Dynamic Copula Quantile Regressions and Tail Area Dynamic Dependence in Forex Markets

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Abstract

We introduce a general approach to nonlinear quantile regression modelling that is based on the specification of the copula function that defines the dependency structure between the variables of interest. Hence we extend Koenker and Bassett's [1978] original statement of the quantile regression problem by determining a distribution for the dependent variable Y conditional on the regressors X and hence the specification of all the quantile regression functions. We use the fact that this multivariate distribution can be split into two parts: the marginals and the dependence function (or copula). We then deduce the form of the non linear conditional quantile relationship implied by the copula. Notice that this can be done with arbitrary distributions assumed for the marginals. Some properties of the copula based quantiles or c-quantiles are then derived. Finally, we develop an empirical application which examines conditional quantile dependency in the foreign exchange market and compare this approach with the standard tail area dependency measures.

1 Introduction

The problem of characterising the dependence between random variables at a given quantile is an important issue, especially if the distributions of the variables involved are fat tailed as is standard with financial returns. Tail area dependency for instance may be quite different to that implied by standard correlation measures and may signal where extreme downside protection may be found if two assets do not show causal dependency in their extreme quantile relationships. One goal of this paper is to introduce a general approach to nonlinear quantile regression modelling that exploits the form of the copula linking the assets involved.

The starting point is the multivariate distribution for the variables, and then working down from this, using the fact that this multivariate distribution can be split into two parts - the margins and the dependence function or copula. The conditional non linear quantile relationship implied by the copula, the c-quantile, as opposed to an empirical quantile, can then be derived.

A second objective of this paper is to apply the c-quantile idea to assess the conditional dependency between foreign exchange rates. It is an important issue in practice as to how exchange rates are dependent when the markets are under stress and by using c-quantiles we can examine the entire conditional distribution rather than the question of asymptotic dependence and independence which is captured by standard tail area dependency measures. These issues have been considered by PATTON (2001) and HARTMANN,STRAETMANS AND DE VRIES(2002) using related but different techniques. We also consider dynamic dependency and how dynamic risk measures may be developed based on c-quantiles and expected shortfall. The c-quantile approach here provides a different approach to that considered by ENGLE AND MANGANELLI (2000) who assumed the form of the dynamic quantile functions whereas the form of the c-quantiles follows from the joint distribution.

In the next section, we briefly review regression quantiles and then the concept of copula is defined and the implications for the assessing the dependence structure between X and Y are presented. Then, we introduce the concept of a copula quantile curve, derive some properties of this c-quantile curve and provide some examples for particular copulae. In the next section, the copula quantile regression model is formally defined and we discuss the estimation issue. Then the application to analysing quantile and tail area dependence in foreign exchange markets is provided.A final section offers some conclusions.

2 Regression Quantiles

Koenker and Bassett intoduced linear quantile regression in *Econometrica* in 1978 [11]. We first review how they define quantile regression and the main properties of their model. Let $(y_1, \ldots, y_T)$ be a random sample on Y and $(x_1, \ldots, x_T)$ a random k-vector sample on X.

Definition 1 *The p-**th quantile regression** is any solution to the following problem:*

$$\min_{\beta \in \mathbb{R}^k} \left(\sum_{t \in \mathcal{T}_p} p \left| y_t - x_t \beta \right| + \sum_{t \in \mathcal{T}_{1-p}} (1-p) \left| y_t - x_t \beta \right| \right)$$

with $\mathcal{T}_p = \{t : y_t \geq x_t \beta\}$ and $\mathcal{T}_{1-p}$ its complement. This can be alternatively expressed as [1]*:*

$$\min_{\beta \in \mathbb{R}^k} \left(\sum_{t=1}^{T} \left(p - \mathbb{I}_{\{y_t \leq x_t \beta\}} \right) (y_t - x_t \beta) \right) \tag{1}$$

Non-linearity in quantile regression was developed by Powell [1986] using a censored model. The consistency of non-linear quantile regression estimation has been investigated by White [1994], Engle and Manganelli [2000]and Kim and White [2002]. For a recent overview of quantile regression see Yu, Lu, and Stander [2001]. As noted by Buchinsky [1998], quantile regression models have useful features: (i) with non-gaussian error terms, quantile regression estimators may be more efficient than least-square estimators, (ii) the entire conditional distribution can be characterized, (iii) different relationships between the regressor and the dependent variable may arise at different quantiles. In this paper, we attempt to resolve one problem with using quantile regression, the question of how to specify the form of the quantile regression function. We achieve this by deriving a distribution for Y conditional on X which then implies the structural form of the quantile regression. For simplicity, our model is developed for the one regressor case, corresponding to a bivariate copula but it may be extended to multiple regressors.

[1] Koenker and Bassett discuss properties of their estimator, especially through the following theorem:

Theorem 1 *Let $\beta^\star(p, y, X) \in \mathcal{B}^\star(p, y, X)$. Then, the following properties hold:*

1. $\beta^\star(p, \kappa y, X) = \begin{cases} \kappa \beta^\star(p, y, X) & \text{for } \kappa \in \mathbb{R}^+ \\ \kappa \beta^\star(1-p, y, X) & \text{for } \kappa \in \mathbb{R}^- \end{cases}$
2. $\beta^\star(p, y + X\delta, X) = \beta^\star(p, y, X)$ *for* $\delta \in \mathbb{R}^k$
3. $\beta^\star(p, y, X\Gamma) = \Gamma^{-1} \beta^\star(p, y, X)$ *with* Γ *non-singular* $(k \times k)$ *matrix*

3 Copulae and dependence

The goal of this preliminary section is to provide a definition of a copula function and Sklar's theorem which ensures the uniqueness of the copula when the bivariate distribution for two random variables (corresponding in our modelling framework to the dependent variable and a regressor) is given and the margins are continuous. Then, we introduce the concepts of positive quadrant dependence and the left tail decreasing property and show how these two concepts are related. These definitions are the starting point to demonstrating that the concavity (respectively convexity) of the copula in its first argument induces a positive (respectively negative) dependence at each quantile level.

Definition 2 *A **bivariate copula** is a function* $\mathbf{C} : [0,1]^2 \to [0,1]$ *such that:*

1. $\forall (u,v) \in [0,1]^2$,
$$\begin{cases} \mathbf{C}(u,0) = \mathbf{C}(0,v) = 0 \\ \mathbf{C}(u,1) = u \text{ and } \mathbf{C}(1,v) = v \end{cases} \tag{2}$$

2. $\forall (u_1, v_1, u_2, v_2) \in [0,1]^4$, $u_1 \leq u_2$ *and* $v_1 \leq v_2$,
$$\mathbf{C}(u_2,v_2) - \mathbf{C}(u_1,v_2) - \mathbf{C}(u_2,v_1) + \mathbf{C}(u_1,v_1) \geq 0 \tag{3}$$

Theorem 2 (Sklar's Theorem) *Let* X *and* Y *be two random variables with joint distribution* $\mathbf{F}$*. Then, there exists a unique copula* $\mathbf{C}$ *satisfying*

$$\mathbf{F}(x,y) = \mathbf{C}(F_X(x), F_Y(y)) \tag{4}$$

if F_X *and* F_Y *are continuous and represent the marginal distribution functions of* X *and* Y *respectively.*

Definition 3 (Order) *Let* $(\mathbf{C}, \mathbf{D}) \in \mathcal{C}^2$ *with* $\mathcal{C}$ *the set of copulae. One says that* $\mathbf{C}$ *is greater than* $\mathbf{D}$ ($\mathbf{C} \succ \mathbf{D}$ *or* $\mathbf{D} \prec \mathbf{C}$) *if*

$$\forall (u,v) \in [0,1]^2, \ \mathbf{C}(u,v) \geq \mathbf{D}(u,v)$$

Theorem 3 (Fréchet Bounds) *Let* $\mathbf{C} \in \mathcal{C}$*. Then,*

$$\mathbf{C}^{-} \prec \mathbf{C} \prec \mathbf{C}^{+}$$

where $\mathbf{C}^-$ *and* $\mathbf{C}^+$ *are such that*

$$\begin{aligned}\mathbf{C}^-(u,v) &= \max(u+v-1,0)\\ \mathbf{C}^+(u,v) &= \min(u,v)\end{aligned}$$

The concept of order for copulae is importnat as it allows us to rank the dependence between random variables. One interesting copula is the product copula $\mathbf{C}^\perp$- that corresponds to independence - so that $\mathbf{C}^\perp(u,v) = uv$.

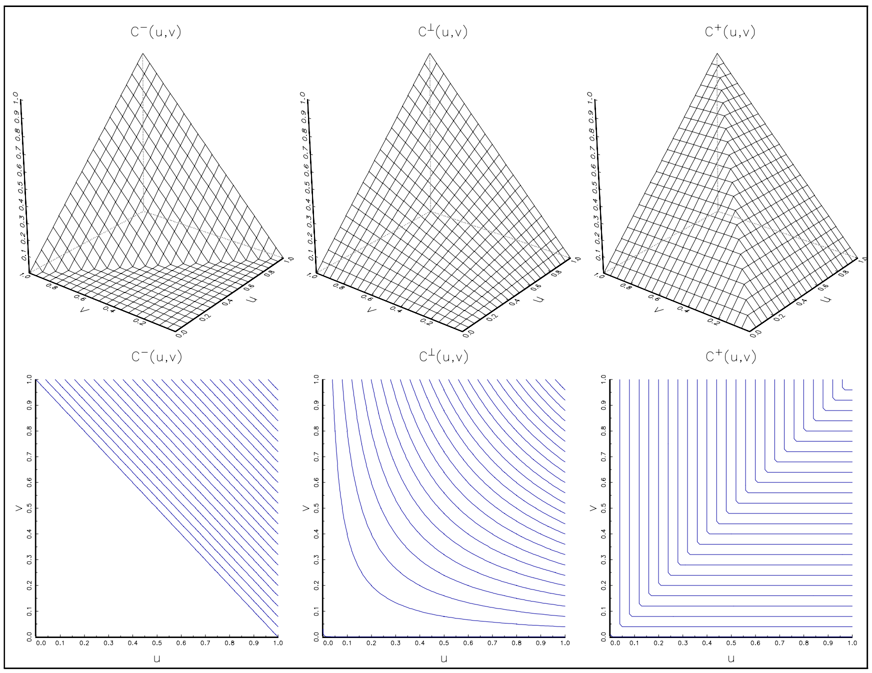

Figure 1: $\mathbf{C}^-$, $\mathbf{C}^\perp$ and $\mathbf{C}^+$

Definition 4 (Lehmann (1966)) *The pair* (X,Y) *is* ***positive quadrant dependent*** $(\mathbf{PQD}(X,Y))$ *if*

$$\Pr\{X \leq x, Y \leq y\} \geq \Pr\{X \leq x\}\Pr\{Y \leq y\} \tag{5}$$

In terms of copulae, this definition can be restated $\mathbf{C}^\perp \prec \mathbf{C}$.

Definition 5 (Esary and Proschan (1972)) Y *is* ***left tail decreasing*** *in* X $(\mathbf{LTD}(Y \mid X))$ *if*

$$\forall y,\ \Pr\{Y \leq y \mid X \leq x\} \ \textit{is a nonincreasing function of } x \tag{6}$$

This definition can be equivalently expressed using copulae as:

Theorem 4 (Nelsen (1998))

$$\begin{aligned}
\mathbf{LTD}(Y \mid X) &\iff \frac{\mathbf{C}(u,v)}{u} \text{ is nonincreasing in } u \\
&\iff \frac{\partial \mathbf{C}(u,v)}{\partial u} \leq \frac{\mathbf{C}(u,v)}{u}
\end{aligned} \tag{7}$$

Theorem 5 *Let $\mathbf{C} \in \mathcal{C}$. The following holds*

$$\text{If } \forall (u,v) \in [0,1]^2, \frac{\partial^2 \mathbf{C}(u,v)}{\partial u^2} \leq 0 \text{ then } \mathbf{C}^\perp \prec \mathbf{C} \tag{8}$$

$$\text{If } \forall (u,v) \in [0,1]^2, \frac{\partial^2 \mathbf{C}(u,v)}{\partial u^2} \geq 0 \text{ then } \mathbf{C} \prec \mathbf{C}^\perp \tag{9}$$

Proof. We refer to NELSEN (1998), p 151-160, for the proof. The first part is based on the fact that $\frac{\partial^2 \mathbf{C}(u,v)}{\partial u^2} \leq 0 \Rightarrow \mathbf{LTD}(Y \mid X) \Rightarrow \mathbf{PQD}(X,Y)$. ■

The previous theorem tells us that if the copula function is concave in the marginal distribution F_X then the random variables X and Y are positively related *i.e.* their copula value is greater than given by the independence copula $\mathbf{C}^\perp$. Conversely, convexity implies a negative relationship *i.e.* the copula linking X and Y lies below the independence copula $\mathbf{C}^\perp$. For simplicity, we still have not introduced the parameter(s) of the copula function which in effect measure the degree and different forms of dependence between the variables, let us denote these parameters by $\delta \in \Delta$. Then, through the family of copula functions, we can distinguish three classes:

1. Copulae that only exhibit negative dependence:

$$\forall \delta \in \Delta, \forall (u,v) \in [0,1]^2, \text{ then } \mathbf{C}(u,v;\delta) \prec \mathbf{C}^\perp(u,v)$$

2. Copulae that only exhibit positive dependence:

$$\forall \delta \in \Delta, \forall (u,v) \in [0,1]^2, \text{ then } \mathbf{C}^\perp(u,v) \prec \mathbf{C}(u,v;\delta)$$

3. Copulae that exhibit both negative and positive dependence depending on the parameter values:

$$\begin{aligned}
\forall \delta &\in \Delta^-, \forall (u,v) \in [0,1]^2, \text{ then } \mathbf{C}(u,v;\delta) \prec \mathbf{C}^\perp(u,v) \\
\forall \delta &\in \Delta^+, \forall (u,v) \in [0,1]^2, \text{ then } \mathbf{C}^\perp(u,v) \prec \mathbf{C}(u,v;\delta)
\end{aligned}$$

In the next section, the concept of a quantile curve of Y conditional on X is defined and we derive several results that are directly deduced from the underlying copula properties outlined above.

4 Quantile curve

First, the copula p-th quantile curve of y conditionally on x or p'th c-quantile curve is defined. Second, its main properties are exhibited. Third, the case of radially symmetric variables is studied. Finally, the quantile curves are developed for three special cases: the Kimeldorf and Sampson, Gaussian and Frank copulae.

4.1 Definitions

We restrict the study to *monotonic* copula for simplicity. Define the probability distribution of y conditional on x by $p(x, y; \delta)$:

$$\begin{aligned} p(x, y; \delta) &= \Pr\{Y \leq y \mid X = x\} \\ &= \mathbf{E}\left(\mathbb{I}_{\{Y \leq y\}} \mid X = x\right) \\ &= \lim_{\varepsilon \to 0} \Pr\{Y \leq y \mid x \leq X \leq x + \varepsilon\} \\ &= \lim_{\varepsilon \to 0} \frac{\mathbf{F}(x + \varepsilon, y; \delta) - \mathbf{F}(x, y; \delta)}{F_X(x + \varepsilon) - F_X(x)} \\ &= \lim_{\varepsilon \to 0} \frac{\mathbf{C}[F_X(x + \varepsilon), F_Y(y); \delta] - \mathbf{C}[F_X(x), F_Y(y); \delta]}{F_X(x + \varepsilon) - F_X(x)} \\ p(x, y; \delta) &= C_{1.}[F_X(x), F_Y(y); \delta] \end{aligned} \tag{10}$$

with $C_{1.}(u, v; \delta) = \frac{\partial}{\partial u} C(u, v; \delta)$. Since the distribution functions F_X and F_Y are nondecreasing, $p(x, y; \delta)$ is nondecreasing in y. Using the same argument, $p(x, y; \delta)$ is nondecreasing in x if $C_{2.}(u, v; \delta) \leq 0$ and nonincreasing in x if $C_{2.}(u, v; \delta) \geq 0$ where $C_{2.}(u, v; \delta) = \frac{\partial^2 \mathbf{C}(u,v;\delta)}{\partial u^2}$.

Definition 6 *For a parametric copula $\mathbf{C}(., .; \delta)$, the **p-th copula quantile curve** of y conditional on x is defined by the following implicit equation*

$$p = C_{1.}[F_X(x), F_Y(y); \delta] \tag{11}$$

where $\delta \in \Delta$ the set of parameters.

Under some conditions[2], equation (11) can be expressed as follows in order to capture the relationship between X and Y:

$$y=\mathbf{q}\left(x,p;\delta\right) \tag{12}$$

where $\mathbf{q}\left(x,p;\delta\right)=F_Y^{[-1]}\left(D\left(F_X\left(x\right),p;\delta\right)\right)$ with D the partial inverse in the second argument of C_1 and $F_Y^{[-1]}$ the pseudo-inverse of F_Y. Note that the relationship (12) can alternatively be expressed using uniform margins as:

$$v=\mathbf{r}\left(u,p;\delta\right). \tag{13}$$

with $u=F_X\left(x\right)$ and $v=F_Y\left(y\right)$.

4.2 Properties

Two properties are demonstrated. The first tells us that the quantile curve shifts up with the quantile level. The second indicates that the quantile curve has a positive (respectively negative) slope if the copula function is concave (respectively convex) in its first argument.

Property 1 *If* $0<p_1\leq p_2<1$ *then* $\mathbf{q}\left(x,p_1;\delta\right)\leq\mathbf{q}\left(x,p_2;\delta\right)$.

Property 2 *Let* $x_1\leq x_2$.

If $\mathbf{C}\left(u,v\right)$ *is concave in* u *then* $\mathbf{q}\left(x_1,p;\delta\right)\leq\mathbf{q}\left(x_2,p;\delta\right)$

If $\mathbf{C}\left(u,v\right)$ *is convex in* u *then* $\mathbf{q}\left(x_1,p;\delta\right)\geq\mathbf{q}\left(x_2,p;\delta\right)$

Proof. Thanks the implicit function theorem, y may be expressed as a function of x and p *i.e.* $y=\mathbf{q}\left(x,p;\delta\right)$. Let us rewrite equation (11) as $F\left(x,p,\mathbf{q}\left(x,p;\delta\right)\right)=0$. Thus,

$$\left\{\begin{array}{ll}\frac{\partial F}{\partial x}\left(x,p,\mathbf{q}\left(x,p;\delta\right)\right)+\frac{\partial F}{\partial y}\left(x,p,\mathbf{q}\left(x,p;\delta\right)\right)\frac{\partial \mathbf{q}}{\partial x}\left(x,p;\delta\right) & =0\\ \frac{\partial F}{\partial p}\left(x,p,\mathbf{q}\left(x,p;\delta\right)\right)+\frac{\partial F}{\partial y}\left(x,p,\mathbf{q}\left(x,p;\delta\right)\right)\frac{\partial \mathbf{q}}{\partial p}\left(x,p;\delta\right) & =0\end{array}\right..$$

Then,

$$\left\{\begin{array}{ll}\frac{\partial \mathbf{q}}{\partial x}\left(x,p;\delta\right) & =-\frac{\frac{\partial F}{\partial x}\left(x,p,\mathbf{q}\left(x,p;\delta\right)\right)}{\frac{\partial F}{\partial y}\left(x,p,\mathbf{q}\left(x,p;\delta\right)\right)}\\ \frac{\partial \mathbf{q}}{\partial p}\left(x,p;\delta\right) & =-\frac{\frac{\partial F}{\partial p}\left(x,p,\mathbf{q}\left(x,p;\delta\right)\right)}{\frac{\partial F}{\partial x}\left(x,p,\mathbf{q}\left(x,p;\delta\right)\right)}\end{array}\right..$$

Just note that $F\left(x,p,y\right)=C_{1.}\left[F_X\left(x\right),F_Y\left(y\right);\delta\right]-p$, it follows that

$$\left\{\begin{array}{ll}\frac{\partial \mathbf{q}}{\partial x}\left(x,p;\delta\right) & =-\frac{f_X(x)C_{2.}[F_X(x),F_Y(y);\delta]}{f_Y(y)C_{11}[F_X(x),F_Y(y);\delta]}\\ \frac{\partial \mathbf{q}}{\partial p}\left(x,p;\delta\right) & =\frac{1}{f_Y(y)C_{11}[F_X(x),F_Y(y);\delta]}\end{array}\right.. \tag{14}$$

As $\forall\left(u,v\right)\in\left[0,1\right]^2$, $C_{11}\left[u,v;\delta\right]\geq0$, $f_X\left(x\right)\geq0$ and $f_Y\left(y\right)\geq0$, this completes the proof. ■

[2]Note that $C_{1.}$ has to be partially invertible in its second argument. If it is not anatically invertible, a numerical root finding procedure can be used.

4.3 Symmetric case

An interesting case concerns the radial symmetry of X and Y. Indeed, in this case, a remarkable relationship exists between the p-th quantile curve and the $(1-p)$-th quantile curve. First, the definition of radial symmetry is given. Then, a theorem is stated and a corollary that informs us about the slopes of the quantile curves is provided.

Definition 7 *Two random variables X and Y are radially symmetric about (a,b) if*

$$\Pr\{X \leq x-a, Y \leq y-b\} = \Pr\{X \geq x+a, Y \geq y+b\} \tag{15}$$

Theorem 6 (Nelsen (1998)) *Let X and Y be respectively symmetric about a and b. They are radially symmetric about (a,b)* ***iff*** *their copula $\mathbf{C}$ satisfies:*

$$\mathbf{C}(u,v) = u+v-1+\mathbf{C}(1-u,1-v) \tag{16}$$

Corollary 7 *If the copula $\mathbf{C}$ satisfies equation (11), then*

$$\begin{cases} C_{1.}(u,v;\delta) = & 1-C_{1.}(1-u,1-v;\delta) \\ C_{2.}(u,v;\delta) = & C_{2.}(1-u,1-v;\delta) \\ C_{11}(u,v;\delta) = & C_{11}(1-u,1-v;\delta) \end{cases}$$

Theorem 8 (Radial symmetry and copula quantile curves) *If two random variables X and Y are radially symmetric about (a,b) then*

$$\mathbf{q}(a-x,p;\delta) + \mathbf{q}(a+x,1-p;\delta) = 2b \tag{17}$$

Proof. From equation (15),

$$\Pr\{Y \leq y-b \mid X \leq x-a\} = \Pr\{Y \geq y+b \mid X \geq x+a\}$$

In terms of copula,

$$\begin{aligned} C_{1.}[F_X(a-x), F_Y(b-y);\delta] &= 1-C_{1.}[F_X(a+x), F_Y(b+y);\delta] \\ p(a-x,b-y) &= 1-p(a+x,b+y) \end{aligned}$$

Then, for $p(a-x,b-y)=p$,

$$\begin{cases} b-y = & \mathbf{q}(a-x,p;\delta) \\ b+y = & \mathbf{q}(a+x,1-p;\delta) \end{cases}$$

and the proof follows. ■

Note that a direct implication of this theorem is $\mathbf{q}\left(a,\frac{1}{2};\delta\right) = b$.

Corollary 9 *If two random variables X and Y are radially symmetric about (a,b) then*

$$\frac{\partial \mathbf{q}}{\partial x}(a-x,p;\delta)=\frac{\partial \mathbf{q}}{\partial x}(a+x,1-p;\delta) \tag{18}$$

4.4 Examples

We first describe a case where the copula quantiles can be derived analytically, this is for the Kimeldorf and Sampson copula. We then describe how to develop c-quantiles for the general class of Archimedean Copulae and the Clayton-Joe Copula in particular (BB7 in JOE[1997]. We then study two specific copulae that allow both positive and negative slopes for the quantile curves, depending on the value of their dependence parameter. These are the Gaussian copula where the dependence pattern is measured by correlation but where the marginal distributions may be non-gaussian. We then show that we have to be careful when choosing copula since some copulae, such as the Frank copula, maay not allow us to adequately capture the full range of behaviour in the distribution of the dependent variable Y.

4.4.1 Kimeldorf and Sampson copula

Consider the copula given by

$$C(u,v)=(u^{-\theta}+v^{-\theta}-1)^{-\frac{1}{\theta}} \text{ for } \theta>0$$

we then have

$$\begin{aligned}
C_2(v|u) &= \frac{\partial C(u,v)}{\partial u} \\
&= -\frac{1}{\theta}(u^{-\theta}+v^{-\theta}-1)^{-\frac{(1+\theta)}{\theta}}(-\theta u^{-(1+\theta)}) \\
&= (1+u^{\theta}(v^{-\theta}-1))^{-\frac{(1+\theta)}{\theta}}
\end{aligned}$$

solving $p=C_2(v|u)$ for v gives

$$C_2^{-1}(v|u)=v=(p^{\frac{-\theta}{1+\theta}}-1)u^{-\theta}+1)^{\frac{-1}{\theta}}$$

which provides us with the c-quantiles relating v and u for different values of p. Using the empirical distribution functions for $u=F_X(x)$ and $v=F_Y(y)$ we can find explicit expressions for the conditional c-quantiles for the variable Y conditional on X.

$$y=F_Y^{-1}\left((p^{\frac{-\theta}{1+\theta}}-1)F_X(x)^{-\theta}+1)^{\frac{-1}{\theta}}\right)$$

4.4.2 Archimedean Copulae

4.4.2.1 General case An archimedean copula is defined as follows

$$C(u,v) = \phi^{-1}\left[\phi\left(u\right) + \phi\left(v\right)\right] \tag{19}$$

with ϕ a continuous and strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\phi\left(1\right) = 0$. ϕ is often called the generator function. From $p = \frac{\partial}{\partial u}C(u,v)$, we obtain

$$\begin{aligned} p &= \frac{\phi\prime\left(u\right)}{\phi\prime\left(C(u,v)\right)} \\ p &= \frac{\phi_{\delta}\prime\left(u\right)}{\phi\prime\left(\phi^{-1}\left[\phi\left(u\right)+\phi\left(v\right)\right]\right)} \end{aligned} \tag{20}$$

and the quantile regression curve for archimedean copulae can in general be deduced as

$$\begin{aligned} v &= \mathbf{r}\left(u,p;\delta\right) \\ v &= \phi^{-1}\left[\phi\left(\phi\prime^{-1}\left(\frac{1}{p}\phi\prime\left(u\right)\right)\right) - \phi\left(u\right)\right] \end{aligned}$$

Introducing $u = F_X\left(x\right)$ and $v = F_Y\left(y\right)$, the equation for the c-quantile above becomes

$$y = F_Y^{-1}\left(\phi^{-1}\left[\phi\left(\phi\prime^{-1}\left(\frac{1}{p}\phi\prime\left(F_X\left(x\right)\right)\right)\right) - \phi\left(F_X\left(x\right)\right)\right]\right)$$

4.4.3 A specific archimedean copula: Clayton Joe

For the copula defined by

$$C_{\delta,\theta}(u,v) = 1 - \left(1 - \left[\left(1-(1-u)^{\theta}\right)^{-\delta} + \left(1-(1-v)^{\theta}\right)^{-\delta} - 1\right]^{-\frac{1}{\delta}}\right)^{\frac{1}{\theta}} \tag{21}$$

with $\theta \geq 1$ and $\delta > 0$, see Joe [1997], p 153). This two parameter copula is archimedean as

$$C_{\delta,\theta}(u,v) = \phi_{\delta,\theta}^{-1}\left[\phi_{\delta,\theta}\left(u\right) + \phi_{\delta,\theta}\left(v\right)\right]$$

with

$$\begin{cases} \phi_{\delta,\theta}\left(s\right) = \left[1-(1-s)^{\theta}\right]^{-\delta} - 1 \\ \phi_{\delta,\theta}^{-1}\left(s\right) = 1 - \left[1-(1+s)^{-\frac{1}{\delta}}\right]^{\frac{1}{\theta}} \\ \phi_{\delta,\theta}\prime\left(s\right) = -\left[1-(1-s)^{\theta}\right]^{-\delta-1}\delta\left[-(1-s)^{\theta}\,\frac{\theta}{-1+s}\right] \end{cases} \tag{22}$$

It only allows positive dependence and we can see that

$$\begin{aligned}
\lim_{\delta\to\infty} C_{\delta,\theta}(u,v) &= C^{+}(u,v) \\
\lim_{\theta\to\infty} C_{\delta,\theta}(u,v) &= C^{+}(u,v)
\end{aligned}$$

An important property is that each parameter respectively measures lower (δ) and upper (θ) tail dependence as we show below. Moreover this copula encompasses two copulae sub-families as for $\theta = 1$ one obtains the Kimeldorf & Sampson (1975) copula:

$$C_{\delta}(u,v) = \left(u^{-\delta} + v^{-\delta} - 1\right)^{-\frac{1}{\delta}},$$

and for $\delta \to 0$ the Joe (1993) copula:

$$C_{\theta}(u,v) = 1 - \left((1-u)^{\theta} + (1-v)^{\theta} - (1-u)^{\theta}(1-v)^{\theta}\right)^{1/\theta}.$$

4.4.4 Gaussian copula

The bivariate copula in this case is written:

$$\mathbf{C}\left(u,v;\rho\right) = \Phi_2\left(\Phi^{[-1]}\left(u\right), \Phi^{[-1]}\left(v\right); \rho\right) \tag{23}$$

with Φ_2 the bivariate gaussian distribution and Φ the univariate distribution.

$$p = \Phi\left(\frac{\Phi^{[-1]}\left(v\right) - \rho\Phi^{[-1]}\left(u\right)}{\sqrt{1-\rho^2}}\right)$$

or equivalently solving for v we find the $p'th$ c-quantile curve to be,

$$v = \mathbf{r}\left(u,p;\rho\right) = \Phi\left(\rho\Phi^{[-1]}\left(u\right) + \sqrt{1-\rho^2}\Phi^{[-1]}\left(p\right)\right).$$

The slope of the p-quantile curve is given by:

$$\frac{\partial\mathbf{r}\left(u,p;\rho\right)}{\partial u} = \rho\frac{\phi\left(\rho\Phi^{[-1]}\left(u\right) + \sqrt{1-\rho^2}\Phi^{[-1]}\left(p\right)\right)}{\phi\left(\Phi^{[-1]}\left(u\right)\right)}.$$

A positive correlation is characterized by a positive slope and conversely for a negative correlation. Moreover,

$$\frac{\partial\mathbf{r}\left(u,p;\rho\right)}{\partial p} = \sqrt{1-\rho^2}\frac{\phi\left(\rho\Phi^{[-1]}\left(u\right) + \sqrt{1-\rho^2}\Phi^{[-1]}\left(p\right)\right)}{\phi\left(\Phi^{[-1]}\left(u\right)\right)}.$$

that is always positive. Then, the higher p the higher the quantile curve. The relationship between y and x for the p-quantile is:

$$y = F_Y^{[-1]}\left[\Phi\left(\rho\Phi^{[-1]}\left(F_X\left(x\right)\right) + \sqrt{1-\rho^2}\Phi^{[-1]}\left(p\right)\right)\right] \tag{24}$$

Let assume that X and Y are jointly bivariate gaussian with $\mu_X = \mathbf{E}\left[X\right]$, $\mu_Y = \mathbf{E}\left[Y\right]$, $\sigma_X^2 = \mathbf{Var}\left[X\right]$, $\sigma_Y^2 = \mathbf{Var}\left[Y\right]$ and $\rho = \mathbf{Corr}\left[X, Y\right]$. Then, given equation (24), the relationship becomes linear and we have

$$y = \mathbf{q}\left(x_t, p; \delta = \rho\right) = a + bx$$

with slope and intercept values determined by the quantile level;

$$\begin{cases} a = \mu_Y + \sigma_Y\sqrt{1-\rho^2}\Phi^{[-1]}\left(p\right) - \rho\frac{\sigma_Y}{\sigma_x}\mu_X \\ b = \rho\frac{\sigma_Y}{\sigma_x} \end{cases}$$

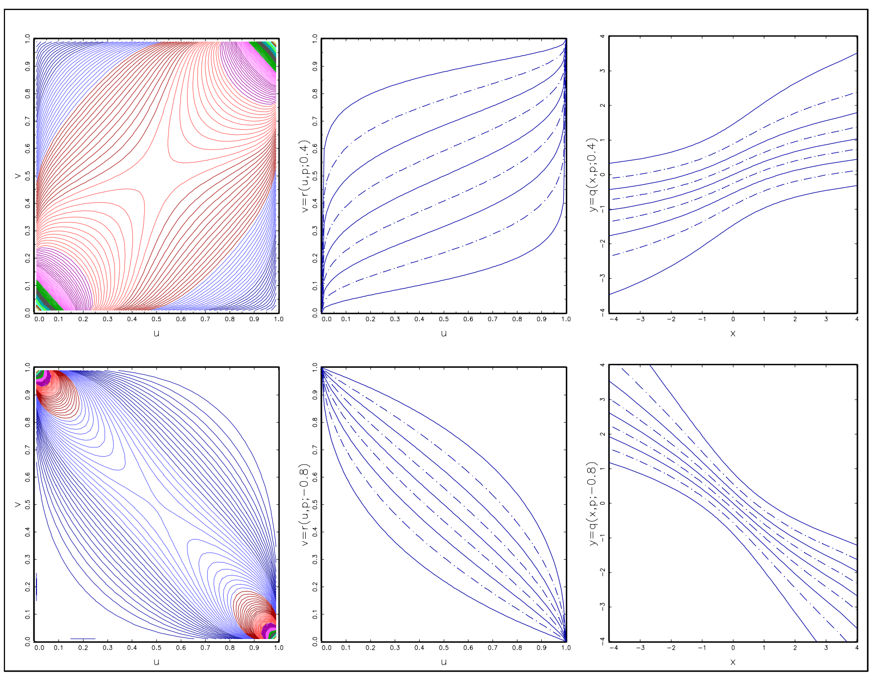

Figure 2: Gaussian copula densities, copula p^{th} quantile curves (for $p = .1, .2, \ldots, .9$) for (u, v) and (x, y) under the hypothesis of Student margins ($\nu = 3$) for $\rho = 0.4$ (upper plots) and $\rho = -0.8$ (lower plots)

4.4.5 Frank copula

This copula is given by

$$\mathbf{C}(u,v;\delta) = -\frac{1}{\delta}\ln\left(1+\frac{(e^{-\delta u}-1)(e^{-\delta v}-1)}{e^{-\delta}-1}\right) \tag{25}$$

By computing its first derivative with respect to u, one obtains the copula p-th quantile curve, $p = \mathbf{C}_{1.}(u,v;\delta)$ as

$$p = e^{-\delta u}\left(\left(1-e^{-\delta}\right)(1-e^{-\delta v})^{-1} - (1-e^{-\delta u})\right)^{-1}$$

or equivalently,

$$v = -\frac{1}{\delta}\ln\left(1-\left(1-e^{-\delta}\right)\left[1+e^{-\delta u}\left(p^{-1}-1\right)\right]^{-1}\right).$$

From the definition of the uniform distribution, one obtains the non-linear relationship between x and y for the p-quantile as:

$$y = F_Y^{[-1]}\left[-\frac{1}{\delta}\ln\left(1-\left(1-e^{-\delta}\right)\left[1+e^{-\delta F_X(x)}\left(p^{-1}-1\right)\right]^{-1}\right)\right] \tag{26}$$

We can see that the Frank copula might not always be a good choice as for $u \in [0,1]$,

$$\begin{aligned} -\frac{1}{\delta}\ln\left(1-\left(1-e^{-\delta}\right)p\right) &\leq \mathbf{r}(u,p;\delta) \leq -\frac{1}{\delta}\ln\left(\frac{1-e^{-\delta}}{1+e^{-\delta}\left(p^{-1}-1\right)}\right) \text{ for } \delta > 0 \\ &\text{and} \\ -\frac{1}{\delta}\ln\left(1-\left(1-e^{-\delta}\right)p\right) &\geq \mathbf{r}(u,p;\delta) \geq -\frac{1}{\delta}\ln\left(\frac{1-e^{-\delta}}{1+e^{-\delta}\left(p^{-1}-1\right)}\right) \text{ for } \delta < 0 \end{aligned}$$

5 Copula quantile regression

5.1 Definition

In this section, we define the concept of copula quantile regression as a special case of non-linear quantile regression. Let $(y_1,\ldots,y_T)$ be a random sample on Y and $(x_1,\ldots,x_T)$ a random k-vector sample on X.

Definition 8 *The p**-th copula quantile regression** $\mathbf{q}(x_t,p;\delta)$ is any solution to the following problem:*

$$\min_{\delta}\left(\sum_{t\in\mathcal{T}_p} p\left|y_t-\mathbf{q}(x_t,p;\delta)\right| + \sum_{t\in\mathcal{T}_{1-p}}(1-p)\left|y_t-\mathbf{q}(x_t,p;\delta)\right|\right) \tag{27}$$

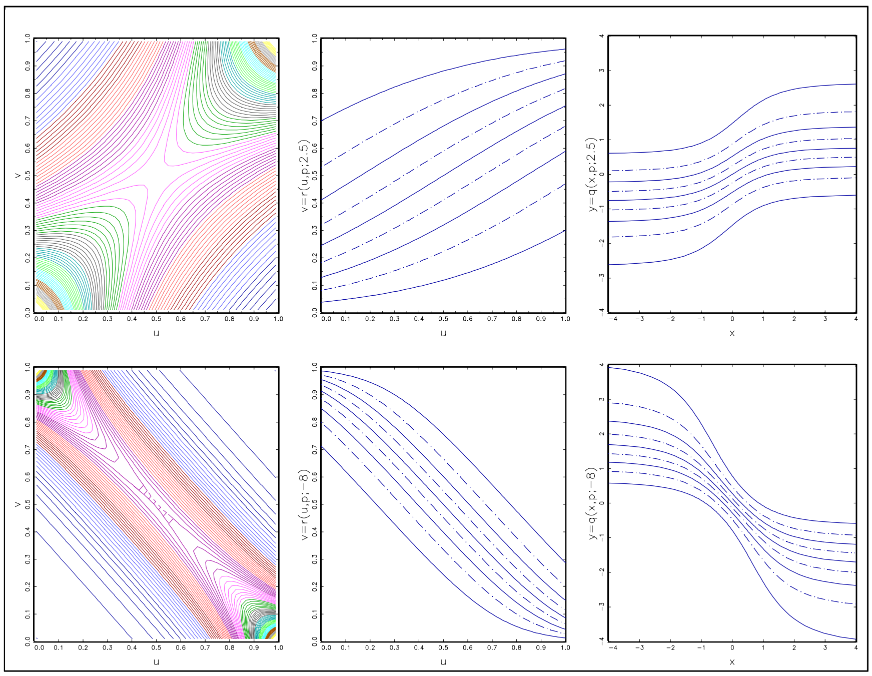

Figure 3: Frank copula densities, copula p^{th} quantile curves (for $p = .1, .2, \ldots, .9$) for (u, v) and (x, y) under the hypothesis of Student margins ($\nu = 3$) for $\delta = 2.5$ (upper plots) and $\delta = -8$ (lower plots)

with $\mathcal{T}_p = \{t : y_t \geq \mathbf{q}(x_t, p; \delta)\}$ *and* $\mathcal{T}_{1-p}$ *its complement. This can be expressed alternatively as:*

$$\min_{\delta} \left(\sum_{t=1}^{T} \left(p - \mathbb{I}_{\{y_t \leq \mathbf{q}(x_t, p; \delta)\}}\right) \left(y_t - \mathbf{q}(x_t, p; \delta)\right) \right) \tag{28}$$

This definition indicates that the estimate of the dependence parameter δ is provided by an L^1 norm estimator. This problem has already been investigated by Koenker and Park [1996] who propose an algorithm for problems with response functions that are non-linear in parameters. We refer the reader to Koenker and Park's original article for a detailed discussion of the development of an interior point algorithm to solve the estimation problem. The main idea is to solve the non-linear L^1 problem by splitting it into a succession of linear L^1 problems.

It might be surprising that the probability level p appears in equation (27) as an argument of the function $\mathbf{q}$ itself. This is simply because we have adopted a top-down strategy in our modelling by first specifying the joint distribution and then deriving the implied quantile function. By postulating given margins for X and Y and their copula, we implicitly assume a specific parametric functional for $\mathbf{q}(x_t, p; \delta)$. In fact, the probability level is implicit in the original quantile regression definition of Koenker and Bassett [1978].

5.2 Application to FX markets and Measuring Tail area Dependency

We now turn to consider the form of dependency between exchange rates. We start by considering the static inter-relationship between the Dollar -Yen, Dollar-Sterling and Dollar- DM rates using 522 weekly returns on the exchange rates from August 1992 to August 2002 as shown in the following figure.We then turn to consider the dynamic evolution of conditional quantiles both within and between these rates. All three exchange rates fail univariate normality tests with excess kurtosis and positive skew except for Sterling -Dollar which shows a negative skew over the sample period.

We then compute the nonlinear quantile regression estimates of $\hat{\rho}(p)$ such that:

$$\hat{\rho}(p) = \arg\min \left(\sum_{t=1}^{T} \left(p - \mathbb{I}_{\left\{S_{1t} \leq \mathbf{q}\left(S_{2t}, p; \rho, \hat{\theta}_1, \hat{\theta}_2\right)\right\}}\right) \left(S_{1t} - \mathbf{q}\left(S_{2t}, p; \rho, \hat{\theta}_1, \hat{\theta}_2\right)\right) \right) \tag{29}$$

Figure 4: Exchange rates levels

Assuming a gaussian copula the relationship between any two exchange rates S_1 and S_2 at the p'th-quantile is:

$$S_1 = \hat{F}_1^{[-1]} \left[\Phi \left(\hat{\rho}\left(p\right) \Phi^{[-1]} \left(\hat{F}_2\left(S_2\right) \right) + \sqrt{1 - \hat{\rho}^2\left(p\right)} \Phi^{[-1]}\left(p\right) \right) \right], \tag{30}$$

with $\hat{F}_1$ and $\hat{F}_2$ the *empirical* marginal distribution functions for exchange rates 1 and 2 respectively. The results for estimates of the dependency parameter, the correlation at each quantile level $\hat{\rho}\left(p\right)$ expressed in percentage terms, are reported in Table 6 below together with their empirical standard deviations. The mean regression results are also reported for information. The lower p the lower the regression curve.

The copula quantile regression results in Tables 1 and 2 above indicate significant dependence using standard inference procedures at all probability levels and for all exchange rates using the Gaussian copula. There is a relatively low degree of association indicated between the Yen:Dollar and the Sterling:Dollar rates and a much higher association indicated at all quantile levels for the Dollar: Sterling and Dollar:DM rates.A fairly symmetric degree of dependence is indicated as we range from the 5% quantile to the 95% quantile with relatively minor differ-

S_1	**USD/Y**	**USD/Y**	**USD/£**
S_2	**USD/£**	**USD/DM**	**USD/DM**
5%	14.2%	37.7%	49.1%
	(5.4%)	*(3.5%)*	*(4.6%)*
10%	16.5%	31.9%	57.2%
	(4.7%)	*(4.2%)*	*(4.0%)*
50%	20.2%	32.9%	72.0%
	(3.8%)	*(4.0%)*	*(3.1%)*
90%	14.1%	28.5%	63.2%
	(5.5%)	*(4.7%)*	*(3.6%)*
95%	13.2%	23.3%	55.8%
	(5.9%)	*(5.7%)*	*(4.1%)*
mean regression	18.3%	32.0%	65.2%
	(4.2%)	*(4.2%)*	*(3.5%)*

Table 1: C- Quantile Regression Estimates baserd on Gaussian Copula

S_1	**USD/£**	**USD/DM**	**USD/DM**
S_2	**USD/Y**	**USD/Y**	**USD/£**
5%	14.4%	21.4%	51.2%
	(5.9%)	*(6.2%)*	*(4.0%)*
10%	17.5%	20.1%	57.7%
	(4.9%)	*(6.6%)*	*(3.6%)*
50%	20.5%	33.4%	64.3%
	(4.1%)	*(4.0%)*	*(3.2%)*
90%	22.8%	37.1%	66.1%
	(3.7%)	*(3.6%)*	*(3.1%)*
95%	16.9%	34.3%	51.2%
	(5.0%)	*(3.9%)*	*(4.0%)*
mean regression	19.2%	32.0%	62.1%
	(4.4%)	*(4.2%)*	*(3.3%)*

Table 2: C-Quantile Regression Estimates based on Gaussian Copula

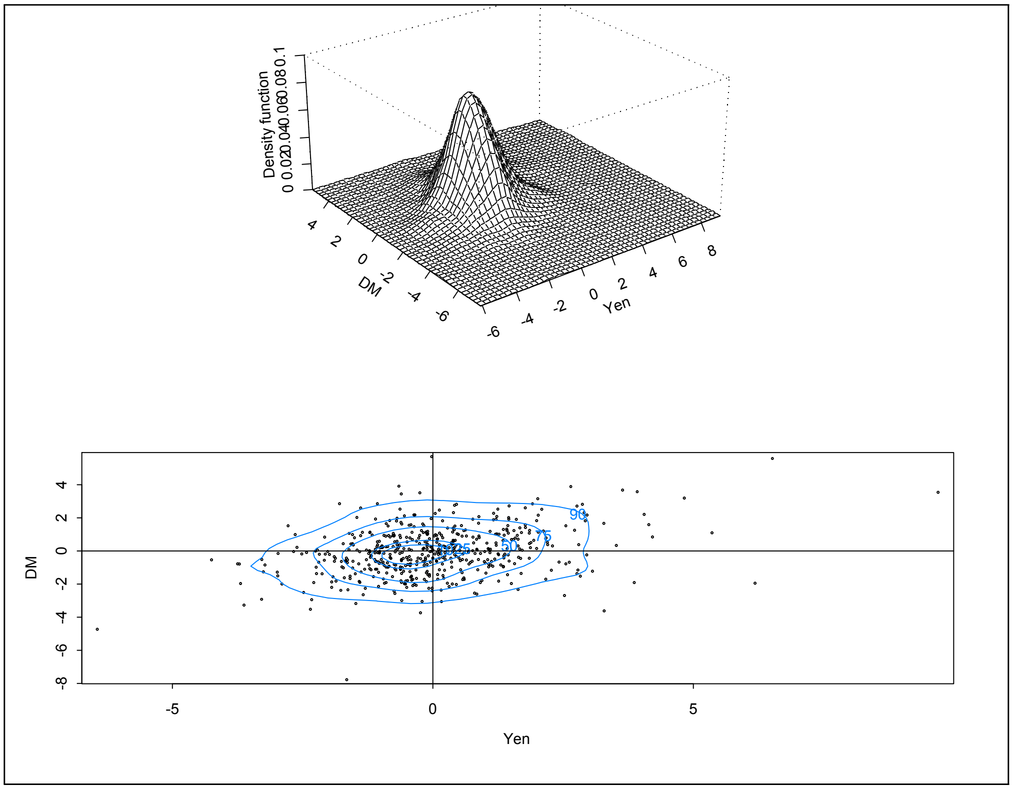

Figure 5: Empirical Yen/DM bivariate return distribution

ences from the mean regression results.We find the same qualitative conclusions in these two cases when we reverse the causality in Table 2. What are striking however are the results for Yen:Dollar and DM:Dollar dependency patterns revealed in the C-quantile regressions. In particular we can see a clear asymmetric structure in the dependency between the lower quantiles and the upper quantiles with much stronger dependency being shown in the lower quantiles when the Yen is the dependent variable (and vice versa in the upper quantiles when the DM is the dependent variable). Use of the mean or median regression in this case could give a substantially misleading idea of the relative joint risks.

Using the Clayton Joe copula Tables 3 and 4 show the some indication of upper tail dependence in the Yen DM dollar rates in levels and Sterling:DM dollarrates in the uppertil in returns. Some lower tail dependence is found for the Yen:Sterling Dollar rates and Sterling Dm Dollar rates in levels and Steringl: DM in returns. Otherwise we find dependence at all.

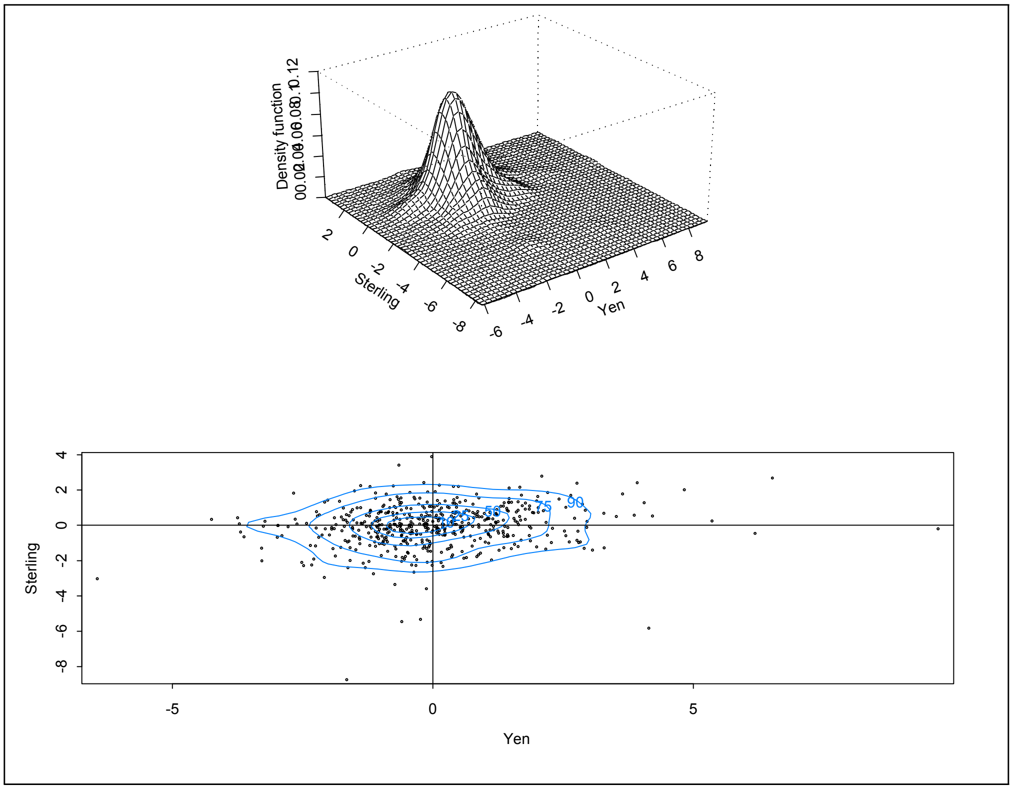

Figure 6: Empirical Yen/Sterling bivariate return distribution

6 Tail Area Dependency

Several dependence measures for extremes, λ - so called *tail dependence measures* have been developed where asymptotic independence is given by $\lambda = 0$ and $\lambda_u \in (0, 1]$ for upper tail dependence and where $\lambda_l \in (0, 1]$ may be similarly defined for lower tail dependency.

λ_u is linked to the asymptotic behaviour of the copula:

$$\begin{aligned}
\lambda_u &= \lim_{\alpha \to 1^-} \Pr\{X_2 > VaR_\alpha(X_2) | X_1 > VaR_\alpha(X_1)\} \\
&= \lim_{\alpha \to 1^-} \frac{1 - 2\alpha + \bar{C}(\alpha, \alpha)}{1 - \alpha}
\end{aligned}$$

or, alternatively (see Embrechts P., A. McNeil and D. Straumann (1999)):

$$\begin{aligned}
\lambda_u &= -\lim_{x \to 1^-} \frac{d(1 - 2x + \bar{C}(x, x))}{dx} \\
&= \lim_{x \to 1^-} \Pr\{U_2 > x | U_1 = x\} + \lim_{x \to 1^-} \Pr\{U_1 > x | U_2 = x\} \\
&= 2 \lim_{x \to 1^-} \Pr\{U_2 > x | U_1 = x\}
\end{aligned}$$

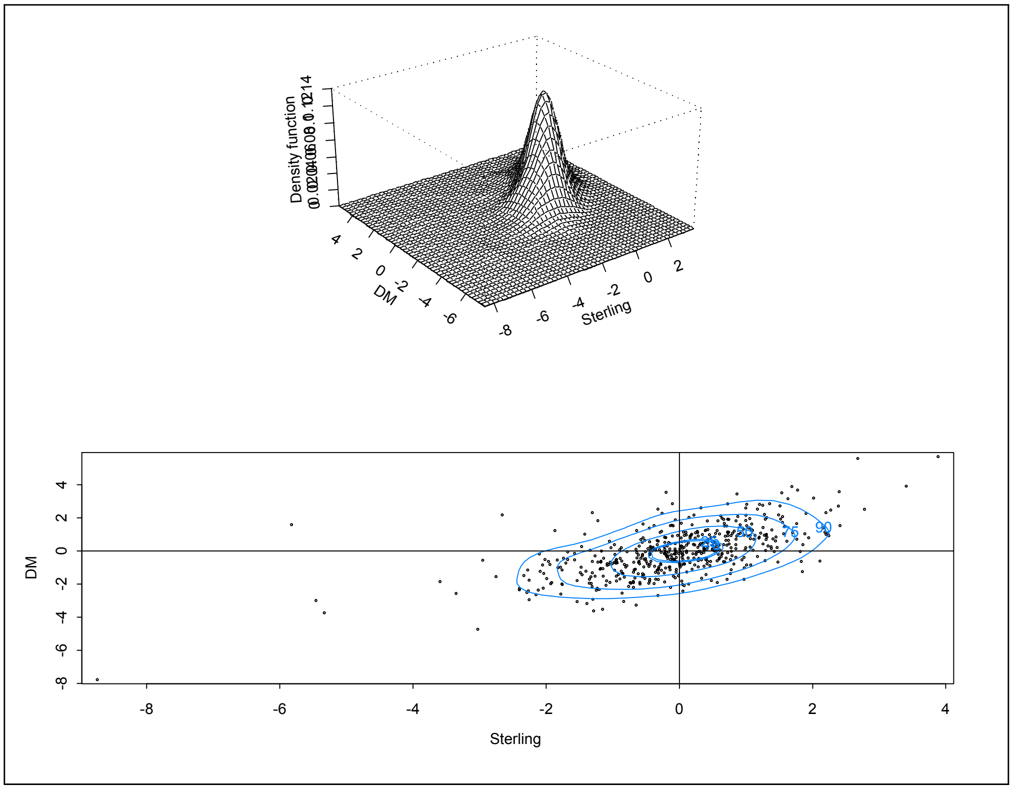

Figure 7: Empirical Sterling/DM bivariate return distribution

Applying the same transformation F_1^{-1} to both marginals, and $(X,Y)^\top \sim C(F_1(x),F_1(y))$,

$$\begin{aligned}\lambda_u &= 2\lim_{x\to\infty} \Pr\{F_1^{-1}(U_2) > x | F_1^{-1}(U_1) = x\} \\ &= 2\lim_{x\to\infty} \Pr\{Y > x | X = x\}\end{aligned}$$

An alternative interpretation is that $\lambda(u)$ may be viewed as a quantile dependent measure of dependence (Coles,Currie and Tawn (Lancaster University, [1999]) .Then

$$\lambda_u(u) = \Pr[U_1 > u | U_2 > u] = \frac{\bar{C}(u,u)}{1-u}$$

and

$$\lambda_u = \lim_{u\to 1}\frac{\bar{C}}{1-u}$$

Where

$$\bar{C}(u_1,u_2) = \tilde{C}(1-u_1,1-u_2)$$

A major problem with interpreting asymptotic tail area dependency is that independence in the sense of *the factorisation of the bivariate distribution* in the tails implies $\lambda_u = 0$ but $\lambda_u = 0$

r_t	**USD/Y**		**USD/Y**		**USD/£**	
r_t	**USD/£**		**USD/DM**		**USD/DM**	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.07	0.00	1.17	0.00	1.42	0.19
10%	1.07	0.00	1.17	0.00	1.39	0.21
50%	1.06	0.03	1.13	0.09	1.21	0.37
90%	1.05	0.08	1.10	0.21	1.07	0.53
95%	1.04	0.09	1.09	0.23	1.06	0.55

Table 3: C-Quantile Regression estimates of the relative returns of the exchange rates $r_t = S_t/S_{t-1} - 1$.

S_t	**USD/Y**		**USD/Y**		**USD/£**	
S_t	**USD/£**		**USD/DM**		**USD/DM**	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.01	0.67	1.39	0.00	1.04	0.44
10%	1.02	0.54	1.39	0.00	1.03	0.44
50%	1.00	0.00	1.37	0.00	1.00	0.35
90%	1.00	0.00	1.25	0.17	1.00	0.15
95%	1.00	0.00	1.24	0.20	1.00	0.11

Table 4: C-Quantile Regression estimates of the levels of the exchange rates S_t.

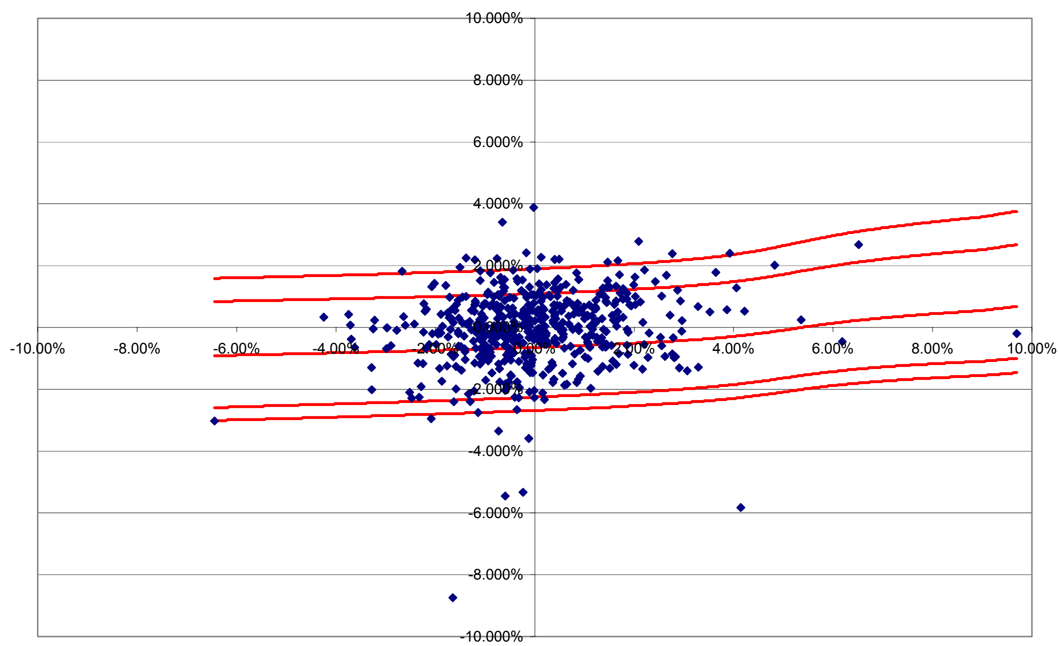

Figure 8: Nonlinear quantile regression of USD/Y on $USD/£$ for 5%, 10%, 50%, 90%, 95% probability levels.

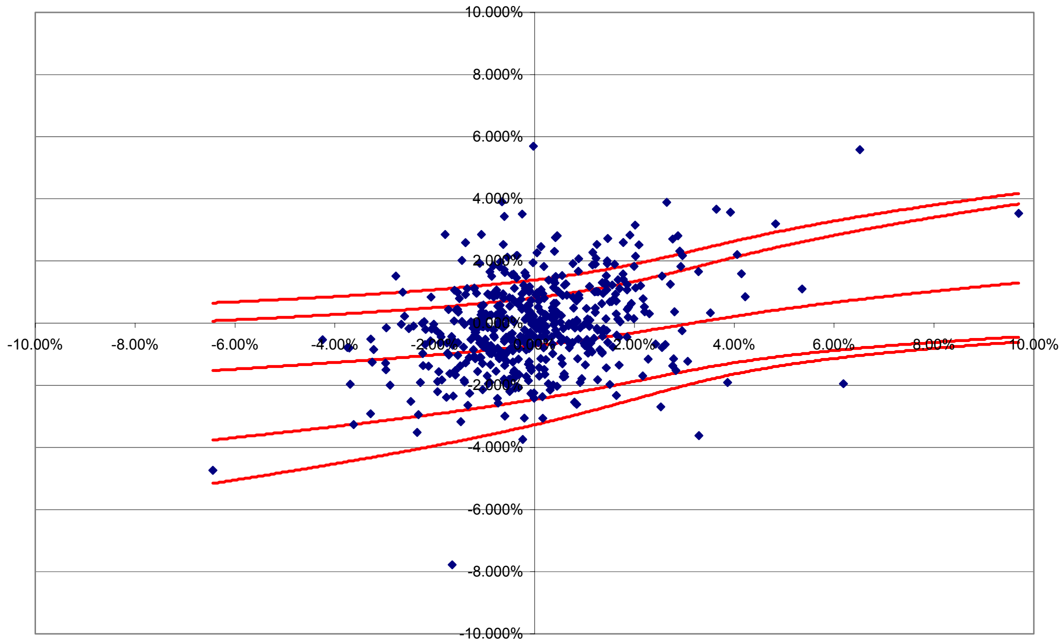

Figure 9: Nonlinear quantile regression of USD/Y on USD/DM for 5%, 10%, 50%, 90%, 95% probability levels.

	Upper Tail Dependency λ_u	Lower Tail Dependency λ_l
	$\alpha = .975$	$\alpha = 0.2$
Yen-Sterling	0.01538	0.926
Yen-DM	0.03846	0.8349
DM-Sterling	0.3076	0.8780

Table 5: Upper and Lower Tail index Estimates

does not imply factorization and hence independence. There may still be dependence in the tails even though $\lambda_u = 0$.Several alternative conditions must be used to ensure factorization, Ledford and Tawn[1998], for instance we also need to satisfy $\bar{\lambda} = 0$ where if $\bar{\lambda} > 0$ large values occur simultaneously more frequently than if they were independent;

$$\bar{\lambda} = \lim_{u \to 1} \frac{2 \log \Pr\{X > F_X^{-1}(u)\}}{\log \Pr\{X > F_X^{-1}(u), Y > F_Y^{-1}(u)\}} - 1$$

Applying these measures to numerically compute the upper and lower tail indices for the three exchange rates we find in Table 5;

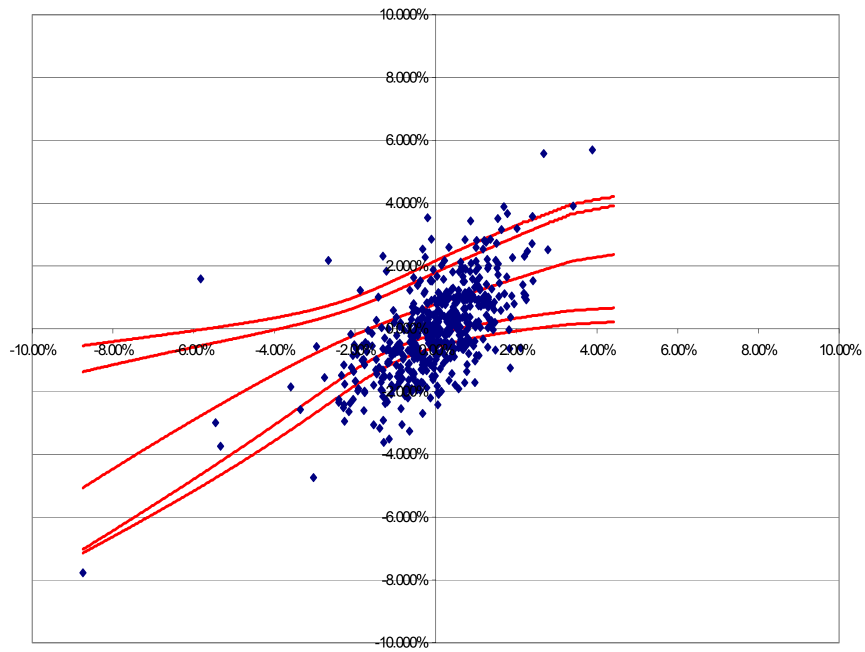

Figure 10: Nonlinear quantile regression of $USD/£$ on USD/DM for 5%, 10%, 50%, 90%, 95% probability levels.

	Upper Tail Dependency λ_u	**Lower Tail Dependency λ_l**
Yen-Sterling	0.08	0.0
Yen-DM	0.122	0.03
DM-Sterling	0.227	0.25

Table 6: Parametric Upper and Lower Tail index Estimates

first it would seem that there is some question regarding the reliability of these estimates of asymptotic dependence based on the empirical copula . We see unreasonably high dependence in the lower tail for all pairs of rates and reasonable low dependence in the upper tail for Yen:Dollar and Sterling :Dollar rates but somewhat higher upper tail dependence for the DM:Dollar and Sterling :Dollar Rates. These results are not immediately consistent with the information provided by the C-Quantile estimates indicating that we are getting distinct information from the two different approaches to measuring conditional dependency.It is clear we get substantially more information regarding the joint risk structure from the C-Quantile regressions and the question that is raised is whether we are really interested in asymptotic dependence or moderate extreme dependence. Rather than using the empirical copula to compute these tail area dependence paramters it may be that more reliable estimates can be found from estimating the parameters of the relevant copula and then calculating the tail dependence measures directly.

The lower and upper tail dependence measures for archimedean copula are defined in general by

$$\begin{cases} \lambda_l = \lim\limits_{\alpha \to 1^-} \frac{1-2\alpha+\phi^{-1}(2\phi(\alpha))}{1-\alpha} \\ \lambda_u = \lim\limits_{\alpha \to 0^+} \frac{\phi^{-1}(2\phi(\alpha))}{\alpha} \end{cases} \tag{31}$$

and for the Clayton Joe Copula specifically are given by

$$\begin{cases} \lambda_l = 2^{-1/\delta} \\ \lambda_u = 2 - 2^{1/\theta} \end{cases} \tag{32}$$

which gives rise to the following parametric estimates

These estimates correspond fairly well with the empirical copula based estimates reported above for the upper tail dependence but differ substantially for the lower asymptotic tail dependence.

	USD/Y		USD/£		USD/DM	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.03	0.00	1.00	0.04	1.00	0.06
	(0.02)	-	-	*(0.02)*	-	*(0.03)*
10%	1.03	0.00	1.00	0.03	1.00	0.05
	(0.02)	-	-	*(0.02)*	-	*(0.03)*
50%	1.04	0.00	1.00	0.00	1.00	0.04
	(0.02)	-	-	-	-	*(0.02)*
90%	1.05	0.00	1.00	0.00	1.00	0.02
	(0.02)	-	-	-	-	*(0.01)*
95%	1.05	0.00	1.00	0.00	1.00	0.02
	(0.02)	-	-	-	-	*(0.01)*

Table 7: C-Quantile Regression estimates of the returns of the exchange rates S_t on S_{t-1}.

6.1 Dynamic c-quantiles

We next compute the nonlinear *dynamic* quantile regression estimates $\left(\hat{\delta}(p), \hat{\theta}(p)\right)$ for the Clayton Joe Copula such that:

$$\left(\hat{\delta}(p), \hat{\theta}(p)\right) = \arg\min \left(\sum_{t=1}^{T} \left(p - \mathbb{I}_{\{S_t \leq \mathbf{q}(S_{t-1},p;\delta,\theta)\}}\right) \left(S_t - \mathbf{q}\left(S_{t-1}, p; \delta, \theta\right)\right) \right) \tag{33}$$

with

$$\mathbf{q}\left(S_{t-1}, p; \delta, \theta\right) = \hat{F}^{[-1]} \left[\phi_{\delta,\theta}^{-1} \left[\phi_{\delta,\theta} \left(\phi_{\delta,\theta}\prime^{-1} \left(\frac{1}{p} \phi_{\delta,\theta}\prime \left(\hat{F}\left(S_{t-1}\right)\right)\right)\right) - \phi_{\delta,\theta}\left(\hat{F}\left(S_{t-1}\right)\right)\right]\right] \tag{34}$$

with $\phi_{\delta,\theta}$ the generator of the copula defined in equation (22) and $\hat{F}$ the empirical distribution function of the exchange rate S_t. The estimates are given below for the levels and relative returns.

The results are summarized in the following table:

	USD/Y	USD/£	USD/DM
model	Joe	none	Kimeldorf & Sampson
lower dependence	no	no	yes
upper dependence	yes	no	no

S_{t-1}	**USD/Y**		**USD/£**		**USD/DM**	
S_t	**USD/Y**		**USD/Y**		**USD/Y**	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	2.95	1.16	1.00	0.01	1.35	0.00
10%	2.91	1.18	1.00	0.02	1.34	0.00
50%	2.57	1.34	1.00	0.10	1.30	0.07
90%	2.30	1.45	1.00	0.27	1.20	0.27
95%	2.27	1.46	1.00	0.30	1.19	0.30

Table 8: C-Quantile Regression estimates of the level of the exchange rate S_t on S_{t-1}.

S_{t-1}	**USD/Y**		**USD/£**		**USD/DM**	
S_t	**USD/£**		**USD/£**		**USD/£**	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.00	0.68	2.38	0.64	1.00	0.47
10%	1.01	0.55	2.33	0.68	1.00	0.46
50%	1.00	0.00	1.94	1.04	1.00	0.35
90%	1.00	0.00	1.57	1.50	1.00	0.22
95%	1.00	0.00	1.53	1.55	1.00	0.19

Table 9: C-Quantile Regression estimates of the level of the exchange rate S_t on S_{t-1}.

S_{t-1}	**USD/Y**		**USD/£**		**USD/DM**	
S_t	**USD/DM**		**USD/DM**		**USD/DM**	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.38	0.00	1.04	0.40	3.43	0.64
10%	1.38	0.00	1.03	0.40	3.34	0.69
50%	1.36	0.00	1.00	0.33	2.53	1.18
90%	1.25	0.17	1.00	0.19	1.82	1.71
95%	1.23	0.20	1.00	0.16	1.75	1.76

Table 10: C-Quantile Regression estimates of the level of the exchange rate S_t on S_{t-1}.

r_{t-1}	**USD/Y**		**USD/£**		**USD/DM**	
r_t	**USD/Y**		**USD/Y**		**USD/Y**	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.00	0.00	1.00	0.00	1.00	0.00
10%	1.01	0.00	1.00	0.00	1.00	0.00
50%	1.03	0.00	1.00	0.00	1.00	0.00
90%	1.05	0.00	1.01	0.00	1.01	0.00
95%	1.05	0.00	1.01	0.00	1.02	0.00

Table 11: C-Quantile Regression estimates of the relative return of the exchange rate $r_t = S_t/S_{t-1} - 1$ on r_{t-1}.

r_{t-1}	**USD/Y**		**USD/£**		**USD/DM**	
r_t	**USD/£**		**USD/£**		**USD/£**	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.02	0.00	1.00	0.08	1.00	0.06
10%	1.02	0.00	1.00	0.07	1.00	0.06
50%	1.02	0.00	1.00	0.01	1.00	0.04
90%	1.02	0.00	1.00	0.00	1.02	0.00
95%	1.02	0.00	1.00	0.00	1.03	0.00

Table 12: C-Quantile Regression estimates of the relative return of the exchange rate $r_t = S_t/S_{t-1} - 1$ on r_{t-1}.

r_{t-1}	**USD/Y**		**USD/£**		**USD/DM**	
r_t	**USD/DM**		**USD/DM**		**USD/DM**	
p	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$	$\hat{\theta}(p)$	$\hat{\delta}(p)$
5%	1.00	0.02	1.00	0.08	1.00	0.06
10%	1.00	0.02	1.00	0.07	1.00	0.06
50%	1.00	0.02	1.00	0.04	1.00	0.04
90%	1.00	0.01	1.00	0.00	1.02	0.00
95%	1.00	0.01	1.00	0.00	1.02	0.00

Table 13: C-Quantile Regression estimates of the relative return of the exchange rate $r_t = S_t/S_{t-1} - 1$ on r_{t-1}.

Tables 7 to 9 show the dynamic dependence between the *levels* of the exchange rates. The Sterling Yen and Sterling DM rates with the Dollar show no upper tail dependence. Own lag dependence is strong and varies consitently with the quantile level. The Yen DM Dollar rates show relatively strong upper tail dynamic dependence but otherwise there seems to be relatively little strong dynamic dependence in the tails of these exchange rates.

Tables 10 to 12 show that there is no dynamic dependence at *any* quantile level between the returns of the exchange rates at least at the weekly level that our data has been recorded. The Clayton Joe parameter estimates indicate independence even in the relative extremes of the joint distribution.

7 Conclusion

These results are indicative of the structure that can be uncovered using copula based quantile regressions. We note that the independence shown between the returns of the own exchange rates applies at all quantiles and hence a much stronger "efficiency" condition seems to apply, even into the tails of the distribution than implied by standard martingale efficiency conditions which involve the conditional mean.

We intend to move from this point to consider dynamic copula based risk measures using derivatives of these c-quantile estimates- in particular following TASCHE[2000] to compute expected shortfall measures.

Copula	DM/Sterling	DM/Yen	Yen/Sterling
Gaussian	136.56	27.09	8.39
Frank	127.93	22.10	7.29
K-Sampson	110.62	18.32	5.66
Gumbel	133.73	32.51	9.77
Galambos	132.68	29.81	7.44
Husler-Reiss	128.96	26.88	6.12
Tawn	133.49	32.96	11.40
bb1	144.76	32.83	9.52
bb2	109.61	17.32	4.66
bb3	171.95*	49.47*	20.26*
bb4	144.00	30.50	7.63
bb5	132.72	31.51	8.77
bb7	141.88	33.40	9.46
Joe	103.29	27.32	6.85

Table 14: Penalised Maximum Likelihood Values

Appendix: Statistical Selection of Copula

Table 13 presents penalised likelihood statistics, similar to Akaike's Information Criterion. The copula best supported by the sample data displays the maximum value in the table.

It seems clear from these calculations that the **bb3** family of copula (see Joe [1997]) is best able to describe the exchange rates we have considered.

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