Bounds on some contingent claims with non-convex payoff based on multiple assets

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Abstract

We propose a copositive relaxation framework to calculate both upper and lower bounds for prices of some European options with non-convex payoffs when first and second moments of underlying assets are known. Computational results shows that these upper and lower bounds are reasonably good for call options on the minimum of multiple assets and put options on the maximum of multiple assets.

1 Introduction

Option valuation is important for a wide variety of hedging and investment purposes. Black and Scholes [3] derive a pricing formula for a European call option on a single asset with no-arbitrage arguments and the lognormal distribution assumption of the underlying asset price. Merton [9] provide bounds on option prices with no assumption on the distribution of the asset price. Given the mean and variance of the asset price, Lo [7] obtains an upper bound for the European option price based on this single asset. This result is generalized in Bertsimas and Popescu [1]. In the case of options written on multiple underlying assets, Boyle and Lim [4] provides upper bounds for European call options on the maximum of several assets. Zuluaga and Peña [13] obtain these bounds using moment duality and conic programming.

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Contributions and Paper Outline

The options considered in these papers have convex payoff functions. Given first and second moments of underlying asset prices, a simple tight lower bound can be calculated using Jensen's inequality. In this paper, we consider a class of European options with non-convex payoff, the call option written on the minimum of several assets. Similarly, put options on the maximum of several assets are also options with non-convex payoff functions. Both upper and lower bounds for prices of European call options on the minimum of several assets calculated using copositive relaxation are considered in Section 2 and 3. Some computational results for these call and put options are reported in Section 4.

2 Upper Bounds

We consider the European call options written on the minimum of n assets. At maturity, these assets have price $X_1, \ldots, X_n$ respectively. If the option strike price is K, then the expected payoff can be calculated as follows:

$$P = \mathbb{E}[(\min_{1\leq k\leq n} X_k - K)^+]. \tag{1}$$

The rational option price can be obtained by discounting this expectation at the risk-free rate under the no-arbitrage assumption. Therefore, we can firstly derive bounds for this expected payoff P without discount factor involvement and obtain bounds for the option price later.

We do not assume any distribution models for the multivariate nonnegative random variable $\boldsymbol{X} = (X_1, \ldots, X_n)$. Given that first and second moments of $\boldsymbol{X}$, $\mathbb{E}[\boldsymbol{X}] = \boldsymbol{\mu}$ and $\mathbb{E}[\boldsymbol{X}\boldsymbol{X}^T] = \boldsymbol{Q}$, we would like to calculate the tight upper bound $P_{\max} = \max_{\boldsymbol{X}\sim(\boldsymbol{\mu},\boldsymbol{Q})^+} \mathbb{E}[(\min_{1\leq k\leq n} X_k - K)^+]$ and lower bound $P_{\min} = \min_{\boldsymbol{X}\sim(\boldsymbol{\mu},\boldsymbol{Q})^+} \mathbb{E}[(\min_{1\leq k\leq n} X_k - K)^+]$. In this section, we focus on upper bounds while lower bounds will be considered in Section 3.

We have, the upper bound $P_{\max}$ is the optimal value of the following optimization problem:

$$\begin{array}{rll} P_{\max} = \max_f & \int_{\mathbb{R}^n_+} (\min_{1\leq k\leq n} x_k - K)^+ f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} & \\ \text{s.t.} & \int_{\mathbb{R}^n_+} x_k f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \mu_k, & \forall\, k = 1, \ldots, n, \\ & \int_{\mathbb{R}^n_+} x_k x_l f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = Q_{kl}, & \forall\, 1 \leq k \leq l \leq n, \\ & \int_{\mathbb{R}^n_+} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 1, & \\ & f(\boldsymbol{x}) \geq 0, & \forall\, \boldsymbol{x} \in \mathbb{R}^n_+, \end{array} \tag{2}$$

where f is a probability density function.

Taking dual of Problem (2) (see Bertsimas and Popescu [2]), we obtain the following dual problem:

$$\begin{aligned} P_u = \min_{\boldsymbol{Y},\boldsymbol{y},y_0} \quad & \boldsymbol{Q}\cdot\boldsymbol{Y} + \boldsymbol{\mu}^T\boldsymbol{y} + y_0 \\ \text{s.t.} \quad & \boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{y} + y_0 \geq (\min_{1\leq k\leq n} x_k - K)^+, \quad \forall \boldsymbol{x}\in\mathbb{R}_+^n, \end{aligned}$$

or equivalently,

$$\begin{aligned} P_u = \min_{\boldsymbol{Y},\boldsymbol{y},y_0} \quad & \boldsymbol{Q}\cdot\boldsymbol{Y} + \boldsymbol{\mu}^T\boldsymbol{y} + y_0 \\ \text{s.t.} \quad & \boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{y} + y_0 \geq 0, \qquad \forall \boldsymbol{x}\in\mathbb{R}_+^n, \\ & \boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{y} + y_0 \geq \min_{1\leq k\leq n} x_k - K, \quad \forall \boldsymbol{x}\in\mathbb{R}_+^n. \end{aligned} \tag{3}$$

Weak duality shows that $P_u \geq P_{\max}$, which means P_u is an upper bound for the expected payoff P. Under a weak Slater condition on moments of X, strong duality holds and $P_u = P_{\max}$, which becomes a tight upper bound (see Bertsimas and Popescu [2] and references therein).

We now attempt to reformulate Problem (3). The first constraint is equivalent to a copositive matrix constraint as shown in the following lemma:

Lemma 1 *$\boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{y} + y_0 \geq 0$ for all $\boldsymbol{x}\in\mathbb{R}_+^n$ if and only if $\bar{\boldsymbol{Y}} = \begin{pmatrix} \boldsymbol{Y} & \frac{\boldsymbol{y}}{2} \\ \frac{\boldsymbol{y}^T}{2} & y_0 \end{pmatrix}$ is copositive.*

Proof. We have:

$$\boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{y} + y_0 = \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix}^T \begin{pmatrix} \boldsymbol{Y} & \frac{\boldsymbol{y}}{2} \\ \frac{\boldsymbol{y}^T}{2} & y_0 \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix}.$$

If the matrix $\bar{\boldsymbol{Y}}$ is copositive, then clearly $\boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{y} + y_0 \geq 0$ for all $\boldsymbol{x}\in\mathbb{R}_+^n$ as $(\boldsymbol{x},1)\in\mathbb{R}_+^{n+1}$ for all $\boldsymbol{x}\in\mathbb{R}_+^n$.

Conversely, if $\boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{y} + y_0 \geq 0$ for all $\boldsymbol{x}\in\mathbb{R}_+^n$, we prove that $\boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x}$ also nonnegative for all $\boldsymbol{x}\in\mathbb{R}_+^n$. Assume that there exists $\boldsymbol{x}\in\mathbb{R}_+^n$ such that $\boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} < 0$ and consider the function $f(k) = (k\boldsymbol{x})^T\boldsymbol{Y}(k\boldsymbol{x}) + (k\boldsymbol{x})^T\boldsymbol{y} + y_0$. We have: $f(k) = (\boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x})k^2 + (\boldsymbol{x}^T\boldsymbol{y})k + y_0$, which is a strictly concave quadratic function. Therefore, $\lim_{k\to+\infty} f(k) = -\infty$, which means there exists $\boldsymbol{z} = k\boldsymbol{x}\in\mathbb{R}_+^n$ such that $\boldsymbol{z}^T\boldsymbol{Y}\boldsymbol{z} + \boldsymbol{z}^T\boldsymbol{y} + y_0 < 0$ (contradiction). Thus we have $\boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} \geq 0$ for all $\boldsymbol{x}\in\mathbb{R}_+^n$. It means that $\boldsymbol{z}^T\bar{\boldsymbol{Y}}\boldsymbol{z} \geq 0$ for all $\boldsymbol{z}\in\mathbb{R}_+^{n+1}$ or $\bar{\boldsymbol{Y}}$ is copositive. □

The reformulation makes it clear that finding the (tight) upper bound P_u is a hard problem. Murty [10] shows that even the problem of determining whether a matrix is not copositive is NP-complete. In order to tractably compute an upper bound for the expected payoff P, we relax this constraint using a well-known copositivity sufficient condition (see Parrilo [11] and references therein):

Remark 1 (Copositivity) *If $\bar{\boldsymbol{Y}} = \boldsymbol{P} + \boldsymbol{N}$, where $\boldsymbol{P}\succeq 0$ and $\boldsymbol{N}\geq\boldsymbol{0}$, then $\bar{\boldsymbol{Y}}$ is copositive.*

According to Diananda [5], this sufficient condition is also necessary if $\bar{\boldsymbol{Y}} \in \mathbb{R}^{m\times m}$ with $m \leq 4$.

Now consider the second constraint, we will relax it using the following lemma:

Lemma 2 *If there exists $\boldsymbol{\mu} \in \mathbb{R}^n_+$, $\sum_{k=1}^n \mu_k = 1$, such that $\boldsymbol{Y}_\mu = \begin{pmatrix} \boldsymbol{Y} & \frac{\boldsymbol{y}-\sum_{k=1}^n \mu_k \boldsymbol{e}_k}{2} \\ \frac{(\boldsymbol{y}-\sum_{k=1}^n \mu_k \boldsymbol{e}_k)^T}{2} & y_0 + K \end{pmatrix}$ is copositive, where $\boldsymbol{e}_k$ is the k-th unit vector in $\mathbb{R}^n$, $k = 1, \ldots, n$, then $\boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x}+\boldsymbol{x}^T\boldsymbol{y}+y_0 \geq \min_{1\leq k\leq n} x_k - K$ for all $\boldsymbol{x} \in \mathbb{R}^n_+$.*

Proof. The second constraint can be written as follows:

$$\min_{\boldsymbol{x}\in\mathbb{R}^n_+} \max_{1\leq k\leq n} \boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{y} + y_0 - x_k + K \geq 0.$$

We have: $\max_{1\leq k\leq n} -x_k = \max_{\boldsymbol{z}\in C} -\boldsymbol{z}^T\boldsymbol{x}$, where C is the convex hull of $\boldsymbol{e}_k$, $k = 1, \ldots, n$. If we define $f(\boldsymbol{x}, \boldsymbol{z}) = \boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{y} + y_0 - \boldsymbol{z}^T\boldsymbol{x} + K$, then the second constraint is

$$\min_{\boldsymbol{x}\in\mathbb{R}^n_+} \max_{\boldsymbol{z}\in C} f(\boldsymbol{x}, \boldsymbol{z}) \geq 0.$$

Applying weak duality for the minmax problem $\min_{\boldsymbol{x}\in\mathbb{R}^n_+} \max_{\boldsymbol{z}\in C} f(\boldsymbol{x}, \boldsymbol{z})$, we have:

$$\min_{\boldsymbol{x}\in\mathbb{R}^n_+} \max_{\boldsymbol{z}\in C} f(\boldsymbol{x}, \boldsymbol{z}) \geq \max_{\boldsymbol{z}\in C} \min_{\boldsymbol{x}\in\mathbb{R}^n_+} f(\boldsymbol{x}, \boldsymbol{z}).$$

Thus if $\max_{\boldsymbol{z}\in C} \min_{\boldsymbol{x}\in\mathbb{R}^n_+} f(\boldsymbol{x}, \boldsymbol{z}) \geq 0$ then the second constraint is satisfied. This relaxed constraint can be written as follows:

$$\exists \boldsymbol{z} \in C : f(\boldsymbol{x}, \boldsymbol{z}) \geq 0, \quad \forall \boldsymbol{x} \in \mathbb{R}^n_+.$$

We have: $C = \left\{\sum_{k=1}^n \mu_k \boldsymbol{e}_k | \boldsymbol{\mu} \in \mathbb{R}^n_+, \sum_{k=1}^n \mu_k = 1\right\}$, thus the constraint above is equivalent to the following constraint:

$$\exists \boldsymbol{\mu} \in \mathbb{R}^n_+, \sum_{k=1}^n \mu_k = 1 : \boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{y} + y_0 - \sum_{k=1}^n \mu_k x_k + K \geq 0, \quad \forall \boldsymbol{x} \in \mathbb{R}^n_+.$$

Using Lemma 1, we obtain the equivalent constraint:

$$\exists \boldsymbol{\mu} \in \mathbb{R}^n_+, \sum_{k=1}^n \mu_k = 1 : \boldsymbol{Y}_\mu = \begin{pmatrix} \boldsymbol{Y} & \frac{\boldsymbol{y}-\sum_{k=1}^n \mu_k \boldsymbol{e}_k}{2} \\ \frac{(\boldsymbol{y}-\sum_{k=1}^n \mu_k \boldsymbol{e}_k)^T}{2} & y_0 + K \end{pmatrix} \text{ is copositive.}$$

Thus we have, $\boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{y} + y_0 \geq \min_{1\leq k\leq n} x_k - K$ for all $\boldsymbol{x} \in \mathbb{R}^n_+$ if there exists $\boldsymbol{\mu} \in \mathbb{R}^n_+$, $\sum_{k=1}^n \mu_k = 1$, such that $\boldsymbol{Y}_\mu$ is copositive. □

From Lemma 1 and 2, and the copositivity sufficient condition in Remark 1, we can calculate an upper bound for the expected payoff P as shown in the following theorem:

Theorem 1 *The optimal value of the following semidefinite programming problem is an upper bound for the expected payoff P:*

$$\begin{array}{rll} P_u^c = \min & \boldsymbol{Q}\cdot\boldsymbol{Y} + \boldsymbol{\mu}^T\boldsymbol{y} + y_0 & \\ \text{s.t.} & \begin{pmatrix} \boldsymbol{Y} & \frac{\boldsymbol{y}}{2} \\ \frac{\boldsymbol{y}^T}{2} & y_0 \end{pmatrix} = \boldsymbol{P}_1 + \boldsymbol{N}_1, & \\ & \begin{pmatrix} \boldsymbol{Y} & \frac{\boldsymbol{y}-\sum_{k=1}^n \mu_k \boldsymbol{e}_k}{2} \\ \frac{(\boldsymbol{y}-\sum_{k=1}^n \mu_k \boldsymbol{e}_k)^T}{2} & y_0 + K \end{pmatrix} = \boldsymbol{P}_2 + \boldsymbol{N}_2, & \\ & \sum_{k=1}^n \mu_k = 1, \boldsymbol{\mu} \geq \boldsymbol{0}, & \\ & \boldsymbol{P}_i \succeq 0, \boldsymbol{N}_i \geq \boldsymbol{0} & i = 1, 2. \end{array} \tag{4}$$

Proof. Consider an optimal solution $(\boldsymbol{Y}, \boldsymbol{y}, y_0, \boldsymbol{P}_1, \boldsymbol{N}_1, \boldsymbol{P}_2, \boldsymbol{N}_2, \boldsymbol{\mu})$ of Problem (4). According to Remark 1, $\bar{\boldsymbol{Y}}$ is a copositive matrix. Therefore, $(\boldsymbol{Y}, \boldsymbol{y}, y_0)$ satisfies the first constraint of Problem (3) following Lemma 1. Similarly, the second constraint of Problem (3) is also satisfied by $(\boldsymbol{Y}, \boldsymbol{y}, y_0)$ according to Lemma 2. Thus, $(\boldsymbol{Y}, \boldsymbol{y}, y_0)$ is a feasible solution of Problem (3), which means

$$P_u^c \geq P_u.$$

We have $P_u \geq P_{\max}$; therefore, $P_u^c \geq P_{\max}$ or P_u^c is an upper bound for the expected payoff P. □

3 Lower Bounds

The tight lower bound of the expected payoff P is $P_{\min} = \min_{\boldsymbol{X}\sim(\boldsymbol{\mu},\boldsymbol{Q})^+} \mathbb{E}[(\min_{1\leq k\leq n} X_k - K)^+]$. However, due to the non-convexity of the payoff function, it is difficult to evaluate $P_{\min}$. Applying Jensen's inequality for the convex function $f(x) = x^+$, we have:

$$\max\{0, \mathbb{E}[\min_{1\leq k\leq n} X_k - K]\} \leq \mathbb{E}[(\min_{1\leq k\leq n} X_k - K)^+].$$

Define $\bar{P}_{\min} = \min_{\boldsymbol{X}\sim(\boldsymbol{\mu},\boldsymbol{Q})^+} \mathbb{E}[\min_{1\leq k\leq n} X_k - K]$, then clearly, $\max\{0, \bar{P}_{\min}\} \leq P_{\min}$ or $\max\{0, \bar{P}_{\min}\}$ is a lower bound for the expected payoff P.

We have, $\bar{P}_{\min}$ can be calculated as follows:

$$\begin{array}{rll} P_{\min} = -\max_f & \int_{\mathbb{R}_+^n} (K - \min_{1\leq k\leq n} x_k) f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} & \\ \text{s.t.} & \int_{\mathbb{R}_+^n} x_k f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \mu_k, & \forall\, k = 1, \ldots, n, \\ & \int_{\mathbb{R}_+^n} x_k x_l f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = Q_{kl}, & \forall\, 1 \leq k \leq l \leq n, \\ & \int_{\mathbb{R}_+^n} f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = 1, & \\ & f(\boldsymbol{x}) \geq 0, & \forall\, \boldsymbol{x} \in \mathbb{R}_+^n, \end{array} \tag{5}$$

where f is a probability density function.

Taking the dual, we obtain the following problem:

$$\begin{aligned} P_l = -\min\nolimits_{\boldsymbol{Y},\boldsymbol{y},y_0} \quad & \boldsymbol{Q}\cdot\boldsymbol{Y} + \boldsymbol{\mu}^T\boldsymbol{y} + y_0 \\ \text{s.t.} \quad & \boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{y} + y_0 \geq K - \min\nolimits_{1\leq k\leq n} x_k, \quad \forall\, \boldsymbol{x}\in\mathbb{R}^n_+, \end{aligned}$$

or equivalently,

$$\begin{aligned} P_l = -\min\nolimits_{\boldsymbol{Y},\boldsymbol{y},y_0} \quad & \boldsymbol{Q}\cdot\boldsymbol{Y} + \boldsymbol{\mu}^T\boldsymbol{y} + y_0 \\ \text{s.t.} \quad & \boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{y} + y_0 + x_k - K \geq 0, \quad \forall\, \boldsymbol{x}\in\mathbb{R}^n_+, k = 1,\ldots,n. \end{aligned} \tag{6}$$

Similarly, $P_l \leq \bar{P}_{\min}$ according to weak duality and if the Slater condition is satisfied, $P_l = \bar{P}_{\min}$.

Now consider the constraints of Problem (6). Using Lemma 1, each constraint of Problem (6) is equivalent to a copositive matrix constraint:

$$\boldsymbol{x}^T\boldsymbol{Y}\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{y} + y_0 + x_k - K \geq 0, \quad \forall\, \boldsymbol{x}\in\mathbb{R}^n_+ \Leftrightarrow \begin{pmatrix} \boldsymbol{Y} & \frac{\boldsymbol{y}+\boldsymbol{e}_k}{2} \\ \frac{\boldsymbol{y}+\boldsymbol{e}_k}{2}^T & y_0 - K \end{pmatrix} \text{ is copositive.}$$

With Remark 1, we can then calculate a lower bound for the expected payoff P as shown in the following theorem:

Theorem 2 $\max\{0, P_l^c\}$ *is a lower bound for the expected payoff* P*, where*

$$\begin{aligned} P_l^c = -\min \quad & \boldsymbol{Q}\cdot\boldsymbol{Y} + \boldsymbol{\mu}^T\boldsymbol{y} + y_0 \\ \text{s.t.} \quad & \begin{pmatrix} \boldsymbol{Y} & \frac{\boldsymbol{y}+\boldsymbol{e}_k}{2} \\ \frac{\boldsymbol{y}+\boldsymbol{e}_k}{2}^T & y_0 - K \end{pmatrix} = \boldsymbol{P}_k + \boldsymbol{N}_k, \quad \forall\, k = 1,\ldots,n \\ & \boldsymbol{P}_k \succeq 0, \boldsymbol{N}_k \geq \boldsymbol{0} \qquad\qquad k = 1,\ldots,n. \end{aligned} \tag{7}$$

Proof. Consider an optimal solution $(\boldsymbol{Y},\boldsymbol{y},y_0,\boldsymbol{P}_k,\boldsymbol{N}_k)$ of Problem 7. According to Remark 1, the matrix $\begin{pmatrix} \boldsymbol{Y} & \frac{\boldsymbol{y}+\boldsymbol{e}_k}{2} \\ \frac{\boldsymbol{y}+\boldsymbol{e}_k}{2}^T & y_0 - K \end{pmatrix}$ is copositive for all $k = 1,\ldots,n$. Lemma 1 shows that $(\boldsymbol{Y},\boldsymbol{y},y_0)$ satisfies all constraints of Problem 6. Thus $(\boldsymbol{Y},\boldsymbol{y},y_0)$ is a feasible solution of Problem 6, which means

$$P_l^c \leq P_l.$$

We have $\bar{P}_{\min} \geq P_l$ and $\max\{0,\bar{P}_{\min}\} \leq P_{\min}$; therefore, $\max\{0,P_l^c\} \leq P_{\min}$ or $\max\{0,P_l^c\}$ is a lower bound for the expected payoff P. $\square$

4 Computational Results

4.1 Call Options on the Minimum of Several Assets

We consider the call option on the minimum of $n = 4$ assets. In order to compare the bounds with the exact option price, we assume that these assets follow a correlated multivariate lognormal distribution. At time t, the price of asset k is calculated as follows:

$$S_k(t) = S_k(0)e^{(r-\delta_k^2/2)t+\delta_k W_k(t)},$$

where $S_k(0)$ is the initial price at time 0, r is the risk-free rate, δ_k is the volatility of asset k, and $(W_k(t))_{k=1}^n$ is the standard correlated multivariate Brownian motion. We use similar parameter values as in Boyle and Lin [4]. The risk-free rate is $r = 10\%$ and the maturity is $T = 1$. The initial prices are set to be $S_k(0) = \$40$ for all $k = 1, \ldots, n$. For each asset k, the price volatility is $\delta_k = 30\%$. The correlation parameters are set to be $\rho_{kl} = 0.9$ for all $k \neq l$ (and obviously, we can define $\rho_{kk} = 1.0$ for all $k = 1, \ldots, n$). These values are used to calculate first and second moments, $\boldsymbol{\mu}$ and $\boldsymbol{Q}$, of $\boldsymbol{X} = (S_k(T))_{k=1}^n$using the following formulae:

$$\mathbb{E}[X_k] = e^{rT}S_k(0), \quad \forall\, k = 1, \ldots, n,$$

and

$$\mathbb{E}[X_k X_l] = S_k(0)S_l(0)e^{2rT}e^{\rho_{kl}\delta_k\delta_j T}, \quad \forall\, k, l = 1, \ldots, n.$$

The rational option price is $e^{-rT}P$, where P is the expected payoff. The exact price is calculated by Monte Carlo simulations of correlated multivariate Brownian motion described in Glasserman [6]. The upper and lower bounds are calculated by solving semidefinite programming problems formulated in Theorem 1 and 2.

In this report, all codes are developed using Matlab 7.4 and semidefinite programming problems are solved with SeduMi solver (Sturm [12]) using YALMIP interface (Löfberg [8]). We vary the strike price from $K = \$20$ to $K = \$50$ in this experiment and the results are shown in Table 1 and Figure 1.

In this example, we obtain valid positive lower bounds when the strike price is less than \$40. The lower and upper bounds are reasonably good in all cases. When the strike price decreases, the lower bound tends to be better (closer to the exact value) than the upper bound.

4.2 Put Options on the Maximum of Several Assets

European put options written on the maximum of several assets also have non-convex payoff. The payoff is calculated as $P = \mathbb{E}[(K - \max_{1\leq k\leq n} X_k)^+]$, where X_k is the price of asset k at the maturity.

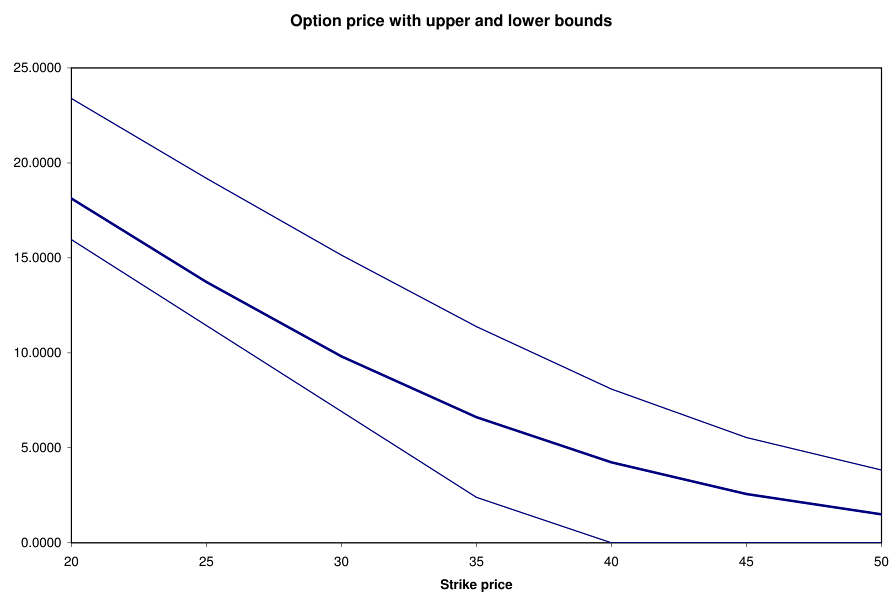

Figure 1: Prices of call options on the minimum of multiple assets and their upper and lower bounds

Strike price	20	25	30	35	40	45	50
Exact option price	18.1299	13.7308	9.8097	6.6091	4.2340	2.5712	1.5011
Upper bound	23.3489	19.1889	15.1476	11.3819	8.0961	5.5452	3.8287
Lower bound	15.9625	11.4383	6.9142	2.3900	0.0000	0.0000	0.0000

Table 1: Call option prices with different strike prices and their upper and lower bounds

Similar to call options on the minimum of multiple assets, upper and lower bounds of this payoff can be calculated by solving the following semidefinite programming problems:

$$\begin{array}{ll} \min & \boldsymbol{Q} \cdot \boldsymbol{Y} + \boldsymbol{\mu}^T \boldsymbol{y} + y_0 \\ \text{s.t.} & \begin{pmatrix} \boldsymbol{Y} & \frac{\boldsymbol{y}}{2} \\ \frac{\boldsymbol{y}}{2}^T & y_0 \end{pmatrix} = \boldsymbol{P}_1 + \boldsymbol{N}_1, \\ & \begin{pmatrix} \boldsymbol{Y} & \frac{\boldsymbol{y}+\sum_{k=1}^n \mu_k \boldsymbol{e}_k}{2} \\ \frac{(\boldsymbol{y}+\sum_{k=1}^n \mu_k \boldsymbol{e}_k)}{2}^T & y_0 - K \end{pmatrix} = \boldsymbol{P}_2 + \boldsymbol{N}_2, \\ & \sum_{k=1}^n \mu_k = 1, \boldsymbol{\mu} \geq \boldsymbol{0}, \\ & \boldsymbol{P}_i \succeq 0, \boldsymbol{N}_i \geq \boldsymbol{0} \qquad\qquad i = 1, 2, \end{array} \tag{8}$$

and

$$\begin{array}{ll} \min & \boldsymbol{Q} \cdot \boldsymbol{Y} + \boldsymbol{\mu}^T \boldsymbol{y} + y_0 \\ \text{s.t.} & \begin{pmatrix} \boldsymbol{Y} & \frac{\boldsymbol{y}-\boldsymbol{e}_k}{2} \\ \frac{\boldsymbol{y}-\boldsymbol{e}_k}{2}^T & y_0 + K \end{pmatrix} = \boldsymbol{P}_k + \boldsymbol{N}_k, \quad \forall\, k = 1, \ldots, n \\ & \boldsymbol{P}_k \succeq 0, \boldsymbol{N}_k \geq \boldsymbol{0} \qquad\qquad k = 1, \ldots, n. \end{array} \tag{9}$$

Solving these two problems using the same data as in the previous section and varying the strike price from \$40 to \$70, we obtain the results for this put option, which are shown in Table 2 and Figure 2.

Strike price	40	45	50	55	60	65	70
Exact option price	1.7419	3.4669	5.8114	8.7931	12.1431	16.0553	20.0943
Upper bound	4.2896	6.2629	9.0706	12.5363	16.4070	20.5079	24.4722
Lower bound	0.0000	0.0000	0.0000	3.8253	8.3495	12.8737	17.3979

Table 2: Put option prices with different strike prices and their upper and lower bounds

We also have valid positive lower bounds when the strike price is higher than \$50. The lower bound is closer to the exact value than the upper bound when the strike price increases. In general, both upper

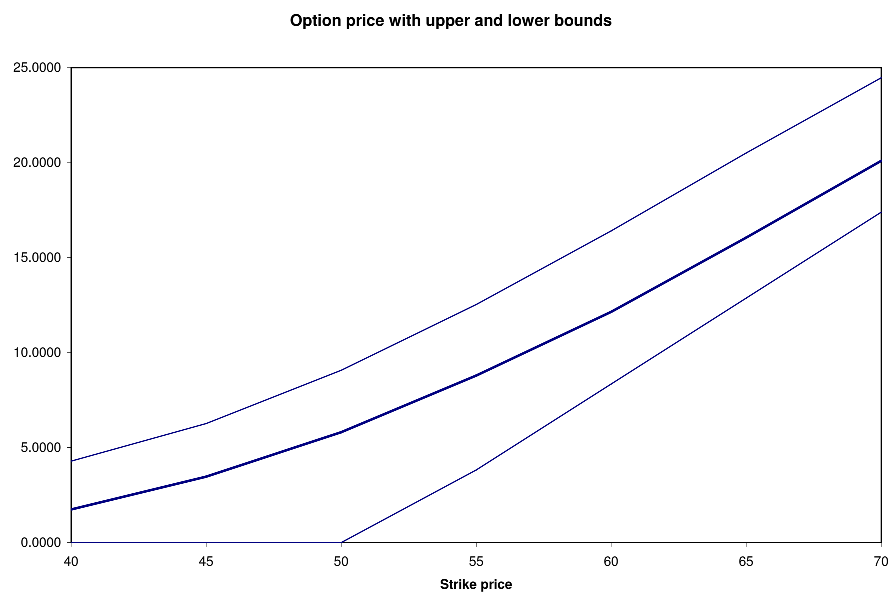

Figure 2: Prices of put options on the maximum of multiple assets and their upper and lower bounds

and lower bounds are significant as compared to the exact option prices.

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