Multivariate exponential integral approximations: a moment approach

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Abstract

We propose a method to approximate a class of exponential multivariate integrals using moment relaxations. Using this approach, both lower and upper bounds of the integrals are obtained and we show that these bound values asymptotically converge to the real value of the integrals when the moment degree r increases. We further demonstrate the method by calculating both hypercubic and order statistic probabilities for multivariate normal distributions.

1 Introduction

Multivariate integrals arise in statistic, physics, engineering and even finance applications. For example, these integrals are needed to calculate probabilities over compact sets for multivariate normal random variables. It is therefore important to compute or approximate multivariate integrals. Usual methods include Monte Carlo schemes (see Niederreiter [5] for details) and some cubature formulae as shown in de la Harpe and Pache [2]. However, there are still many open problems currently and research on multivariate integrals is very much active due to its importance as well as its difficulties.

Contributions and Paper Outline

In this paper, we attempt to approximate a class of exponential integrals, which is useful to calculate probabilities of multivariate normal random variables. Specifically, our contributions and structure of the paper are as follows:

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(1) In Section 2, we provide a general framework to calculate lower and upper bounds for the exponential integrals mentioned above. These bounds are calculated by solving specific semidefinite programming problems constructed from appropriate sequences of moments.

(2) In Section 3, we prove that the two monotone sequences of lower and upper bounds generated by these semidefinite programming problems will asymptotically converge to the real value of the integral. The proof is due to some results from the problem of moments.

(3) In Section 4, computational results are reported for probabilities of multivariate random variables and their order statistics over a hypercube. These results show that the proposed method is indeed applicable for this class of integrals.

2 General Framework

2.1 Recursive Formula

The class of multivariate exponential integrals considered in this paper has the form

$$\rho = \int_{\Omega} g(\boldsymbol{x}) e^{h(\boldsymbol{x})} \mathrm{d}\boldsymbol{x} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $g, h \in \mathbb{R}[\boldsymbol{x}]$, the ring of real polynomials and $\Omega \subset \mathbb{R}^n$ is a compact set defined as

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^n : b_1^{(1)} \leq x_1 \leq b_1^{(2)}, b_i^{(1)}(\boldsymbol{x}[i-1]) \leq x_i \leq b_i^{(2)}(\boldsymbol{x}[i-1]) \quad \forall i = 2, \ldots, n\} \tag{2}$$

where $\boldsymbol{x}[i] \in \mathbb{R}^i$ is the vector of first i elements of $\boldsymbol{x}$, $i = 1, \ldots, n$, $b_i^{(1)}, b_i^{(2)} \in \mathbb{R}[\boldsymbol{x}[i-1]]$, $i = 2, \ldots, n$, and $b_1^{(1)}, b_1^{(2)}$ are constants.

Define $\Omega_k = \{\boldsymbol{x} \in \mathbb{R}^k : b_1^{(1)} \leq x_1 \leq b_1^{(2)}, b_i^{(1)}(\boldsymbol{x}[i-1]) \leq x_i \leq b_i^{(2)}(\boldsymbol{x}[i-1]) \quad \forall i = 2, \ldots, k\}$, $k = 1, \ldots, n$, we have: $\Omega_k \in \mathbb{R}^k$ and $\Omega_n = \Omega$. Clearly, these Ω_k sets are also compact in $\mathbb{R}^k$ for all $k = 1, \ldots, n$. Let us consider the measure $\mu_h^{(i)}$ on $\mathbb{R}^i$ defined by

$$\mu_h^{(i)}(B) = \int_{\Omega_i \cap B} e^{h(\boldsymbol{x}[i])} \mathrm{d}\boldsymbol{x}[i] \tag{3}$$

where $h \in \mathbb{R}[\boldsymbol{x}[i]]$, $B \in \mathcal{B}(\mathbb{R}^i)$, and its sequence of moments $\boldsymbol{z}_h^{(i)} = \{z_h^{(i)}(\boldsymbol{\alpha})\}$:

$$z_h^{(i)}(\boldsymbol{\alpha}) = \int_{\Omega_i} (\boldsymbol{x}[i])^{\boldsymbol{\alpha}} e^{h(\boldsymbol{x}[i])} \mathrm{d}\boldsymbol{x}[i] \tag{4}$$

for all $\boldsymbol{\alpha} \in \mathbb{N}^i$.

We then have $\rho = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} g_{\boldsymbol{\alpha}} z_h^{(n)}(\boldsymbol{\alpha}) = \langle \boldsymbol{g}, \boldsymbol{z}_h^{(n)}\rangle$, where $g_{\boldsymbol{\alpha}}$ is the coefficient of monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$ and $\boldsymbol{g}$ is the sequence of those coefficients. Therefore, what we need to do is to calculate $z_h^{(n)}(\boldsymbol{\alpha})$ for all necessary $\boldsymbol{\alpha} \in \mathbb{N}^i$. Using integration by parts, we have:

$$z_h^{(n)}(\boldsymbol{\alpha}) = \frac{1}{\alpha_n + 1}(A_{\boldsymbol{\alpha}} - B_{\boldsymbol{\alpha}})$$

where

$$A_{\boldsymbol{\alpha}} = \int_{\Omega_{n-1}} (\boldsymbol{x}[n-1])^{\boldsymbol{\alpha}[n-1]} \left[x_n^{\alpha_n+1} e^{h(\boldsymbol{x})}\right]_{b_n^{(1)}(\boldsymbol{x}[n-1])}^{b_n^{(2)}(\boldsymbol{x}[n-1])} \mathrm{d}\boldsymbol{x}[n-1]$$

and

$$B_{\boldsymbol{\alpha}} = \int_{\Omega} (\boldsymbol{x}[n-1])^{\boldsymbol{\alpha}[n-1]} x_n^{\alpha_n+1} \frac{\partial h(\boldsymbol{x})}{\partial x_n} e^{h(\boldsymbol{x})} \mathrm{d}\boldsymbol{x}$$

We have:

$$\frac{\partial h(\boldsymbol{x})}{\partial x_n} = \sum_{\boldsymbol{\beta}\in\mathbb{N}^n} \beta_n h_{\boldsymbol{\beta}} (\boldsymbol{x}[n-1])^{\boldsymbol{\beta}[n-1]} x_n^{\beta_n - 1}$$

Therefore,

$$B_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}\in\mathbb{N}^n} \beta_n h_{\boldsymbol{\beta}} \int_{\Omega} (\boldsymbol{x}[n-1])^{\boldsymbol{\alpha}[n-1]} x_n^{\alpha_n+1} (\boldsymbol{x}[n-1])^{\boldsymbol{\beta}[n-1]} x_n^{\beta_n-1} e^{h(\boldsymbol{x})} \mathrm{d}\boldsymbol{x}$$

or

$$B_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}\in\mathbb{N}^n} \beta_n h_{\boldsymbol{\beta}} \int_{\Omega} \boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\beta}} e^{h(\boldsymbol{x})} \mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{\beta}\in\mathbb{N}^n} \beta_n h_{\boldsymbol{\beta}} z_h^{(n)}(\boldsymbol{\alpha}+\boldsymbol{\beta})$$

Now consider $A_{\boldsymbol{\alpha}}$, we have: $A_{\boldsymbol{\alpha}} = A_{\boldsymbol{\alpha}}^2 - A_{\boldsymbol{\alpha}}^1$ where

$$A_{\boldsymbol{\alpha}}^i = \int_{\Omega_{n-1}} (\boldsymbol{x}[n-1])^{\boldsymbol{\alpha}[n-1]} [b_n^{(i)}(\boldsymbol{x}[n-1])]^{\alpha_n+1} e^{h(\boldsymbol{x}[n-1], b_n^{(i)}(\boldsymbol{x}[n-1]))} \mathrm{d}\boldsymbol{x}[n-1], \quad i = 1, 2$$

Let

$$g_{i,\boldsymbol{\alpha}}^{(n-1)}(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{\alpha}[n-1]} [b_n^{(i)}(\boldsymbol{x})]^{\alpha_n+1}, \quad \boldsymbol{x} \in \mathbb{R}^{n-1}, i = 1, 2$$

and

$$h_i^{(n-1)}(\boldsymbol{x}) = h(\boldsymbol{x}, b_n^{(i)}(\boldsymbol{x})), \quad \boldsymbol{x} \in \mathbb{R}^{n-1}, i = 1, 2$$

All of these four functions are polynomials in $\mathbb{R}[\boldsymbol{x}[n-1]]$. Define two measures $\mu_1^{(n-1)} \equiv \mu_{h_1^{(n-1)}}^{(n-1)}$ and $\mu_2^{(n-1)} \equiv \mu_{h_2^{(n-1)}}^{(n-1)}$ over Ω_{n-1} on $\mathbb{R}^{n-1}$ as defined in (3). With their two respective sequences of moments $\boldsymbol{z}_1^{(n-1)} \equiv \boldsymbol{z}_{h_1^{(n-1)}}^{(n-1)}$ and $\boldsymbol{z}_2^{(n-1)} \equiv \boldsymbol{z}_{h_2^{(n-1)}}^{(n-1)}$ as defined in (4), we have:

$$A_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta}\in\mathbb{N}^{n-1}} \left(g_{2,\boldsymbol{\alpha}}^{(n-1)}\right)_{\boldsymbol{\beta}} z_2^{(n-1)}(\boldsymbol{\beta}) - \sum_{\boldsymbol{\beta}\in\mathbb{N}^{n-1}} \left(g_{1,\boldsymbol{\alpha}}^{(n-1)}\right)_{\boldsymbol{\beta}} z_1^{(n-1)}(\boldsymbol{\beta})$$

Thus we have:

$$z_h^{(n)}(\boldsymbol{\alpha}) = \sum_{\boldsymbol{\beta}\in\mathbb{N}^{n-1}} \frac{\left(g_{2,\boldsymbol{\alpha}}^{(n-1)}\right)_{\boldsymbol{\beta}}}{\alpha_n+1} z_2^{(n-1)}(\boldsymbol{\beta}) - \sum_{\boldsymbol{\beta}\in\mathbb{N}^{n-1}} \frac{\left(g_{1,\boldsymbol{\alpha}}^{(n-1)}\right)_{\boldsymbol{\beta}}}{\alpha_n+1} z_1^{(n-1)}(\boldsymbol{\beta}) - \sum_{\boldsymbol{\beta}\in\mathbb{N}^{n}} \frac{\beta_n h_{\boldsymbol{\beta}}}{\alpha_n+1} z_h^{(n)}(\boldsymbol{\alpha}+\boldsymbol{\beta}) \tag{5}$$

Equation (5) shows that in order to calculate $\boldsymbol{z}_h^{(n)}$, we need to calculate $\boldsymbol{z}_i^{(n-1)}$, $i = 1, 2$, which can then be calculated by some other moment sequences in lower dimensions. Let denote $h_1^{(n)}$ to be the function h and define

$$h_i^{(k)}(\boldsymbol{x}) = h_{\lceil i/2 \rceil}^{(k+1)}(\boldsymbol{x}, b_{k+1}^{(2-\lceil i/2\rceil + \lfloor i/2 \rfloor)}(\boldsymbol{x})), \quad \boldsymbol{x} \in \mathbb{R}^k, k = 1, \ldots, n-1, i = 1, \ldots, 2^{n-k} \tag{6}$$

For each function $h_i^{(k)}$, a measure $\mu_i^{(k)}$ and its moments sequence $\boldsymbol{z}_i^{(k)}$ are also defined over Ω_k in $\mathbb{R}^k$ as in (3) and (4) respectively. We also need to define the function $g_{i,\boldsymbol{\alpha}}^{(k)}$. Let define

$$g_{i,\boldsymbol{\alpha}}^{(k)}(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{\alpha}[k]}[b_{k+1}^{(i)}(\boldsymbol{x})]^{\alpha_{k+1}+1}, \quad \boldsymbol{x} \in \mathbb{R}^k, \boldsymbol{\alpha} \in \mathbb{N}^k, i = 1, 2 \tag{7}$$

Then we have:

$$z_i^{(k)}(\boldsymbol{\alpha}) = \sum_{\boldsymbol{\beta}\in\mathbb{N}^{k-1}} \frac{\left(g_{2,\boldsymbol{\alpha}}^{(k-1)}\right)_{\boldsymbol{\beta}}}{\alpha_k+1} z_{2i}^{(k-1)}(\boldsymbol{\beta}) - \sum_{\boldsymbol{\beta}\in\mathbb{N}^{k-1}} \frac{\left(g_{1,\boldsymbol{\alpha}}^{(k-1)}\right)_{\boldsymbol{\beta}}}{\alpha_k+1} z_{2i-1}^{(k-1)}(\boldsymbol{\beta}) - \sum_{\boldsymbol{\beta}\in\mathbb{N}^{k}} \frac{\beta_k (h_i^{(k)})_{\boldsymbol{\beta}}}{\alpha_k+1} z_i^{(k)}(\boldsymbol{\alpha}+\boldsymbol{\beta}) \tag{8}$$

or $z_i^{(k)}(\boldsymbol{\alpha}) = R_i^{(k)}(\boldsymbol{z}_i^{(k)}, \boldsymbol{z}_{2i}^{(k-1)}, \boldsymbol{z}_{2i-1}^{(k-1)}, \boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbb{R}^k$, $k = 2, \ldots, n$, and $i = 1, \ldots, 2^{n-k}$.

For $k = 1$ and $i = 1, \ldots, 2^{n-1}$, we have:

$$z_i^{(1)}(\alpha) = \frac{(b_1^{(2)})^{\alpha+1}}{\alpha+1} e^{h_i^{(1)}(b_1^{(2)})} - \frac{(b_1^{(1)})^{\alpha+1}}{\alpha+1} e^{h_i^{(1)}(b_1^{(1)})} - \sum_{\beta\in\mathbb{N}} \frac{\beta (h_i^{(1)})_\beta}{\alpha+1} z_i^{(1)}(\alpha+\beta) = R_i^{(1)}(\boldsymbol{z}_i^{(1)}, \alpha) \tag{9}$$

With the general recursive formula presented in (8) and (9), the sequence of moments $\boldsymbol{z}_h^{(n)}$ (or $\boldsymbol{z}_1^{(n)}$) can now be calculated.

2.2 Moment Relaxation

Let consider the measure $\mu_i^{(k)}$ and its corresponding sequence of moments $\boldsymbol{z}_i^{(k)}$. For any nonnegative integer r, the r-moment matrix associated with $\mu_i^{(k)}$ (or equivalently, with $\boldsymbol{z}_i^{(k)}$) $M_r(\mu_i^{(k)}) \equiv M_r(\boldsymbol{z}_i^{(k)})$ is a matrix of size $\binom{k+r}{r}$. Its rows and columns are indexed in the canonical basis $\{(\boldsymbol{x}[k])^{\boldsymbol{\alpha}}\}$ of $\mathbb{R}[\boldsymbol{x}[k]]$, and its elements are defined as follows:

$$M_r(\boldsymbol{z}_i^{(k)})(\boldsymbol{\alpha}, \boldsymbol{\beta}) = z_i^{(k)}(\boldsymbol{\alpha}+\boldsymbol{\beta}), \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^k, |\boldsymbol{\alpha}|, |\boldsymbol{\beta}| \leq r. \tag{10}$$

Similarly, given $\theta \in \mathbb{R}[\boldsymbol{x}[k]]$, the localizing matrix $M_r(\boldsymbol{\theta} \boldsymbol{z}_i^{(k)})$ associated with $\boldsymbol{z}_i^{(k)}$ and θ is defined by

$$M_r(\boldsymbol{\theta} \boldsymbol{z}_i^{(k)})(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \sum_{\boldsymbol{\gamma} \in \mathbb{N}^k} \theta_{\boldsymbol{\gamma}} z_i^{(k)}(\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma}), \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^k, |\boldsymbol{\alpha}|, |\boldsymbol{\beta}| \le r, \tag{11}$$

where $\boldsymbol{\theta} = \{\theta_{\boldsymbol{\gamma}}\}$ is the vector of coefficients of θ in the canonical basis $\{(\boldsymbol{x}[k])^{\boldsymbol{\alpha}}\}$.

If we define the matrix $M_r^{\boldsymbol{\gamma}}(\boldsymbol{z}_i^{(k)})$ with elements

$$M_r^{\boldsymbol{\gamma}}(\boldsymbol{z}_i^{(k)})(\boldsymbol{\alpha}, \boldsymbol{\beta}) = z_i^{(k)}(\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma}), \quad \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{N}^k, |\boldsymbol{\alpha}|, |\boldsymbol{\beta}| \le r,$$

then the localizing matrix can be expressed as $M_r(\boldsymbol{\theta} \boldsymbol{z}_i^{(k)}) = \sum_{\boldsymbol{\gamma} \in \mathbb{N}^k} \theta_{\boldsymbol{\gamma}} M_r^{\boldsymbol{\gamma}}(\boldsymbol{z}_i^{(k)})$.

Note that for every polynomial $f \in \mathbb{R}[\boldsymbol{x}[k]]$ of degree at most r with its vector of coefficients denoted by $\boldsymbol{f} = \{f_{\boldsymbol{\gamma}}\}$, we have:

$$\langle \boldsymbol{f}, M_r(\boldsymbol{z}_i^{(k)}) \boldsymbol{f} \rangle = \int f^2 \mathrm{d}\mu_i^{(k)}, \quad \langle \boldsymbol{f}, M_r(\boldsymbol{\theta} \boldsymbol{z}_i^{(k)}) \boldsymbol{f} \rangle = \int \theta f^2 \mathrm{d}\mu_i^{(k)}. \tag{12}$$

This property shows that necessarily, $M_r(\boldsymbol{z}_i^{(k)}) \succeq \mathbf{0}$ and $M_r(\boldsymbol{\theta} \boldsymbol{z}_i^{(k)}) \succeq \mathbf{0}$, whenever $\mu_i^{(k)}$ has its support contained in the level set $\left\{\boldsymbol{x} \in \mathbb{R}^k : \theta(\boldsymbol{x}) \ge 0\right\}$. If the sequence of moments is restricted to those moments used to construct the moment matrix $M_r(\boldsymbol{z}_i^{(k)})$ (up to moments of degree $2r$), then the second necessary condition is reduced to $M_{r - \lceil d/2 \rceil}(\boldsymbol{\theta} \boldsymbol{z}_i^{(k)}) \succeq \mathbf{0}$, where d is the degree of the polynomial θ. For more details on moment matrices, local matrices, and these necessary conditions, please refer to Laurent [4] and references therein.

Let us define

$$\theta_i^{(k)}(\boldsymbol{x}) = (b_i^{(2)}(\boldsymbol{x}[i-1]) - x_i)(x_i - b_i^{(1)}(\boldsymbol{x}[i-1])), \boldsymbol{x} \in \mathbb{R}^k \quad \forall k \ge i \tag{13}$$

for all $i = 2, \ldots, n$. Similarly, define

$$\theta_1^{(k)}(\boldsymbol{x}) = (b_1^{(2)} - x_1)(x_1 - b_1^{(1)}), \boldsymbol{x} \in \mathbb{R}^k \quad \forall k \ge 1. \tag{14}$$

For a fixed i ($i = 1, \ldots, n$), the polynomials $\theta_i^{(k)}$ depends on the first i variables $x_1, \ldots, x_i$ exactly the same for all $k \ge i$. Thus they all have the same degree d_i for all $k \ge i$.

We also have $\Omega_k = \{\boldsymbol{x} \in \mathbb{R}^k : \theta_i^{(k)}(\boldsymbol{x}) \ge 0 \quad \forall i = 1, \ldots, k\}$ for all $k = 1, \ldots, n$. Thus necessary conditions for moment matrices of all measures $\mu_i^{(k)}$ can be written as follows:

$$M_r(\boldsymbol{z}_i^{(k)}) \succeq 0, \quad M_{r - \lceil d_j/2 \rceil}(\theta_j^{(k)} \boldsymbol{z}_i^{(k)}) \succeq 0 \quad \forall k = 1, \ldots, n, \forall i = 1, \ldots, 2^{n-k}, \forall j = 1, \ldots, k \tag{15}$$

Combining these necessary conditions and the recursive formulae for $\boldsymbol{z}_i^{(k)}$ in (8) and (9), we can find lower and upper bounds for ρ by solving the two following semidefinite programming problems:

$$\mathcal{P}_r^l \triangleq \left[\begin{array}{lll} \inf & \langle \boldsymbol{g}, \boldsymbol{z}_1^{(n)} \rangle & \\ \text{s.t.} & M_r(\boldsymbol{z}_i^{(k)}) \succeq 0 & k = 1, \ldots, n, i = 1, \ldots, 2^{n-k} \\ & M_{r-\lceil d_j/2 \rceil}(\theta_j^{(k)} \boldsymbol{z}_i^{(k)}) \succeq 0 & k = 1, \ldots, n, i = 1, \ldots, 2^{n-k}, j = 1, \ldots, k \\ & z_i^{(1)}(\alpha) = R_i^{(1)}(\boldsymbol{z}_i^{(1)}, \alpha) & i = 1, \ldots, 2^{n-1}, \alpha \in A_{i,r}^{(1)} \\ & z_i^{(k)}(\boldsymbol{\alpha}) = R_i^{(k)}(\boldsymbol{z}_i^{(k)}, \boldsymbol{z}_{2i}^{(k-1)}, \boldsymbol{z}_{2i-1}^{(k-1)}, \boldsymbol{\alpha}) & k = 2, \ldots, n, i = 1, \ldots, 2^{n-k}, \boldsymbol{\alpha} \in A_{i,r}^{(k)} \end{array}\right] \tag{16}$$

and

$$\mathcal{P}_r^u \triangleq \left[\begin{array}{lll} \sup & \langle \boldsymbol{g}, \boldsymbol{z}_1^{(n)} \rangle & \\ \text{s.t.} & M_r(\boldsymbol{z}_i^{(k)}) \succeq 0 & k = 1, \ldots, n, i = 1, \ldots, 2^{n-k} \\ & M_{r-\lceil d_j/2 \rceil}(\theta_j^{(k)} \boldsymbol{z}_i^{(k)}) \succeq 0 & k = 1, \ldots, n, i = 1, \ldots, 2^{n-k}, j = 1, \ldots, k \\ & z_i^{(1)}(\alpha) = R_i^{(1)}(\boldsymbol{z}_i^{(1)}, \alpha) & i = 1, \ldots, 2^{n-1}, \alpha \in A_{i,r}^{(1)} \\ & z_i^{(k)}(\boldsymbol{\alpha}) = R_i^{(k)}(\boldsymbol{z}_i^{(k)}, \boldsymbol{z}_{2i}^{(k-1)}, \boldsymbol{z}_{2i-1}^{(k-1)}, \boldsymbol{\alpha}) & k = 2, \ldots, n, i = 1, \ldots, 2^{n-k}, \boldsymbol{\alpha} \in A_{i,r}^{(k)} \end{array}\right] \tag{17}$$

where $A_{i,r}^{(k)}$ is the set of all $\boldsymbol{\alpha} \in \mathbb{R}^k$ that the recursive formulae can be expressed by moments of degree up to $2r$ (used to construct the moment matrices M_r).

Clearly, $Z(\mathcal{P}_r^l) \leq \rho \leq Z(\mathcal{P}_r^u)$ and we will prove that these lower and upper bounds asymptotically converge to ρ when r tends to infinity in the next section.

3 Convergence

In order to prove the convergence of $Z(\mathcal{P}_r^l)$ and $Z(\mathcal{P}_r^u)$, we need to prove the recursive formulae in (8) and (9) define moment sequences for all measures $\mu_i^{(k)}$.

Lemma 1 *Let $h_i^{(k)}$ and $g_{i,\boldsymbol{\alpha}}^{(k)}$ be defined as in (6) and (7) respectively, and let $\boldsymbol{z}_i^{(k)}$ be the moment sequences of some Borel measures $\psi_i^{(k)}$ on the compact sets Ω_k, which satisfy (8) and (9). Then $\mathrm{d}\psi_i^{(k)} = \mathrm{d}\mu_i^{(k)}$ for all $k = 1, \ldots, n$ and $i = 1, \ldots, 2^{n-k}$.*

Proof. We will prove the lemma by induction. Let consider the case $k = 1$, we have, according to (9):

$$z_i^{(1)}(\alpha) = \int x^\alpha \mathrm{d}\psi_i^{(1)} = \left[\frac{x^{\alpha+1}}{\alpha+1} e^{h_i^{(1)}(x)}\right]_{b_1^{(1)}}^{b_1^{(2)}} - \int \frac{x^{\alpha+1}}{\alpha+1} (h_i^{(1)})'(x) \mathrm{d}\psi_i^{(1)} \tag{18}$$

We also have:

$$\int x^\alpha \mathrm{d}\mu_i^{(1)} = \left[\frac{x^{\alpha+1}}{\alpha+1} e^{h_i^{(1)}(x)}\right]_{b_1^{(1)}}^{b_1^{(2)}} - \int \frac{x^{\alpha+1}}{\alpha+1} (h_i^{(1)})'(x) \mathrm{d}\mu_i^{(1)} \tag{19}$$

Let consider the *signed* measure $\phi_i^{(1)}$ on Ω_1 that satisfies $\mathrm{d}\phi_i^{(1)} = \mathrm{d}\psi_i^{(1)} - \mathrm{d}\mu_i^{(1)}$, from (18) and (19), we have:

$$\int x^\alpha \mathrm{d}\phi_i^{(1)} = -\int \frac{x^{\alpha+1}}{\alpha+1}(h_i^{(1)})'(x)\mathrm{d}\phi_i^{(1)} \tag{20}$$

Consider the polynomial $p(x) = \sum_{j=1}^d f_j x^j$, we have: $p'(x) = \sum_{j=0}^{d-1} f_{j+1}(j+1)x^j$. From Equation (20), we obtain the following equation:

$$\int \left[p'(x) + p(x)(h_i^{(1)})'(x)\right] \mathrm{d}\phi_i^{(1)} = 0 \tag{21}$$

We now prove that Equation (21) is also true for all continuous funtion $f(x) = xg(x)$, where g is continuously differentiable on Ω_1. We have, polynomials are dense in the space of continuously differentiable functions on Ω_1 under the sup-norm $\max\{\sup_{x\in\Omega_1}|f(x)|, \sup_{x\in\Omega_1}|f'(x)|\}$ (see Coatmélec [1] for details on simultaneous approximation). Therefore for any $\epsilon > 0$, there exist $p_\epsilon \in \mathbb{R}[x]$ such that $\sup_{x\in\Omega_1}|g(x) - p_\epsilon(x)| \le \epsilon$ and $\sup_{x\in\Omega_1}|g'(x) - p'_\epsilon(x)| \le \epsilon$. Equation (21) is true for the polynomial $p(x) = xp_\epsilon(x)$, thus

$$\int \left[f'(x) + f(x)(h_i^{(1)})'(x)\right] \mathrm{d}\phi_i^{(1)} = \int \left[x(g(x) - p_\epsilon(x))\right]' \mathrm{d}\phi_i^{(1)} + \int x\left[g(x) - p_\epsilon(x)\right](h_i^{(1)})'(x)\mathrm{d}\phi_i^{(1)}$$

We have: $[x(g(x) - p_\epsilon(x))]' = (g(x) - p_\epsilon(x)) + x(g'(x) - p'_\epsilon(x))$, thus

$$|\left[x(g(x) - p_\epsilon(x))\right]'| \le (1 + \sup_{x\in\Omega_1}|x|)\epsilon \quad \forall x \in \Omega_1$$

Similarly,

$$|x\left[g(x) - p_\epsilon(x)\right](h_i^{(1)})'(x)| \le \sup_{x\in\Omega_1}|x(h_i^{(1)})'(x)|\epsilon \quad \forall x \in \Omega_1$$

So we have:

$$|\int \left[f'(x) + f(x)(h_i^{(1)})'(x)\right] \mathrm{d}\phi_i^{(1)}| \le \epsilon(1 + \sup_{x\in\Omega_1}|x| + \sup_{x\in\Omega_1}|x(h_i^{(1)})'(x)|)\int |\mathrm{d}\phi_i^{(1)}|$$

The constant term $M = (1+\sup_{x\in\Omega_1}|x|+\sup_{x\in\Omega_1}|x(h_i^{(1)})'(x)|)\int|\mathrm{d}\phi_i^{(1)}|$ is finite and the above equation is true for all $\epsilon > 0$, thus we have for all $f(x) = xg(x)$:

$$\int \left[f'(x) + f(x)(h_i^{(1)})'(x)\right] \mathrm{d}\phi_i^{(1)} = 0 \tag{22}$$

For an arbitrary polynomial $g(x) = \sum_{j=0}^d g_j x^j$, define $G(x) = \sum_{j=0}^d \frac{g_j}{j+1}x^{j+1}$, we have $G'(x) = g(x)$. Let consider $f(x) = G(x)e^{-h_i^{(1)}(x)}$, we have: $\frac{f(x)}{x}$ is a continuously differentiable function and $f'(x) = g(x)e^{-h_i^{(1)}(x)} - f(x)(h_i^{(1)})'(x)$. Using Equation (22), we obtain the following equation

$$\int g(x)e^{-h_i^{(1)}(x)}\mathrm{d}\phi_i^{(1)} = 0, \quad \forall g \in \mathbb{R}[x] \tag{23}$$

If we define $\mathrm{d}\nu_i^{(1)} = e^{-h_i^{(1)}(x)}\mathrm{d}\phi_i^{(1)}$, we then have $\int f(x)\mathrm{d}\nu_i^{(1)} = 0$ for all continuous function f in Ω_1 since polynomials are dense in the space of continuous functions. This implies that $\nu_i^{(1)}$ is a zero measure. In addition, $e^{-h_i^{(1)}(x)} > 0$ for all $x \in \mathbb{R}$, thus $\phi_i^{(1)}$ is also a zero measure or $\mathrm{d}\psi_i^{(1)} = \mathrm{d}\mu_i^{(1)}$ for all $i = 1, \ldots, 2^{n-1}$.

Now assume that $\mathrm{d}\psi_i^{(k-1)} = \mathrm{d}\mu_i^{(k-1)}$, $2 \leq k \leq n$, for all $i = 1, \ldots, 2^{n-k+1}$. We will prove that $\mathrm{d}\psi_i^{(k)} = \mathrm{d}\mu_i^{(k)}$ for all $i = 1, \ldots, 2^{n-k}$. We have: $\mathrm{d}\psi_{2i}^{(k-1)} = \mathrm{d}\mu_{2i}^{(k-1)}$ and $\mathrm{d}\psi_{2i-1}^{(k-1)} = \mathrm{d}\mu_{2i-1}^{(k-1)}$ for $i : 1 \leq i \leq 2^{n-k}$. Thus $\boldsymbol{z}_{2i}^{(k-1)}$ and $\boldsymbol{z}_{2i-1}^{(k-1)}$ are moment sequences of two measures $\mu_{2i}^{(k-1)}$ and $\mu_{2i-1}^{(k-1)}$ respectively.

According to (8), we have:

$$\int \boldsymbol{x}^{\alpha}\mathrm{d}\psi_i^{(k)} = \sum_{\boldsymbol{\beta}\in\mathbb{N}^{k-1}} \frac{\left(g_{2,\boldsymbol{\alpha}}^{(k-1)}\right)_{\boldsymbol{\beta}}}{\alpha_k+1} z_{2i}^{(k-1)}(\boldsymbol{\beta}) - \sum_{\boldsymbol{\beta}\in\mathbb{N}^{k-1}} \frac{\left(g_{1,\boldsymbol{\alpha}}^{(k-1)}\right)_{\boldsymbol{\beta}}}{\alpha_k+1} z_{2i-1}^{(k-1)}(\boldsymbol{\beta}) - \int \frac{\boldsymbol{x}^{\boldsymbol{\alpha}}x_k}{\alpha_k+1}\frac{\partial h_i^{(k)}(\boldsymbol{x})}{\partial x_k}\mathrm{d}\psi_i^{(k)} \tag{24}$$

Similarly,

$$\int \boldsymbol{x}^{\alpha}\mathrm{d}\mu_i^{(k)} = \sum_{\boldsymbol{\beta}\in\mathbb{N}^{k-1}} \frac{\left(g_{2,\boldsymbol{\alpha}}^{(k-1)}\right)_{\boldsymbol{\beta}}}{\alpha_k+1} z_{2i}^{(k-1)}(\boldsymbol{\beta}) - \sum_{\boldsymbol{\beta}\in\mathbb{N}^{k-1}} \frac{\left(g_{1,\boldsymbol{\alpha}}^{(k-1)}\right)_{\boldsymbol{\beta}}}{\alpha_k+1} z_{2i-1}^{(k-1)}(\boldsymbol{\beta}) - \int \frac{\boldsymbol{x}^{\boldsymbol{\alpha}}x_k}{\alpha_k+1}\frac{\partial h_i^{(k)}(\boldsymbol{x})}{\partial x_k}\mathrm{d}\mu_i^{(k)} \tag{25}$$

Then if we consider the *signed* measure $\phi_i^{(k)}$ on Ω_k that satisfies $\mathrm{d}\phi_i^{(k)} = \mathrm{d}\psi_i^{(k)} - \mathrm{d}\mu_i^{(k)}$, we have:

$$\int \boldsymbol{x}^{\boldsymbol{\alpha}}\mathrm{d}\phi_i^{(k)} = -\int \frac{\boldsymbol{x}^{\boldsymbol{\alpha}}x_k}{\alpha_k+1}\frac{\partial h_i^{(k)}(\boldsymbol{x})}{\partial x_k}\mathrm{d}\phi_i^{(k)} \tag{26}$$

Using similar arguments with simultaneous approximation of continuous functions, we obtain the following equation for all functions $f(\boldsymbol{x}) = x_k g(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^k$, where g and $\frac{\partial g}{\partial x_k}$ are continuous:

$$\int \left[\frac{\partial f(\boldsymbol{x})}{\partial x_k} + f(\boldsymbol{x})\frac{\partial h_i^{(k)}(\boldsymbol{x})}{\partial x_k}\right]\mathrm{d}\phi_i^{(k)} = 0 \tag{27}$$

For any polynomial $g \in \mathbb{R}[\boldsymbol{x}]$, define $G(\boldsymbol{x}) = x_k P(\boldsymbol{x})$, where $P(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ and $\frac{\partial G(\boldsymbol{x})}{\partial x_k} = g(\boldsymbol{x})$, then with $f(\boldsymbol{x}) = G(\boldsymbol{x})e^{-h_i^{(k)}(\boldsymbol{x})}$, we again obtain

$$\int g(\boldsymbol{x})e^{-h_i^{(k)}(\boldsymbol{x})}\mathrm{d}\phi_i^{(k)} = 0, \quad \forall g \in \mathbb{R}[\boldsymbol{x}] \tag{28}$$

Similar arguments are again used and we have the measure $\phi_i^{(k)}$ is a zero measure or $\mathrm{d}\psi_i^{(k)} = \mathrm{d}\mu_i^{(k)}$ for all $i = 1, \ldots, 2^{n-k}$. So the results are true for all $k = 1, \ldots, n$. □

With this lemma, we can now prove the following convergence theorem:

Theorem 1 *Let $g, h \in \mathbb{R}[\boldsymbol{x}]$, Ω be the compact set defined in (2), and consider the semidefinite programming problems $\mathcal{P}_r^l$ and $\mathcal{P}_r^u$ defined in (16) and (17) respectively. Then* *(i) Optimal values $Z(\mathcal{P}_r^l)$ and $Z(\mathcal{P}_r^u)$ are finite and in addition, both problems $\mathcal{P}_r^l$ and $\mathcal{P}_r^u$ are solvable for r large enough.*

(ii) As $r \to \infty$, $Z(\mathcal{P}_r^l) \uparrow \rho$ and $Z(\mathcal{P}_r^u) \downarrow \rho$.

Proof. Clearly, the collection of truncated sequences of moments of $\mu_i^{(k)}$ is a feasible solution for both problems $\mathcal{P}_r^l$ and $\mathcal{P}_r^u$. We have:

$$|z_i^{(k)}(\boldsymbol{\alpha})| \le \int_{\Omega_k} |\,(\boldsymbol{x}[k])^{\boldsymbol{\alpha}}\, e^{h_i^{(k)}(\boldsymbol{x}[k])}|\mathrm{d}\boldsymbol{x}[k] \le \sup_{\boldsymbol{x}\in\Omega_k} |\boldsymbol{x}^{\boldsymbol{\alpha}} e^{h_i^{(k)}(\boldsymbol{x})}|\mathrm{vol}(\Omega_k)$$

Ω_k is compact in $\mathbb{R}^k$; therefore, we have $u_i^{(k)}(\boldsymbol{\alpha}) = \sup_{\boldsymbol{x}\in\Omega_k} |\boldsymbol{x}^{\boldsymbol{\alpha}} e^{h_i^{(k)}(\boldsymbol{x})}| = \max_{\boldsymbol{x}\in\Omega_k} |\boldsymbol{x}^{\boldsymbol{\alpha}} e^{h_i^{(k)}(\boldsymbol{x})}|$ is finite. Now consider the problems $\mathcal{P}_r^l$ and $\mathcal{P}_r^u$ with additional bound constraints $-u_i^{(k)}(\boldsymbol{\alpha}) - 1 \le z_i^{(k)}(\boldsymbol{\alpha}) \le u_i^{(k)}(\boldsymbol{\alpha}) + 1$ for all k, i and $\boldsymbol{\alpha}$, we have:

(i) The feasible sets of these two modified problems are bounded and closed. The objective functions are linear and both problems are feasible. Therefore, they are both solvable and their optimal values are finite.

(ii) Let $\{\boldsymbol{z}_{i,r}^{(k)}\}$ be the optimal solution of $\mathcal{P}_r^l$ (with bound constraints), we extend these truncated sequences with zeros to make them become infinite sequences. According to Lasserre [3], there is a subsequence $\{r_m\}$ and infinite sequences $\{\boldsymbol{z}_{i,*}^{(k)}\}$ such that pointwise convergence holds with respect to the usual sup-norm. Therefore, we have:

$$M(\boldsymbol{z}_{i,*}^{(k)}) \succeq 0, \quad M(\theta_j^{(k)} \boldsymbol{z}_{i,*}^{(k)}) \succeq 0 \quad \forall k, i, j$$

We also have Ω_k is compact in $\mathbb{R}^k$; therefore, according to Putinar [6], there exist measures $\psi_i^{(k)}$ supported on Ω_k such that $\boldsymbol{z}_{i,*}^{(k)}$ are their moment sequences assuming the representation condition holds. In addition, $\{\boldsymbol{z}_{i,*}^{(k)}\}$ satisfy the recursive formulae (8) and (9) from the pointwise convergence. From Lemma 1, we obtain that $\mathrm{d}\psi_i^{(k)} = \mathrm{d}\mu_i^{(k)}$ for all $k = 1, \ldots, n$ and $i = 1, \ldots, 2^{n-k}$. Thus we have:

$$\lim_{m\to\infty} \langle \boldsymbol{g}, \boldsymbol{z}_{1,r_m}^{(n)} \rangle = \langle \boldsymbol{g}, \boldsymbol{z}_{1,*}^{(n)} \rangle = \rho$$

Due to the construction of truncated moment matrices and localizing matrices as well as the sets $A_{i,r}^{(k)}$, clearly, a feasible solution of $\mathcal{P}_{r+1}^l$ generates a feasible solution of $\mathcal{P}_r^l$. Thus with $r \ge \deg(g)$, then $\langle \boldsymbol{g}, \boldsymbol{z}_{1,r}^{(n)} \rangle \le \langle \boldsymbol{g}, \boldsymbol{z}_{1,r+1}^{(n)} \rangle$ or we have:

$$\langle \boldsymbol{g}, \boldsymbol{z}_{1,r}^{(n)} \rangle \uparrow \rho$$

Similar arguments can be applied for the modified problem $\mathcal{P}_r^u$.

So far, the results obtained are for problems $\mathcal{P}_r^l$ and $\mathcal{P}_r^u$ with additional bound constraints. However, at the limit with respect to the usual sup-norm, none of these bound constraints are tight. Thus, these constraints can be removed when r is large enough. In other words, the problems $\mathcal{P}_r^l$ and $\mathcal{P}_r^u$ have finite optimal solutions and solvable when r is large enough and

$$Z(\mathcal{P}_r^l) \uparrow \rho, \quad Z(\mathcal{P}_r^u) \downarrow \rho$$

$\square$

4 Computational Results

In order to demonstrate the method, we use two different kinds of Ω sets: hypercubes and order statistic integrals. The computational results are obtained for integrals over these sets of random multivariate normal distribution.

4.1 Integrals over Hypercubes

If Ω is a hypercube, then $b_k^{(i)}(\boldsymbol{x}) = b_k^{(i)}$ are constants for all $k = 2, \ldots, n$. We have:

$$g_{i,\boldsymbol{\alpha}}^{(k)}(\boldsymbol{x}) = [b_{k+1}^{(i)}]^{\alpha_{k+1}+1} \boldsymbol{x}^{\boldsymbol{\alpha}[k]}, \quad \boldsymbol{x} \in \mathbb{R}^k, \boldsymbol{\alpha} \in \mathbb{N}^k, i = 1, 2 \tag{29}$$

The functions $h_i^{(k)}$ are still formulated as in (6):

$$h_i^{(k)}(\boldsymbol{x}) = h_{\lceil i/2 \rceil}^{(k+1)}(\boldsymbol{x}, b_{k+1}^{(2-\lceil i/2 \rceil + \lfloor i/2 \rfloor)}), \quad \boldsymbol{x} \in \mathbb{R}^k, k = 1, \ldots, n-1, i = 1, \ldots, 2^{n-k}$$

The recursive formula simply becomes

$$z_i^{(k)}(\boldsymbol{\alpha}) = \frac{[b_k^{(2)}]^{\alpha_k+1}}{\alpha_k + 1} z_{2i}^{(k-1)}(\boldsymbol{\alpha}[k-1]) - \frac{[b_k^{(1)}]^{\alpha_k+1}}{\alpha_k + 1} z_{2i-1}^{(k-1)}(\boldsymbol{\alpha}[k-1]) - \sum_{\boldsymbol{\beta} \in \mathbb{N}^k} \frac{\beta_k (h_i^{(k)})_{\boldsymbol{\beta}}}{\alpha_k + 1} z_i^{(k)}(\boldsymbol{\alpha} + \boldsymbol{\beta}) \tag{30}$$

for all $k = 2, \ldots, n$, $\boldsymbol{\alpha} \in \mathbb{R}^k$, $i = 1, \ldots, 2^{n-k}$ while for $k = 1$, it remains the same as in (9):

$$z_i^{(1)}(\alpha) = \frac{(b_1^{(2)})^{\alpha+1}}{\alpha + 1} e^{h_i^{(1)}(b_1^{(2)})} - \frac{(b_1^{(1)})^{\alpha+1}}{\alpha + 1} e^{h_i^{(1)}(b_1^{(1)})} - \sum_{\beta \in \mathbb{N}} \frac{\beta (h_i^{(1)})_\beta}{\alpha + 1} z_i^{(1)}(\alpha + \beta), \quad \forall i = 1, \ldots, 2^{n-1}$$

The polynomials $\theta_i^{(k)}(\boldsymbol{x}) = (b_i^{(2)} - x_i)(x_i - b_i^{(1)}) = -b_i^{(1)} b_i^{(2)} + (b_i^{(1)} + b_i^{(2)}) x_i - x_i^2$, $\boldsymbol{x} \in \mathbb{R}^k$ have the degree $d_i = 2$ for all $k \geq i$, $i = 1, \ldots, n$ in this case.

4.2 Order Statistic Integrals

Order statistic integrals are calculated over the set $\Omega = \{\boldsymbol{x} \in \mathbb{R}^n : b^{(1)} \leq x_1 \leq \ldots \leq x_n \leq b^{(2)}\}$. So we have: $b_k^{(2)} = b^{(2)}$ are constant for all $k = 1, \ldots, n$ while $b_k^{(1)}(\boldsymbol{x}) = x_{k-1}$ for all $k = 2, \ldots, n$ and $b_1^{(1)} = b^{(1)}$. The functions $g_{2,\boldsymbol{\alpha}}^{(k)}$ are defined as in (29) while $g_{1,\boldsymbol{\alpha}}^{(k)}$ are formulated as follows

$$g_{1,\boldsymbol{\alpha}}^{(k)}(\boldsymbol{x}) = x_k^{\alpha_{k+1}+1} \boldsymbol{x}^{\boldsymbol{\alpha}[k]} = \boldsymbol{x}^{\boldsymbol{\alpha}[k]+(\alpha_{k+1}+1)\boldsymbol{e}_k}, \quad \boldsymbol{x} \in \mathbb{R}^k, \boldsymbol{\alpha} \in \mathbb{N}^k, \tag{31}$$

where $\boldsymbol{e}_k$ is the k^{th} unit vector in $\mathbb{R}^k$.

The recursive formula is then

$$z_i^{(k)}(\boldsymbol{\alpha}) = \frac{[b^{(2)}]^{\alpha_k+1}}{\alpha_k+1} z_{2i}^{(k-1)}(\boldsymbol{\alpha}[k-1]) - \frac{1}{\alpha_k+1} z_{2i-1}^{(k-1)}(\boldsymbol{\alpha}[k-1] + (\alpha_k+1)\boldsymbol{e}_k) - \sum_{\boldsymbol{\beta} \in \mathbb{N}^k} \frac{\beta_k (h_i^{(k)})_{\boldsymbol{\beta}}}{\alpha_k+1} z_i^{(k)}(\boldsymbol{\alpha}+\boldsymbol{\beta}) \tag{32}$$

for all $k = 2, \ldots, n$, $\boldsymbol{\alpha} \in \mathbb{R}^k$, $i = 1, \ldots, 2^{n-k}$. For $k = 1$, the formula remains the same as in (9).

Finally, the polynomials $\theta_i^{(k)}(\boldsymbol{x}) = (b^{(2)} - x_i)(x_i - x_{i-1}) = -b^{(2)}x_{i-1} + b^{(2)}x_i + x_{i-1}x_i - x_i^2$, $\boldsymbol{x} \in \mathbb{R}^k$ also have degree $d_i = 2$ for all $k \geq i$, $i = 2, \ldots, n$. For $i = 1$, the polynomial $\theta_1^{(k)}(\boldsymbol{x})$ is the same as in the previous section with degree $d_1 = 2$.

4.3 Computational Results for Normal Distributions

Multivariate normal distributions are used to obtain computational results for integrals over hypercubes and order statistic integrals. The density function of a multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A}'$ is

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2} \det(\boldsymbol{\Sigma})^{1/2}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})} \tag{33}$$

Thus, in this case, $g(\boldsymbol{x}) = \left[(2\pi)^{n/2} \det(\boldsymbol{\Sigma})^{1/2}\right]^{-1}$, a constant, and $h(\boldsymbol{x}) = -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})$, a quadratic polynomials.

Algorithms for general g and h are implemented in Matlab for both hypercubic and order statistic integrals. Semidefinite programming problems are solved using SeDuMi routines. For the case of normal distributions, means and covariance matrices are created randomly using random vectors and matrices with elements within the range of $[-1, 1]$. The two Ω sets are defined on the hypercube $\{\boldsymbol{x} \in \mathbb{R}^n : -1 \leq x_i \leq 1 \quad \forall i = 1, \ldots, n\}$.

The algorithms are run for different normal distributions in $n = 1$, 2, and 3-dimension space. The moment degree starts at $r = 2$ and increases until the tolerance is met. The tolerance $\epsilon =$

$\frac{1}{2}\left[Z(\mathcal{P}_r^u) - Z(\mathcal{P}_r^l)\right]$ is set to be 5×10^{-5} (4-digit accuracy). The results will be averaged from those different distributions in each case. All computations are done under a Windows environment on a Pentium III 1GHz with 256MB RAM.

Dimension n	Min. degree $r_{\min}$	Max. degree $r_{\max}$	Average degree $\bar{r}$	Computational time (seconds)			
				$r=2$	$r=3$	$r=4$	$r=5$
1	2	5	3.5	1.332	1.631	1.505	1.437
2	3	5	4.3	1.994	3.215	11.036	70.107
3	-	-	-	5.7	131.1	2289.2	-

Table 1: Computational results for hypercubic integrals

Dimension n	Min. degree $r_{\min}$	Max. degree $r_{\max}$	Average degree $\bar{r}$	Computational time (seconds)			
				$r=2$	$r=3$	$r=4$	$r=5$
1	2	5	3.5	1.332	1.631	1.505	1.437
2	3	5	4.3	2.079	3.413	12.294	79.418
3	-	-	-	4.8	134.6	2622.5	-

Table 2: Computational results for order statistic integrals

Table 1 and 2 show computational results for hypercubic and order statistic integrals respectively. Minimum degree $r_{\min}$ is the smallest moment degree we need to use to meet the tolerance setting while $r_{\max}$ is the largest moment degree. $\bar{r}$ is the average value obtained from all distributions tested for each value of n. In the case $n = 1$, there is not much difference between computational times for different moment degrees and we need roughly $r = 3$ or $r = 4$ on average to obtain the desired results. These observations are for both hypercubic and order statistic integrals even though the latter need slightly more time to be calculated. For $n = 2$, the average moment degree is $\bar{r} = 4.3$ and computational times are significantly varied from 2 seconds for $r = 2$ up to approximately 80 seconds for $r = 5$. Due to the limit of memory needed for SeDuMi routine, we cannot run the algorithms with $r = 5$ in the case $n = 3$. The computational times are not very favorable in this case with more than 2000 seconds needed when $r = 4$.

Table 3 shows the real gaps between lower and upper bounds of integral values for some specific normal distributions, both hypercubic and order statistic integrals. These values decrease when the

Integral	Dimension n	Value $\bar{\rho}$	Error $\frac{1}{2}\left[Z(\mathcal{P}_r^u) - Z(\mathcal{P}_r^l)\right]$ $r=2$	$r=3$	$r=4$	$r=5$
Hypercubic	1	0.2665	1.4047E-04	8.1177E-07	-	-
	2	0.1085	1.0376E+01	1.1200E-02	6.7765E-04	4.7855E-05
	3	0.0736	4.7420E-01	8.0400E-02	2.2000E-03	-
Order statistic	2	0.0670	7.8969E+00	6.3000E-03	4.2279E-04	3.2933E-05
	3	0.0274	4.9390E-01	5.4000E-03	4.2253E-04	-

Table 3: Real errors for specific normal distributions

moment order increases. For example, when $n = 3$, we cannot solve the semidefinite programming problems for $r = 5$ due to the limit of memory; however, the gap between lower and upper bounds for $r = 4$ is small enough. From these results, we can see that this method is plausible to be implemented even though the computational times for large n and r are yet very promising.

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