the patterns which can also be achieved from the economy without intermediate consumption.
Figure 2. Mean reversion in a stationary economy. Assume that the total investment horizon is 100 years and $\tau$ is the time to horizon date. Panel A shows that the dividend yield is linear to the risk premium. Applying the parameter values in Table I, Panel B shows that the price of risk decreases as the market level increases. The price of risk also varies in a homogeneous manner through time. Panel C indicates consumption smoothing in the sense that the change in consumption is less sensitive than the change in wealth.
Notes

1 See, for example, Hodges and Carverhill (1993) and He and Leland (1993). These authors also characterize the equilibrium price processes by exploiting this property.

2 Their result can be justified if we define \( S \) as the relative asset price (that is, the risky asset price normalized by the bond price). The drift term \( \mu \) in PDE (15) should then be interpreted as the risk premium, provided that the risk premium is a deterministic function of the relative price and time. This is similar to the setup in Bick (Bick 1990).

3 Note that taking a negative \( c_2 \) will not change the properties of the solutions.

4 It is also interesting to derive a \( y \) function which depends on \( S \) only. This can be achieved by letting \( \theta = -\left(r - \frac{\sigma^2}{2}\right) \) so that the resulting \( y \) can be expressed as

\[
y = \frac{(c_2 + c_1)B \cdot S^A + c_1 - c_2}{B \cdot S^A + 1}, \quad \text{for } y \in (c_1 - c_2, c_1 + c_2),
\]

where \( A = \frac{2(1 - 2p_1)c_2}{\sigma} \) and \( B \) is a positive real. Again, provided \( p_0 < -\theta \) and \( p_1 > 1/2 \), we can obtain a decreasing \( y \).