Traces, Homotopy Theory, and Motivic Galois Groups

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Abstract. This thesis consists of two independent parts. In the first part we ask how traces in monoidal categories behave under homotopical operations. In order to investigate this question we define traces in closed monoidal derivators and establish some of their properties. In the stable setting we derive an explicit formula for the trace of the homotopy colimit over finite categories in which every endomorphism is invertible.

In the second part, we study motives of algebraic varieties over a subfield of the complex numbers, as defined by Nori on the one hand and by Voevodsky, Levine, and Hanamura on the other. Ayoub attached to the latter theory a motivic Galois group using the Betti realization, based on a weak Tannakian formalism. Our main theorem states that Nori's and Ayoub's motivic Galois groups are isomorphic. In the process of proving this result we construct well-behaved functors relating the two theories which are of independent interest.

Zusammenfassung. Diese Dissertation besteht aus zwei voneinander unabhängigen Teilen. Im ersten Teil widmen wir uns der Frage, wie sich die Spur in monoidalen Kategorien unter homotopischen Operationen verhält. Dazu definieren wir Spuren in abgeschlossenen monoidalen Derivatoren und beweisen einige ihrer Eigenschaften. Im stabilen Fall leiten wir schliesslich eine explizite Formel her für die Spur eines Homotopie-Kolimes über einer endlichen Kategorie, in der alle Endomorphismen invertierbar sind.

Im zweiten Teil behandeln wir Motive algebraischer Varietäten über einem Unterkörper der komplexen Zahlen, wie sie von Nori einerseits, und von Voevodsky, Levine, und Hanamura andererseits definiert wurden. Ayoub hat der letzteren der beiden Theorien eine motivische Galoisgruppe zugeordnet, mit Hilfe der Betti-Realisierung und basierend auf einem schwachen Tannakaformalismus. Unser Hauptresultat lautet, dass die motivischen Galoisgruppen von Nori und von Ayoub isomorph sind. Im Laufe des Beweises konstruieren wir auch Funktoren, welche die beiden Theorien miteinander in Verbindung setzen und die unabhängig von Interesse sind.

CONTENTS

Manuscript information and contribution of co-authors

- Chapter II of this thesis is almost identical to the published article:
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- Chapter III is joint work with Utsav Choudhury and incorporates appendix C from our preprint:
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I

INTRODUCTION

Since this thesis is composed of mathematical papers on different subjects we felt the need to include this short introduction in which we explore how the chapters to come are related. The answer, as we will try to explain, is that the relation doesn't lie in the results but in some of the methods used in obtaining them. These methods belong to the field which one could reasonably call *homotopy theory*, and we start by outlining some aspects of what this field, as we see it, is supposed to achieve. Then we sketch some of its methods, and how they will be employed in the main body of the thesis. When writing this introduction we had in mind a reader who might not be familiar with the terminology and context of the later chapters. Since each of the subsequent chapters starts with a more detailed account of its content, the reader may safely jump directly to page 11.

Contents

1. Homotopy theory

1.1. Localization. Arguably, the *classification problem* is one of the fundamental mathematical questions. It asks, given a collection of objects, structured in some way, and a notion of equivalence between them, for a useful description of the equivalence classes. This description usually takes the form of a complete set of *invariants* which permit to decide whether or not two objects fall into the same equivalence class.

In a number of mathematical fields such a collection of objects is nowadays often organized in a *category* $\mathcal{C}$, and the equivalence is induced by a distinguished class of morphisms $\mathcal{W}$ in $\mathcal{C}$, often called *weak equivalences*. The classification problem then turns into the related task of understanding the *homotopy category* $\mathcal{C}[\mathcal{W}^{-1}]$ obtained from $\mathcal{C}$ by "inverting" the morphisms in $\mathcal{W}$, a process known as *localization*. Thus there is a functor $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ which maps $\mathcal{W}$ to isomorphisms and it is universal for this property, in the sense that given any functor $\mathcal{C} \to \mathcal{D}$ which maps $\mathcal{W}$ to isomorphisms, there is a unique functor $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ making the triangle

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \\ \mathcal{C}[\mathcal{W}^{-1}] \arrow[ur] & \end{tikzcd}$$

commute. More often than not, invariants in this setting are functors defined on $\mathcal{C}$ which factor through the homotopy category.

Example 1.1

(1) One of the fundamental examples arises in topology when trying to classify spaces "up to homotopy". Topological spaces assemble into a category (with continuous maps as morphisms) and the equivalence is induced by weak homotopy equivalences, i. e. continuous maps inducing a bijection in π_0 and isomorphisms of homotopy groups for varying base points.[1] The associated homotopy category hTop is sometimes called simply *the* homotopy category. Clearly, π_0 and the homotopy groups are most useful invariants in this context.

Besides ordinary homotopy groups, topologists are also interested in *stable* homotopy groups of spaces. These are the fundamental invariants of the stable homotopy category shTop obtained by localizing spectra with respect to stable weak equivalences. Other important invariants in the context of both unstable and stable homotopy theory include various cohomology theories.

(2) Many interesting functors $f : \mathcal{B} \to \mathcal{A}$ between abelian categories turn out to preserve only cokernels but not kernels (or vice-versa). Evaluation of such f at a short exact sequence in $\mathcal{B}$ does not in general produce another short exact sequence in $\mathcal{A}$ but only a *long* exact sequence involving so-called higher derived functors $\mathrm{L}^n f$. It was an important insight that these objects are only shadows of a structurally richer object, well-defined up to some indeterminacy. More precisely, given an object $b \in \mathcal{B}$ one associates to it under suitable assumptions a chain complex $\mathrm{L}f(b)$ in $\mathcal{A}$, well-defined up to quasi-isomorphism. Here, a quasi-isomorphism is a morphism of chain complexes inducing isomorphisms in homology. Thus one is naturally led to consider the category of chain complexes in $\mathcal{A}$ and localize it with respect to quasi-isomorphisms. The resulting homotopy category $\mathbf{D}(\mathcal{A})$ is called the derived category of $\mathcal{A}$. Under suitable assumptions f induces a "total" derived functor $\mathrm{L}f : \mathbf{D}(\mathcal{B}) \to \mathbf{D}(\mathcal{A})$ and the higher derived functor $\mathrm{L}^n f$ is recovered as the n-th homology object $\mathrm{H}_n \mathrm{L}f$. (An analogous story of course applies to *left*-exact functors.)

It is not difficult to see how $\mathcal{C}[\mathcal{W}^{-1}]$ can be constructed:[2] It can be taken to have the same objects as $\mathcal{C}$ and morphisms in $\mathcal{C}[\mathcal{W}^{-1}]$ to be represented by zig-zags of morphisms in $\mathcal{C}$, in general of arbitrary (finite) length, where the arrows pointing in the wrong direction lie in $\mathcal{W}$. However, certain zig-zags need to be identified and the whole thing quickly becomes intractable. Thus unfortunately and, given the nature of the task, not unexpectedly, the localization process and consequently the resulting homotopy category are difficult to control in general.

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[1] Another possibility is to consider the equivalence induced by homotopy equivalences.

[2] Ignoring set-theoretic issues, that is.

And even when there is a better understanding of the morphisms, $\mathcal{C}[\mathcal{W}^{-1}]$ typically behaves badly from a categorical point of view. Categorical operations (e. g. colimits and limits), even if possible in $\mathcal{C}$, often cannot be performed on the level of the homotopy category. In the same vein, categorical structures present in $\mathcal{C}$ often fail to descend to $\mathcal{C}[\mathcal{W}^{-1}]$. The reason is of course that localization is a process of identifications and there is no reason for these to be compatible with the operations or the structure.

Example 1.2

(1) Except in trivial cases the derived category $\mathbf{D}(\mathcal{A})$ is not abelian anymore. In fact, it does not even possess kernels and cokernels. Instead, one has to content oneself with the structure of a *triangulated* category where short exact sequences are replaced by distinguished triangles

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to A[1],$$

where $[1]$ denotes the shift of a complex. α (called the cocone) is a weak kernel for β, and conversely β (called the cone) is a weak cokernel for α. This means that the "universal" property is formulated as an existence statement without the uniqueness part. It is one of the fundamental problems in the theory of triangulated categories that (co)cones are not functorial.

(2) Similarly, while hTop possesses (co)products, many other (co)limits do not exist. One of the reasons why topologists prefer to work in the stable homotopy category is that it at least possesses the structure of a triangulated category (the shift in the case of complexes is replaced by the suspension).

1.2. Models and homotopy theory. *Homotopy theory*, at least in one interpretation, is the mathematician's tool kit to deal with the problems just mentioned. As a general rule, it assumes that the pair $(\mathcal{C}, \mathcal{W})$ comes with additional data which allow the construction of a *model* for the homotopy category. Depending on the situation different types of additional data are available and accordingly different types of models can be constructed. But they all share the basic property that the homotopy category can be obtained from the model so that the localization breaks into two steps:

$$(\mathcal{C}, \mathcal{W}) + \text{ data } \rightsquigarrow \text{ model } \rightsquigarrow \mathcal{C}[\mathcal{W}^{-1}]$$

Moreover, the second step is usually well-understood and therefore allows for a better grasp of the homotopy category.

While such a model cannot magically create a (co)limit in the homotopy category which did not exist before, it does allow to produce so-called homotopy (co)limits which often represent a satisfactory substitute. Loosely speaking, the homotopy (co)limit of a diagram $F : I \to \mathcal{C}$ is the (co)limit where one is allowed to replace diagrams by weakly equivalent ones. If it exists it is an object in $\mathcal{C}[\mathcal{W}^{-1}]$, well-defined up to unique isomorphism. Although its definition makes sense without a model, the latter is often needed in order to prove that the object satisfies the universal property. Similarly, the model will not produce by itself some structure on the homotopy category but it is often needed in order to prove the existence of such a structure. In particular, it is often preferable to pass through a model to show that a categorical structure present in $\mathcal{C}$ descends to $\mathcal{C}[\mathcal{W}^{-1}]$. At the risk of oversimplification one could say that while $(\mathcal{C}, \mathcal{W})$ contains all the information lost in the localization process, it is only the model which both has enough information *and* is organized in a homotopically meaningful way.

1.3. Model categories. By now, there is a large number of models available in the literature, many of them coming in different variants to accomodate as many situations as possible. In this thesis we will employ two of them. The most prominent and influential model is a *model category*. Two additional classes of morphisms in $\mathcal{C}$ are assumed to be given, called cofibrations and fibrations, respectively, which together with the pair $(\mathcal{C}, \mathcal{W})$ have to satisfy a number of axioms. For example, one of the axioms requires $\mathcal{C}$ to possess small (co)limits, and another states that every morphism can be factored into a cofibration followed by a fibration, one of which may additionally be chosen to be a weak equivalence. From these data (called a model structure), it is possible to mimick classical homotopical constructions such as cylinder and path objects, and to define homotopies between morphisms. An object is called cofibrant if the unique morphism it receives from the initial object is a cofibration, and fibrant if the unique morphism to the terminal object is a fibration. The second axiom above implies that every object c can be replaced, up to weak equivalence, by a cofibrant and fibrant object $\overline{c}$. One finally shows that in the homotopy category $\mathcal{C}[\mathcal{W}^{-1}]$ the morphisms from c_1 to c_2 may be identified with homotopy classes of morphism from $\overline{c}_1$ to $\overline{c}_2$.

For several interesting choices of categories I, the category $\mathcal{C}^I$ of I-diagrams in $\mathcal{C}$ admits model structures such that the homotopy (co)limit of a diagram $F : I \to \mathcal{C}$ may be computed as the ordinary (co)limit of a (co)fibrant replacement.[3] Moreover, there are established theories of monoidal or stable model categories (among others) which under certain assumptions permit to put a natural monoidal or triangulated structure on the homotopy category.

Example 1.3

(1) Topological spaces with weak homotopy equivalences admit a model structure whose fibrations are Serre fibrations. With the cartesian product this category also possesses a monoidal structure but this does not behave well for general spaces. However, if one restricts to compactly generated spaces then the monoidal and the model structures are compatible and endow the localization (which is equivalent to hTop) with a monoidal structure.

   The category of spectra with stable weak equivalences inherits a model structure from the one on topological spaces. Moreover it is a pointed category and the suspension acts as an equivalence on the homotopy category hence the model structure is stable and the homotopy category possesses an induced triangulated structure. To endow it with a monoidal structure one needs another model though, e. g. the category of *symmetric* spectra with the smash product.

(2) Chain complexes in an abelian Grothendieck category with quasi-isomorphisms admit a model structure whose cofibrations are monomorphisms. Since it has a zero object and the shift functor is an autoequivalence this model structure is stable and the derived category inherits a triangulated structure. If $\mathcal{A}$ is a monoidal abelian category then the category of chain complexes is so as well but the model structure described above and the monoidal structure rarely interact well.[4] Of course, in many cases of interest there are other model structures which *are* compatible with the monoidal product thus rendering the derived category a monoidal triangulated category. This is true, for example, in the case of modules over a commutative ring and more generally sheaves on a ringed space.

1.4. Derivators. A rather different model is provided by the theory of *derivators*. For any category I one defines $\mathcal{W}^I$ to be the class of morphisms η in $\mathcal{C}^I$ which are objectwise weak equivalences, i. e. $\eta(i) \in \mathcal{W}$ for all i. Given a functor $f : J \to I$, precomposition with f

[3] Finding such a replacement is sometimes challenging though.

[4] In fact, we will have to confront this problem in the case of comodule categories over a bialgebra in chapter IV.

provides a functor $\mathcal{C}^I \to \mathcal{C}^J$ which preserves weak equivalences. By the universal property of the localization, it thus induces a functor on the level of homotopy categories, denoted by f^*. As a first approximation, the derivator $\mathbb{D}$ belonging to $(\mathcal{C}, \mathcal{W})$ can be thought of as the association

$$I \mapsto \mathcal{C}^I[(\mathcal{W}^I)^{-1}], \qquad\qquad f \mapsto f^*.$$

Of course, the homotopy category is obtained by simply evaluating $\mathbb{D}$ at the terminal category $\star$. The functors f^* are assumed to possess both left and right adjoints which are denoted by $f_!$ and f_*, respectively. If $p_I : I \to \star$ denotes the unique functor to the terminal category then $p_{I!}, p_{I*} : \mathbb{D}(I) \to \mathbb{D}(\star)$ are nothing but the homotopy colimit and limit functors. As before there is a theory of monoidal and stable derivators which endows the homotopy category with monoidal and triangulated structures, respectively; in fact, all categories $\mathbb{D}(I)$ are then monoidal and triangulated, with all f^* monoidal and exact, respectively.

For this simplicity in the formalism there is of course a price to pay. First one is often forced to take recourse to another model in order to establish the existence[5] and the axioms of the derivator. And secondly, the formalism does not lend itself to explicit computations in specific contexts. But, as we hope to illustrate in chapter II, it is very well suited to prove something which is true in *all* (monoidal, stable, …) derivators.

Example 1.4 If $(\mathcal{C}, \mathcal{W})$ admits a model structure then the associated derivator exists and satisfies the required axioms. Moreover, if the model category is monoidal or stable then so is the associated derivator. In particular there are derivators associated topological spaces (monoidal), spectra (stable monoidal), and chain complexes in Grothendieck abelian categories (stable, frequently also monoidal).

2. About this thesis

In §1 we described some of the goals of homotopy theory as we see it, and sketched two models which are used in order to achieve these goals. Now we would like to relate this story to the subsequent chapters of this thesis. "Doing homotopy theory" can mean either enlarging the homotopy theoretic tool kit, or using homotopy theoretic tools to prove statements outside of homotopy theory. The reader will find instances of both types in this document although the focus is on the latter.

2.1. Traces. The starting point for chapter II is the simple observation that there is something common to the following elementary mathematical statements.

(1) If V and W are finite-dimensional vector spaces, then the dimension of their direct sum is given by
$$\dim(V \oplus W) = \dim(V) + \dim(W).$$
(2) If $X \leftarrow Y \to Z$ are maps of finite CW-complexes then the Euler characteristic of the homotopy pushout of X and Z along Y satisfies
$$\chi(X \cup^h_Y Z) = \chi(X) - \chi(Y) + \chi(Z).$$
(3) If G is a finite group, and A a finite G-set, then Burnside's lemma states that the number of G-orbits admits the following description:
$$\#(A/G) = \frac{1}{\#G} \sum_{g \in G} \#(A^g).$$
Here, $\#(A^g)$ is the number of fixed points of g acting on A.

[5] Again, the existence of the homotopy categories is set-theoretically problematic.

Indeed, what seems to be common is that in each case the "size" of the result of some operation is expressed in terms of the "sizes" of the data on which the operation is performed. Often such formulas provide a means to reduce "size measurements" of more complicated objects to simpler ones, and are therefore a very handy tool. Now, it had been realized long before that the different "size" invariants appearing in the formulas above can be interpreted as instances of a single mathematical notion, called the *trace*, which is defined in any (symmetric unitary) monoidal category (cf. §II.3).

Definition 2.1 Let $(\mathcal{C}, \otimes)$ be a (symmetric) monoidal category with unit $\mathbb{1}$.

(1) An object c in $\mathcal{C}$ is called *dualizable with dual* c^* if there are morphisms $\eta : \mathbb{1} \to c \otimes c^*$ and $\varepsilon : c^* \otimes c \to \mathbb{1}$ such that $(c \otimes \varepsilon)(\eta \otimes c)$ is the identity on c, and $(\varepsilon \otimes c^*)(c^* \otimes \eta)$ is the identity on c^*.
(2) Given a dualizable object c with data as above, and an endomorphism $f : c \to c$, we define its *trace* to be the composition

$$\mathrm{tr}(f) : \mathbb{1} \xrightarrow{\eta} c \otimes c^* \xrightarrow{f \otimes \mathrm{id}} c \otimes c^* \xrightarrow{\sim} c^* \otimes c \xrightarrow{\varepsilon} \mathbb{1}.$$

(3) The *Euler characteristic* $\chi(c)$ of a dualizable object c is the trace of the identity endomorphism.

Example 2.2

(1) The category of k-vector spaces with the tensor product is monoidal. An object V is dualizable if and only if it is finite dimensional, and in this case the dual can be taken to be the dual vector space V^* with maps

$$\begin{aligned} k &\longrightarrow V \otimes V^* & \qquad V^* \otimes V &\longrightarrow k \\ 1 &\longmapsto \sum_{i=1}^{n} v_i \otimes v_i^* & \qquad f \otimes v &\longmapsto f(v) \end{aligned}$$

where $v_1, \ldots, v_n$ is a basis of V with dual basis $v_1^*, \ldots, v_n^*$. An easy computation shows that under the identification $\mathrm{End}(k) \cong k$, the trace of an endomorphism on V coincides with the usual trace from linear algebra. In particular, the Euler characteristic of V is given by $\dim(V)$.

More generally, in the category of Λ-modules over a commutative ring Λ with the usual tensor product, the dualizable objects are the finitely generated projective modules, and the Euler characteristic is identified with the rank of a module. In both these cases we can also consider these objects in the derived category which inherits a monoidal structure (via the derived tensor product). The functor $\mathbf{Mod}(\Lambda) \to \mathbf{D}(\Lambda)$ clearly preserves dualizability and traces.
(2) While the category of topological spaces is monoidal, its only dualizable object is the 1-point space. However, we can consider spaces as objects in shTop which, as we saw above, is monoidal. Any compact ENR (in particular any compact manifold or finite CW-complex) is dualizable with dual the Thom space of its stable normal bundle. The endomorphism ring of the unit spectrum is isomorphic to $\mathbb{Z}$ and the trace is given by the fixed point index. In particular, the Euler characteristic coincides with the usual Euler characteristic from topology.
(3) Finally, as the category of sets is cartesian monoidal, its only dualizable object is again the terminal object. However, to a set we can associate the k-vector space with the set as basis, and so the G-set A gives rise to a k-vector space V with a G-action. Notice that the dimension of V/G is precisely $\#(A/G)$ while for any $g \in G$, the trace of $g : V \to V$ is

equal to $\#(A^g)$. Hence we find that if $\#G$ is invertible in k then Burnside's lemma gives a formula for the Euler characteristic of the quotient of V by G in terms of other traces.

Summarizing, we find that the three statements we started with can all be interpreted as being about traces of endomorphisms in (stable) monoidal homotopy categories. Starting with some objects in such a homotopy category, in each example we perform some homotopical operation (direct sum, homotopy pushout, (homotopy) quotient) and we obtain a formula for the Euler characteristic of the resulting object. Moreover, the formula is in terms of traces or Euler characteristics of the objects we started with. Hence the simple belief which was our starting point turns into the quest for a general formula telling us how traces behave under homotopical operations.

In a famous article, May essentially proved that the trace is additive in distinguished triangles whenever the homotopy category comes from a stable monoidal model category. He did so by laying down a list of rather complicated axioms which express some form of compatibility between the monoidal and the triangulated structure. He then proved on the one hand that these axioms imply the additivity result, and on the other that homotopy categories of stable monoidal model categories satisfy the axioms. This recovers of course the first formula above, and can be shown to recover also the second. However, it is not clear how to recover the third or further formulas for other homotopical operations.

As we mentioned in section 1, the theory of derivators is well suited to prove formal properties of all homotopical operations. Thus in chapter II we work with stable monoidal derivators. Our main result gives an explicit formula for the trace of homotopy colimits over any finite EI-category (i. e. a category in which every endomorphism is invertible), of which we state here two prominent instances.

Theorem 2.3 *Let $\mathbb{D}$ be a stable closed monoidal derivator, and I a finite poset. For any fiberwise dualizable I-diagram A in $\mathbb{D}$, and any endomorphism f of A, the following identity holds:*

$$\mathrm{tr}(\mathrm{hocolim}_I f) = \sum_{i\in I} \lambda_i \mathrm{tr}(f_i)$$

where

$$\lambda_i = \sum_{n\geq 0} (-1)^n \#\{\textit{chains of length } n \textit{ in } I \textit{ starting at } i\}.$$

(Fiberwise dualizable means that for any $i \in I$, the "fiber" $A_i := i^* A \in \mathbb{D}(\star)$ is dualizable.)

Example 2.4

(1) If we take I to be the discrete category on two objects a and b, then the theorem gives back our first formula above:

$$\mathrm{tr}(f_a \oplus f_b) = \mathrm{tr}(f_a) + \mathrm{tr}(f_b).$$

(2) If we take I to be the span $a \leftarrow b \rightarrow c$, we find the coefficients $\lambda_a = \lambda_c = 1$, and $\lambda_b = -1$ hence we recover the formula for the homotopy pushout above.

The last formula, Burnside's lemma, is a consequence of the second instance of the theorem we would like to record here.

Theorem 2.5 *Let G be a finite group and $\mathbb{D}$ a stable closed monoidal derivator in which $\#G$ is invertible. For any G-representation on a dualizable object $A \in \mathbb{D}(\star)$, the following identity holds:*

$$\chi(A/G) = \frac{1}{\#G} \sum_{g\in G} \mathrm{tr}(g : A \rightarrow A).$$

The proof proceeds by defining the notion of a trace in such derivators, lifting the one in the homotopy category, and establishing its basic properties. This first step is surprisingly intricate and forms the technical heart of the chapter. In the second step we show that the derivators we are interested in admit a particularly simple description when evaluated at a finite EI-category. Finally, deducing the sought after formula is a beautiful exercise in what one could call "categorical combinatorics". For more details we refer the reader to the introductory pages of chapter II.

2.2. Motivic Galois groups. We now pass to chapters III and IV of the thesis. One of the basic objects in algebraic geometry are varieties over a field, and one powerful method to study them is to compute their cohomology in different cohomology theories. Grothendieck predicted the existence of a *universal* cohomology theory, called the theory of *(mixed) motives*, from which all the others should be derived. Understanding this universal cohomological invariant is therefore of utmost interest in algebraic geometry.

More formally, there should exist, for any field k, an abelian category $\mathcal{M}(k)$ of motives over k together with a functor $M : (\mathrm{Var}/k)^{\mathrm{op}} \to \mathcal{M}(k)$ associating to any variety over k its motive. Every cohomology theory $h : (\mathrm{Var}/k)^{\mathrm{op}} \to \mathcal{A}$ for varieties over k should factor through a realization functor $R_h : \mathcal{M}(k) \to \mathcal{A}$, i. e. $h(X) = R_h(M(X))$. If the characteristic of k is 0, one expects the realization functor for some cohomology theories h to present $\mathcal{M}(k)$ as a neutral Tannakian category with dual an affine pro-algebraic group called the *motivic Galois group* $\mathcal{G}(k)$. In other words, $\mathcal{M}(k)$ is the category of $\mathcal{G}(k)$-representations. The main practical advantage of this viewpoint is that it allows to translate geometric and arithmetic questions about k-varieties into questions about (pro-)algebraic groups and their representations.

Although the picture drawn above is still conjectural, there are candidates for these objects and related constructions in specific situations. If k is a subfield of the complex numbers, then there are essentially two existing approaches to motives, one due to Nori and another due to several mathematicians, including Voevodsky.[6] The difference between the two approaches is extreme: Nori's category of motives is provably abelian and even Tannakian, whereas Voevodsky's category has the structure of a triangulated category, and is in fact a candidate not for $\mathcal{M}(k)$ but its derived category. And while Nori relies in his construction essentially on the Betti realization for varieties (and therefore on transcendental input), Voevodsky's category of motives is defined purely in terms of algebro-geometric data. Because of this, the morphisms and extensions in Nori's category are highly intractable but can be related to known algebro-geometric invariants in Voevodsky's case. It is therefore one of the ultimate goals in the theory of motives to create a bridge connecting the two approaches in order to combine their individual advantages.

While this goal seems out of reach at the moment, our main result in chapter IV can be seen as providing a weak link while sidestepping the more difficult and deep issues of the theory. Ayoub defined a motivic Galois group for Voevodsky motives using the Betti realization and a weak Tannakian formalism for monoidal categories, and we prove:

Theorem 2.6 *Nori's and Ayoub's motivic Galois groups are isomorphic.*

Of course this leaves open the question of the relation between Voevodsky motives and representations of the motivic Galois group in the theorem. Conjecturally the former live in the derived category of the latter but as we already mentioned, this is difficult to prove. Still, we believe that the theorem strengthens our belief in the correctness of the two approaches to

[6] Both approaches have by now been extended to more general bases.

motives. And we also expect to draw useful consequences from the isomorphism of groups because their constructions differ in interesting ways.

We have now described the main result proved in chapter IV. It shouldn't come as a surprise that homotopy theory enters the proof since, as we mentioned, Voevodsky's category is a candidate for the derived category $\mathbf{D}(\mathcal{M}(k))$. It is a homotopy category which can be modeled by a model category. (Since we want to prove statements about a specific model, derivators would be of limited use.) We will need good models for certain objects in this homotopy category and we want to be able to manipulate them efficiently, for example we want to compute their image under certain derived functors. In other words, we will need cofibrant and fibrant replacements which we understand sufficiently well, and for this it is necessary to study the model category giving rise to Voevodsky motives.

Although there are different variants of the construction, the basic idea is always to start with a category of geometric objects (smooth schemes over k, say) and to consider chain complexes of presheaves on it. The geometric objects will give rise to "motives" in the homotopy category, and moreover they should "generate" all motives. Taking presheaves functions as both a linearization and a (co)completion of the geometric objects. One then imposes relations one would like the candidate theory of motives to satisfy: descent with respect to a well-chosen topology (étale or Nisnevich), $\mathbb{A}^1$-invariance and invertibility of the "Tate twist". It turns out that for our purposes the first relation is the crucial one. It can be understood as a localization giving rise to the derived category of sheaves, and we will perform a close analysis of its homotopy theory in chapter III. Although most of these results are probably known to experts (even if they haven't all appeared in print) we believe that our mode of presentation might be interesting for emphasizing the universality of the constructions as well as for its level of generality.

With these homotopy theoretic tools under our belt, we will, in chapter IV, relate the two types of motives and representations by first constructing realization functors in both directions using among other things the six functors formalism for motives without transfer developed by Ayoub, as well as Nori's and Beilinson's "Basic Lemma". Apart from the application in our proof the existence of these well-behaved realizations may be used in other contexts as well, for example to deduce well-behaved (derived) mixed Hodge realizations for Voevodsky motives. We will see that the realization functors induce morphisms between the two motivic Galois groups, and the hard part is to show that these are inverses to each other. This relies on a close analysis of Ayoub's model for the object representing Betti cohomology on the one hand, and on our understanding of the good models and their image under the Betti realization mentioned above on the other. For more details we refer to the introductory pages of chapter IV.

II

TRACES IN MONOIDAL DERIVATORS

The additivity of traces. Let $\mathcal{C}$ be a symmetric monoidal category which in addition is triangulated. Examples include various "stable homotopy categories" (such as the classical and equivariant in algebraic topology, the motivic in algebraic geometry) or all kinds of "derived categories" (of modules, of perfect complexes on a scheme, etc.). Let X, Y and Z be dualizable objects in $\mathcal{C}$,

$$D: \qquad X \to Y \to Z \to^{+}$$

a distinguished triangle, and f an endomorphism of D. The *additivity of traces* is the statement that the following relation holds among the traces of the components of f:

$$\operatorname{tr}(f_Y) = \operatorname{tr}(f_X) + \operatorname{tr}(f_Z). \tag{0.7}$$

Well-known examples are the additivity of the Euler characteristic of finite CW-complexes ($\chi(Y) = \chi(X) + \chi(Y/X)$ for $X \subset Y$ a subcomplex) or the additivity of traces in short exact sequences of finite dimensional vector spaces. The additivity of traces should be considered as a *principle*: Although incorrect as it stands, it embodies an important idea. One should therefore try to find the right context to formulate this idea precisely and prove it.

In [55], J. Peter May made an important step in this direction. He gave a list of axioms expressing a compatibility of the monoidal and the triangulated structure, and proved that if they are satisfied, then one can always replace f by an endomorphism f' with $f'_X = f_X$ and $f'_Y = f_Y$ such that (0.7) holds for the components of f'. This result has two drawbacks though: Firstly, there is this awkwardness of f' replacing f, and secondly, the axioms are rather complicated.

As noted in [27], both these drawbacks are related to the well-known deficiencies of triangulated categories. Since the foremost example of a situation in which May's compatibility axioms hold, is when $\mathcal{C}$ is the homotopy category of a stable monoidal model category, it should not come as a surprise that May's result can be reproved in the setting of stable derivators. Moreover, since stable derivators eliminate some of the problems encountered in triangulated categories, a more satisfying formulation of the additivity of traces should be available. We will describe it now.

Let $\mathbb{D}$ be a closed symmetric monoidal stable derivator[1], and $\ulcorner$ the free category on the following graph:

$$\begin{tikzcd} (1,1) & (0,1) \arrow[l] \\ (1,0). \arrow[u] & \end{tikzcd} \tag{0.8}$$

Let A be an object of $\mathbb{D}(\ulcorner)$ with underlying diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ 0, & \end{tikzcd}$$

and suppose that both X and Y are dualizable objects of $\mathbb{D}(\star)$, $\star$ denoting the terminal category. Let f be an endomorphism of A, and denote by $p_\ulcorner$ the unique functor $\ulcorner \to \star$. Then there is a distinguished triangle

$$X \to Y \to p_{\ulcorner !}A \to^{+}$$

in $\mathbb{D}(\star)$, $p_{\ulcorner !}A$ is also dualizable, and the following relation holds:

$$\mathrm{tr}(f_{(0,1)}) = \mathrm{tr}(f_{(1,1)}) + \mathrm{tr}(p_{\ulcorner !}f).$$

This is the main theorem of [27].

The trace of the homotopy colimit. Another advantage of the formulation in the context of derivators is that it immediately invites us to consider the additivity of traces as a mere instance of a more general principle. As a first step, we see that the condition $A_{(1,0)} = 0$ is not essential. Indeed, if A is an object of $\mathbb{D}(\ulcorner)$ whose fibers are all dualizable objects in $\mathbb{D}(\star)$ and if f is an endomorphism of A then the formula above generalizes to

$$\mathrm{tr}(p_{\ulcorner !}f) = \mathrm{tr}(f_{(0,1)}) + \mathrm{tr}(f_{(1,0)}) - \mathrm{tr}(f_{(1,1)}). \tag{0.9}$$

And now, in the second step, it is natural to replace the category $\ulcorner$ by other categories I and try to see whether there still is an explicit formula for $\mathrm{tr}(p_{I!}f)$. The main result of this chapter states that this is the case for finite EI-categories, i. e. finite categories in which all endomorphisms are invertible (such as groups or posets), provided that the derivator is $\mathbb{Q}$-linear. For each of these categories the trace of the homotopy colimit of an endomorphism of a fiberwise dualizable object can be computed as a linear combination of "local traces" (depending only on the fibers of the endomorphism and the action of the automorphisms of the objects in the category) with coefficients which depend only on the category and can be computed combinatorially.

As for the proof of this result, the idea is to define the trace of endomorphisms of objects not only living in $\mathbb{D}(\star)$ but in $\mathbb{D}(I)$ for general categories I. This trace should contain enough information to relate the trace of the homotopy colimit to the local traces of the endomorphism. The naive approach of considering $\mathbb{D}(I)$ as a monoidal category and taking the usual notion of the trace doesn't lead too far though since few objects in $\mathbb{D}(I)$ will be dualizable in general even if in $\mathbb{D}(\star)$ all of them are; in other words, being fiberwise dualizable does not imply being dualizable.

This is why we will replace the "internal" tensor product by an "external product"

$$\boxtimes : \mathbb{D}(I) \times \mathbb{D}(I) \to \mathbb{D}(I \times I)$$

[1]See §1 for the definition of this notion.

and the internal hom by an "external hom"

$$\langle -,- \rangle : \mathbb{D}(I)^{\circ} \times \mathbb{D}(I) \to \mathbb{D}(I^{\circ} \times I),$$

which has the property that for any object A of $\mathbb{D}(I)$ and objects i, j of I

$$\langle A, A \rangle_{(i,j)} = [A_i, A_j]$$

(implying that fiberwise dualizable objects will be "dualizable with respect to the external hom") and which also contains enough information to compute $[A, A]$ (among other desired formal properties). As soon as this bifunctor is available we can mimic the usual definition of the "internal" trace in a closed symmetric monoidal category to define an "external" trace for any endomorphism of a fiberwise dualizable object, replacing the internal by the external hom everywhere. It will turn out that this new trace encodes all local traces, and in good cases allows us to relate these to the trace of the homotopy colimit, thus yielding the sought after formula.

After this chapter had been accepted for publication ([22]), Kate Ponto and Michael Shulman independently obtained the same result in [60]. However, their proof is quite different from the one presented here, relying on the technology of bicategorical traces.

Outline of this chapter. We do not include an introduction to the theory of derivators (see for this the references given in §1.3). However, as the definition of a derivator varies in the literature we give in §1.3 the axioms we use. Moreover, the few results on derivators we need are either proved or justified by a reference to where a proof can be found. In §1.4 we define the notion of a (closed) monoidal derivator and describe its relation to the axiomatization available in the literature. We also discuss briefly linear structures on derivators (1.5), stable derivators (1.6), and the interplay between stability and monoidal structures on derivators (1.7). Apart from this, §1 is intended to fix the notation used in the remainder of the chapter.

The main body of the text starts with §2 where the construction of the external hom mentioned above is given. The proofs of the desired formal properties of this bifunctor are lengthy and not needed in the sequel so they are deferred to appendix A. Next we define the external trace (§3) and prove its functoriality (§4). As a corollary we deduce that this trace encodes all local traces.

The main result is to be found in §5. First we prove that in good cases the trace of the homotopy colimit is a function of the external trace (again, the uninteresting part of the story is postponed to the appendix; specifically to appendix B). In the case of finite EI-categories and a $\mathbb{Q}$-linear stable derivator, this function can be made explicit, and this leads to the formula for the trace of the homotopy colimit in terms of the local traces. Some technical hypotheses used to prove this result will be eliminated in §6.

At several points in the text the need arises for an explicit description of an additive derivator evaluated at a finite group. Although this description is certainly well-known, we haven't been able to find it in the literature and have thus included it as appendix C.

Contents

1. Conventions and preliminaries

In this section, we recall some notions and facts (mostly related to derivators) and fix the notation used in the remainder of the chapter.

1.1. By a 2-category we mean a strict 2-category. The 2-category of (small) categories is denoted by **CAT** (**Cat**). Given a 2-category $\mathcal{C}$ (encompassing the special case of a category), $\mathcal{C}^\circ$ denotes the 2-category with the same objects, and $\mathcal{C}^\circ(x,y) = \mathcal{C}(y,x)$ for all objects x, y. The 2-category $\mathcal{C}^{\circ,\circ}$ also has the same objects as $\mathcal{C}$ but $\mathcal{C}^{\circ,\circ}(x,y) = \mathcal{C}(y,x)^\circ$ (see [45, p. 82]). The (possibly large) sets of objects, 1-morphisms and 2-morphisms in a 2-category $\mathcal{C}$ are sometimes denoted by $\mathcal{C}_0$, $\mathcal{C}_1$, and $\mathcal{C}_2$ respectively.

By a 2-functor we mean a *strict* 2-functor between 2-categories. Modifications are morphisms of lax natural transformations between 2-functors (see [45, p. 82]). For fixed 2-categories $\mathcal{C}$ and $\mathcal{D}$, the 2-functors from $\mathcal{C}$ to $\mathcal{D}$ together with lax natural transformations and modifications form a 2-category $\mathbf{Fun}_{\mathrm{lax}}(\mathcal{C},\mathcal{D})$.

1.2. Counits and units of adjunctions are usually denoted by adj. Given a functor $u : I \to J$, and an object $j \in J_0$, the category of *objects u-under j* is (abusively) denoted by $j\backslash I$ and the category of *objects u-over j* by I/j (see [50, 2.6]). We also need the following construction ([50, p. 223]): Given a category I, we define the *twisted arrow category* associated to I, denoted by $\mathrm{tw}(I)$, as having objects the arrows of I and as morphisms from $i \to j$ to $i' \to j'$ pairs of morphisms making the following square in I commutative:

$$\begin{tikzcd} i \arrow[r] & j \arrow[d] \\ i' \arrow[u] \arrow[r] & j'. \end{tikzcd}$$

There is a canonical functor $\mathrm{tw}(I) \to I^\circ \times I$. In fact, this extends canonically to a functor $\mathrm{tw}(-) : \mathbf{Cat} \to \mathbf{Cat}$ together with a natural transformation $\mathrm{tw}(-) \to (-)^\circ \times (-)$.

1.3. Let us recall the notion of a derivator. For the basic theory we refer to [52], [16], [26]. For an outline of the history of the subject see [16, p. 1385].

A full sub-2-category **Dia** of **Cat** is called a *diagram category* if:

(Dia1) **Dia** contains the totally ordered set $\underline{2} = \{0 < 1\}$;

(Dia2) **Dia** is closed under finite products and coproducts, and under taking the opposite category and subcategories;

(Dia3) if $I \in \mathbf{Dia}_0$ and $i \in I_0$, then $I/i \in \mathbf{Dia}_0$;

(Dia4) if $p : I \to J$ is a fibration (to be understood in the sense of [29, exposé VI]) whose fibers are all in **Dia**, and if $J \in \mathbf{Dia}_0$, then also $I \in \mathbf{Dia}_0$.

By (Dia2), the initial category $\varnothing$ and the terminal category $\star$ are both in **Dia**. We will often use that **Dia** is closed under pullbacks (as follows from (Dia2)). The smallest diagram category consists of finite posets, other typical examples include finite categories, finite-dimensional categories, all small posets or **Cat** itself.

A *prederivator (of type* **Dia***)* is a 2-functor $\mathbb{D} : \mathbf{Dia}^{\circ,\circ} \to \mathbf{CAT}$ from a diagram category **Dia** to **CAT**. If $\mathbb{D}$ is fixed in a context, **Dia** always denotes the domain of $\mathbb{D}$. Given a prederivator $\mathbb{D}$, categories $I, J \in \mathbf{Dia}_0$ and a functor $u : I \to J$, we denote by $u^* : \mathbb{D}(J) \to \mathbb{D}(I)$ the value of $\mathbb{D}$ at u; if u is clear from the context, we sometimes denote u^* by $|_I$. Its left and right adjoint (if they exist) are denoted by $u_!$ and u_* respectively. The unique functor $I \to \star$ is denoted by p_I. Given an object $i \in I_0$, we denote also by $i : \star \to I$ the functor pointing i. Thus for an object $A \in \mathbb{D}(I)_0$ and a morphism $f \in \mathbb{D}(I)_1$, their *fiber over* i is i^*A and i^*f, respectively, sometimes also denoted by A_i and f_i, respectively. Given a natural transformation $\eta : u \to v$ in **Dia**, we denote by η^* the value of $\mathbb{D}$ at η. It is a natural transformation from v^* to u^*. In particular, if $h : i \to j$ is an arrow in I then we can consider it as a natural transformation from the functor $i : \star \to I$ to $j : \star \to I$, and therefore it makes sense to write h^*; evaluated at an object $A \in \mathbb{D}(I)_0$, it yields a morphism of the fibers $A_j \to A_i$. The canonical "underlying diagram" functor $\mathbb{D}(I) \to \mathbf{CAT}(I^\circ, \mathbb{D}(\star))$ is denoted by dia_I. Finally, if $\mathbb{D}$ is a prederivator and $J \in \mathbf{Dia}_0$, we denote by $\mathbb{D}_J$ the prederivator $\mathbb{D}_J(-) = \mathbb{D}(- \times J)$.

A *derivator (of type* **Dia***)* is a prederivator (of type **Dia**) $\mathbb{D}$ satisfying the following list of axioms:

(D1) $\mathbb{D}$ takes arbitrary coproducts to products up to equivalence of categories.

(D2) For every $I \in \mathbf{Dia}_0$, the family of functors $i^* : \mathbb{D}(I) \to \mathbb{D}(\star)$ indexed by I_0 is jointly conservative.

(D3) For all functors $u \in \mathbf{Dia}_1$, the left and right adjoints $u_!$ and u_* to u^* exist.

(D4) Given a functor $u : I \to J$ in **Dia** and an object $j \in J_0$, the "Beck-Chevalley" transformations associated to both comma squares

$$\begin{tikzcd} j\backslash I \arrow[r, "t"] \arrow[d, "p_{j\backslash I}"'] \arrow[dr, phantom, "\nearrow"] & I \arrow[d, "u"] \\ \star \arrow[r, "j"'] & J \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} I/j \arrow[r, "s"] \arrow[d, "p_{I/j}"'] \arrow[dr, phantom, "\swarrow"] & I \arrow[d, "u"] \\ \star \arrow[r, "j"'] & J \end{tikzcd}$$

are invertible: $p_{j\backslash I!}t^* \xrightarrow{\sim} j^*u_!$ and $j^*u_* \xrightarrow{\sim} p_{I/j*}s^*$.

The derivator $\mathbb{D}$ is called *strong* if in addition

(D5) For every $J \in \mathbf{Dia}_0$, the functor $\mathrm{dia}_{\underline{2}} : \mathbb{D}_J(\underline{2}) \to \mathbf{CAT}(\underline{2}^\circ, \mathbb{D}_J(\star))$ is full and essentially surjective.

As an important example, if $\mathcal{M}$ is a model category then the association

$$\begin{aligned} \mathbb{D}^{\mathcal{M}} : \mathbf{Cat}^{\circ,\circ} &\longrightarrow \mathbf{CAT} \\ I &\longmapsto \mathcal{M}^{I^\circ}[\mathcal{W}_I^{-1}] \end{aligned}$$

defines a strong derivator, where $\mathcal{M}^{I^\circ}[\mathcal{W}_I^{-1}]$ denotes the category obtained from $\mathcal{M}^{I^\circ}$ by formally inverting those morphisms of presheaves which are objectwise weak equivalences. This result is due to Denis-Charles Cisinski (see [14]). If $\mathbb{D}$ is a (strong) derivator and $J \in \mathbf{Dia}_0$ then also $\mathbb{D}_J$ is a (strong) derivator.

One consequence of the axioms we shall often have occasion to refer to is the following result on (op)fibrations:

Fact 1.1 *Given a derivator $\mathbb{D}$ of type* **Dia** *and given a pullback square*

$$\begin{tikzcd} \cdot \arrow[r, "w"] \arrow[d, "v"'] & \cdot \arrow[d, "u"] \\ \cdot \arrow[r, "x"'] & \cdot \end{tikzcd}$$

in **Dia** *with either u a fibration or x an opfibration, the canonical "Beck-Chevalley" transformation*

$$v_! w^* \longrightarrow x^* u_! \qquad \textit{(or, equivalently, } u^* x_* \longrightarrow w_* v^* \textit{)}$$

is invertible.

For a proof see [26, 1.30] or [32, 2.7].

1.4. By a *monoidal category* we always mean a symmetric unitary monoidal category. *Monoidal functors* between monoidal categories are functors which preserve the monoidal structure up to (coherent) natural isomorphisms; in the literature, these are sometimes called *strong* monoidal functors. *Monoidal transformations* are natural transformations preserving the monoidal structure in an obvious way. We thus arrive at the 2-category of monoidal categories **MonCAT**. The monoidal product is always denoted by $\otimes$ and the unit by $\mathbb{1}$. If internal hom functors exist, we arrive at its closed variant **ClMonCAT**. (Notice that functors between closed categories are not required to be closed. In other words, **ClMonCAT** is a *full* sub-2-category of **MonCAT**.) The internal hom functor is always denoted by $[-,-]$.

A *(closed) monoidal prederivator (of type* **Dia***)* is a prederivator with a factorization

$$\mathbb{D} : \mathbf{Dia}^{\circ,\circ} \to (\mathbf{Cl})\mathbf{MonCAT} \to \mathbf{CAT},$$

where $(\mathbf{Cl})\mathbf{MonCAT} \to \mathbf{CAT}$ is the forgetful functor. (Closed) monoidal prederivators were also discussed in [2, 2.1.6] and [25].

Let us now define the "external product" mentioned in the introduction. Given a monoidal prederivator $\mathbb{D}$ and categories I, J in **Dia** we define the bifunctor

$$\begin{aligned} \boxtimes : \mathbb{D}(I) \times \mathbb{D}(J) &\to \mathbb{D}(I \times J) \\ (A, B) &\mapsto A|_{I\times J} \otimes B|_{I\times J}. \end{aligned}$$

Given two functors $u : I' \to I$ and $v : J' \to J$ in **Dia**, $A \in \mathbb{D}(I)_0$ and $B \in \mathbb{D}(J)_0$, we define a morphism

$$(u \times v)^*(A \boxtimes B) \to u^* A \boxtimes v^* B \tag{1.2}$$

as the composition

$$\begin{aligned} (u\times v)^*(A|_{I\times J} \otimes B|_{I\times J}) &\xrightarrow{\sim} (u\times v)^* A|_{I\times J} \otimes (u\times v)^* B|_{I\times J} \\ &= (u^*A)|_{I'\times J'} \otimes (v^*B)|_{I'\times J'}. \end{aligned}$$

Hence we see that (1.2) is in fact an isomorphism, and it is clear that it is also natural in A and B. Putting these and similar properties together one finds that the external product defines a pseudonatural transformation of 2-functors

$$\mathbb{D} \times \mathbb{D} \to \mathbb{D} \circ (- \times -),$$

i. e. a 1-morphism in $\mathbf{Fun}_{\mathrm{lax}}(\mathbf{Dia}^{\circ,\circ} \times \mathbf{Dia}^{\circ,\circ}, \mathbf{CAT})$ with invertible 2-cell components.

Now fix a monoidal prederivator $\mathbb{D}$, a functor $u : I \to J \in \mathbf{Dia}_1$, and $A \in \mathbb{D}(I)_0$, $B \in \mathbb{D}(J)_0$. We can define the *projection morphism*

$$u_!(A \otimes u^* B) \to u_! A \otimes B \tag{1.3}$$

by adjunction as the composition

$$A \otimes u^* B \xrightarrow{\text{adj}} u^* u_! A \otimes u^* B \xleftarrow{\sim} u^*(u_! A \otimes B).$$

It is clearly natural in A and B. Fix a second functor $v : I' \to J'$ in $\mathbf{Dia}$ and consider the following morphism ($A \in \mathbb{D}(I)_0$, $B \in \mathbb{D}(I')_0$):

$$(u \times v)_!(A \boxtimes B) \to u_! A \boxtimes v_! B, \tag{1.4}$$

obtained by adjunction from

$$A \boxtimes B \xrightarrow{\text{adj}} u^* u_! A \boxtimes v^* v_! B \xleftarrow{\sim} (u \times v)^*(u_! A \boxtimes v_! B).$$

Of course, it is also natural in A and B.

Lemma 1.5 *Let $\mathbb{D}$ be a monoidal prederivator which satisfies the axioms of a derivator. Then the following conditions are equivalent:*

(1) The projection morphism (1.3) is invertible for all $u = p_I$, $I \in \mathbf{Dia}_0$.
(2) The projection morphism (1.3) is invertible for all fibrations u in $\mathbf{Dia}$.
(3) (1.4) is invertible for all $u, v \in \mathbf{Dia}_1$.

If $\mathbb{D}$ is a closed monoidal prederivator then condition (2) is also equivalent to each of the following ones:

(4) $u^*[B, B'] \to [u^* B, u^* B']$ is invertible for all fibrations u in $\mathbf{Dia}$.
(5) $[u_! A, B] \to u_*[A, u^* B]$ is invertible for all fibrations u in $\mathbf{Dia}$.

Definition 1.6 A *(closed) monoidal derivator* is a (closed) monoidal prederivator which satisfies the axioms of a derivator as well as the equivalent conditions of Lemma 1.5.

Proof of Lemma 1.5. Assume condition (1). Let $u : I \to J$ be a fibration in $\mathbf{Dia}$ and consider, for any $j \in J_0$, the following pullback square:

$$\begin{tikzcd} I_j \arrow[r, "w"] \arrow[d, "p_{I_j}"'] & I \arrow[d, "u"] \\ \star \arrow[r, "j"'] & J. \end{tikzcd}$$

Since u is a fibration the base change morphism $p_{I_j!} w^* \to j^* u_!$ is an isomorphism, by Fact 1.1. Hence for any $A \in \mathbb{D}(I)_0$, $B \in \mathbb{D}(J)_0$, all vertical morphisms in the commutative diagram below are invertible:

$$\begin{tikzcd}
j^*u_!(A\otimes u^*B) \arrow[r] & j^*(u_!A\otimes B) \arrow[d] \\
p_{I_j!}w^*(A\otimes u^*B) \arrow[u] \arrow[d] & j^*u_!A\otimes j^*B \\
p_{I_j!}(w^*A\otimes w^*u^*B) \arrow[d, equal] & p_{I_j!}w^*A\otimes j^*B \arrow[u] \arrow[d, equal] \\
p_{I_j!}(w^*A\otimes p_{I_j}^*j^*B) \arrow[r] & p_{I_j!}w^*A\otimes j^*B.
\end{tikzcd}$$

By assumption, the bottom horizontal arrow is an isomorphism hence so is the top one. Condition (2) now follows from (D2).

For condition (3), write $u\times v=(u\times 1)\circ(1\times v)$ hence by symmetry of the monoidal product we reduce to the case where $u=1_I$, $v:J'\to J$. We use (D2), thus let $i\in I_0$, $j\in J_0$. The fiber of (1.4) over (i,j) is easily seen to be the following composition (w denotes the fibration $i\backslash I\times j\backslash J'\to i\backslash I$):

$$\begin{aligned}
(i,j)^*(1_I\times v)_!(A|_{I\times J'}\otimes B|_{I\times J'}) &\xleftarrow{\sim} p_{i\backslash I!}w_!(A|_{i\backslash I\times j\backslash J'}\otimes B|_{i\backslash I\times j\backslash J'})\\
&\xrightarrow{\sim} p_{i\backslash I!}(A|_{i\backslash I}\otimes w_!B|_{i\backslash I\times j\backslash J'})\\
&\xrightarrow{\sim} p_{i\backslash I!}(A|_{i\backslash I}\otimes p_{i\backslash I}^*p_{j\backslash J'!}B|_{j\backslash J'})\\
&\xrightarrow{\sim} p_{i\backslash I!}A|_{i\backslash I}\otimes p_{j\backslash J'!}B|_{j\backslash J'}\\
&\xrightarrow{\sim} i^*A\otimes j^*v_!B\\
&\xleftarrow{\sim} (i,j)^*(A|_{I\times J}\otimes(v_!B)|_{I\times J})
\end{aligned}$$

The first, the third and the fifth arrows come from (D4), while the second and the fourth are invertible by condition (2), the last one is clearly invertible.

Putting $u=p_I$, $v=1_\star$ in condition (3), one obtains precisely condition (1). This finishes the proof of the first statement in the lemma.

From now on we assume that $\mathbb{D}$ is a closed monoidal prederivator. For condition (4), notice that $u^*\circ[B,-]\to[u^*B,-]\circ u^*$ corresponds via the adjunctions

$$u_!\circ(-\otimes u^*B)\dashv[u^*B,-]\circ u^*\quad\text{and}\quad(-\otimes B)\circ u_!\dashv u^*\circ[B,-]$$

to the projection morphism

$$u_!(-\otimes u^*B)\to u_!-\otimes B.$$

And similarly, the morphism $[u_!A,-]\to u_*\circ[A,-]\circ u^*$ corresponds via the adjunctions

$$u_!A\otimes-\dashv[u_!A,-]\quad\text{and}\quad u_!\circ(A\otimes-)\circ u^*\dashv u_*\circ[A,-]\circ u^*$$

to the projection morphism

$$u_!(A\otimes u^*-)\to u_!A\otimes-.$$

Hence conditions (4) and (5) are both equivalent to condition (2). (For more details, see [2, 2.1.144, 2.1.146].) $\square$

In contrast to this, in a closed monoidal prederivator, the canonical morphism

$$[A, u_* B] \to u_*[u^* A, B] \tag{1.7}$$

is *always* invertible, even if u is not a fibration.

If $\mathbb{D}$ is a (strong) derivator of type **Cat** then precomposition with the 2-functor $(-)^\circ$: $\mathbf{Cat}^\circ \to \mathbf{Cat}^{\circ,\circ}$ defines a (strong) derivator $\overline{\mathbb{D}}$ in the sense of [27], and conversely starting with a (strong) derivator in their sense, precomposition with $(-)^\circ$ yields a (strong) derivator of type **Cat**. By Lemma 1.5, $\mathbb{D}$ being monoidal corresponds to $\overline{\mathbb{D}}$ being symmetric monoidal. By [27, 8.8] then, $\mathbb{D}$ being closed monoidal corresponds to $\overline{\mathbb{D}}$ being closed symmetric monoidal. In particular, [27, 9.13] establishes that if $\mathcal{M}$ is a symmetric monoidal cofibrantly generated model category then the induced derivator $\mathbb{D}^{\mathcal{M}}$ is a closed monoidal, strong derivator (of type **Cat**).

Again, if $\mathbb{D}$ is a (closed) monoidal derivator, then so is $\mathbb{D}_J$ for any $J \in \mathbf{Dia}_0$.

1.5. A few words on linear structures on derivators (see [25, section 3] for the details). An *additive derivator* is a derivator $\mathbb{D}$ such that $\mathbb{D}(\star)$ is an additive category. It follows that $\mathbb{D}(I), u^*, u_*, u_!$ are additive for all $I \in \mathbf{Dia}_0$, $u \in \mathbf{Dia}_1$. We define $R_{\mathbb{D}}$ to be the unital ring $\mathbb{D}(\star)(\mathbb{1}, \mathbb{1})$.

If $\mathbb{D}$ is additive and monoidal, then $R_{\mathbb{D}}$ is a commutative ring and $\mathbb{D}(I)$ is canonically endowed with an $R_{\mathbb{D}}$-linear structure for any $I \in \mathbf{Dia}_0$, making $u^*, u_*, u_!$ all $R_{\mathbb{D}}$-linear functors, $u \in \mathbf{Dia}_1$. Given $f \in \mathbb{D}(I)(A, B)$, $\lambda \in R_{\mathbb{D}}$, λf is defined by

$$A \xleftarrow{\sim} p_I^* \mathbb{1} \otimes A \xrightarrow{p_I^* \lambda \otimes f} p_I^* \mathbb{1} \otimes B \xrightarrow{\sim} B.$$

1.6. We now recall the notion of a stable derivator. Let $\square$ be the partially ordered set considered as a category:

$$\begin{tikzcd}
(1,1) & (0,1) \arrow[l] \\
(1,0) \arrow[u] & (0,0), \arrow[l] \arrow[u]
\end{tikzcd}$$

and $\lrcorner$ the full subcategory defined by the complement of $(1,1)$. Thus there are two canonical embeddings $i_\lrcorner : \lrcorner \to \square$ and $i_\ulcorner : \ulcorner \to \square$. We say that an object $A \in \mathbb{D}(\square)_0$ is cartesian (resp. cocartesian) if the unit

$$A \to i_{\lrcorner *} i_\lrcorner^* A \qquad \text{(resp. the counit } i_{\ulcorner !} i_\ulcorner^* A \to A\text{)}$$

is an isomorphism.

A *stable derivator* is a strong derivator $\mathbb{D}$ such that $\mathbb{D}(\star)$ is pointed and objects in $\mathbb{D}(\square)$ are cartesian if and only if they are cocartesian. If $\mathcal{M}$ is a stable model category, then the derivator $\mathbb{D}^{\mathcal{M}}$ associated to $\mathcal{M}$ is stable. Also, if $\mathbb{D}$ is a stable derivator then so is $\mathbb{D}_J$ for any $J \in \mathbf{Dia}_0$. Any stable derivator factors canonically through the forgetful functor $\mathbf{TrCAT} \to \mathbf{CAT}$ from triangulated categories to **CAT**. In particular, every stable derivator is additive. This result is due to Georges Maltsiniotis ([53, Théorème 1]; see also [26, 4.15, 4.19]), and the triangulated structure is given explicitly. We will need the description of it on $\mathbb{D}(\star)$. (The description on $\mathbb{D}(I)$ can then be deduced by replacing $\mathbb{D}$ by $\mathbb{D}_I$.) Thus given an object

$A \in \mathbb{D}(\star)_0$ one defines canonically an object in $\mathbb{D}(\square)$ with underlying diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma A, \end{tikzcd}$$

some $\Sigma A \in \mathbb{D}(\star)_0$, as $i_{\ulcorner !}(1,1)_* A$. Then we can define the suspension functor $\Sigma : \mathbb{D}(\star) \to \mathbb{D}(\star)$ as $(0,0)^* i_{\ulcorner !}(1,1)_*$. Moreover, if we denote by $\boxminus$ the partial order considered as a category

$$\begin{tikzcd} (2,1) & (1,1) \arrow[l] & (0,1) \arrow[l] \\ (2,0) \arrow[u] & (1,0) \arrow[l] \arrow[u] & (0,0), \arrow[l] \arrow[u] \end{tikzcd}$$

there are three canonical embeddings $i : \square \to \boxminus$ and we say that an object $A \in \mathbb{D}(\boxminus)_0$ is a triangle if $A_{(2,0)} \cong A_{(0,1)} \cong 0$ and $i^* A$ is (co)cartesian for all three embeddings. It then follows that one has a canonical isomorphism $A_{(0,0)} \cong \Sigma A_{(2,1)}$ (see the proof of Lemma 1.10 below) and therefore a triangle in $\mathbb{D}(\star)$:

$$A_{(2,1)} \to A_{(1,1)} \to A_{(1,0)} \to \Sigma A_{(2,1)}. \tag{1.8}$$

The distinguished triangles are those isomorphic to one of the form of (1.8).

1.7. We are also interested in some aspects of the interplay between monoidal and triangulated structures on derivators.

Definition 1.9 A *(closed) monoidal stable derivator* is a (closed) monoidal and stable derivator.

Under the correspondence $\mathbb{D} \leftrightsquigarrow \overline{\mathbb{D}}$ above, a closed monoidal stable derivator of type **Cat** corresponds to a "closed symmetric monoidal, strong, stable derivator" in [27]. Translating the results in [27] back to our setting we see that every such derivator factors canonically through **ClMonTrCAT**, the 2-category of closed monoidal categories with a "compatible" triangulation (in the sense of [55]), such that the following diagram commutes:

$$\begin{tikzcd} & & \mathbf{TrCAT} \arrow[dr] & \\ \mathbf{Dia}^{\circ,\circ} \arrow[r] & \mathbf{ClMonTrCAT} \arrow[ur] \arrow[dr] & & \mathbf{CAT}. \\ & & \mathbf{ClMonCAT} \arrow[ur] & \end{tikzcd}$$

Here, it is understood that following the path on the upper part of the diagram yields the canonical factorization of the stable derivator, while the path through the lower part yields the factorization of the monoidal prederivator. All we will need from this statement is the following lemma.

Lemma 1.10 ([27, 4.1, 4.8]) *Let $\mathbb{D}$ be a monoidal stable derivator and $I \in \mathbf{Dia}_0$. Then the monoidal product $\otimes : \mathbb{D}(I) \times \mathbb{D}(I) \to \mathbb{D}(I)$ is canonically triangulated in both variables.*

Proof. First of all, replacing $\mathbb{D}$ by $\mathbb{D}_I$ we reduce to the case $I = \star$. Moreover, by symmetry of the monoidal product we may fix $B \in \mathbb{D}(\star)_0$ and only prove $- \otimes B$ to be triangulated. Then the condition that the projection morphism $p_{J!}(A \otimes p_J^* B) \to p_{J!} A \otimes B$ be invertible for all $A \in \mathbb{D}(J)_0$ in the case of a finite discrete category J says precisely that $- \otimes B$ is additive.

The following claim is also a consequence of our definition of a monoidal derivator:

(*) Let $A \in \mathbb{D}(\square)_0$ be a cocartesian object. Then also $A \boxtimes B$ is cocartesian.

Indeed, this follows from the following factorization of the counit morphism:

$$\begin{aligned} i_{\ulcorner !} i_{\ulcorner}^* (A \boxtimes B) &\xrightarrow{\sim} i_{\ulcorner !}(i_{\ulcorner}^* A \boxtimes B) \\ &\xrightarrow[\sim]{(1.4)} i_{\ulcorner !} i_{\ulcorner}^* A \boxtimes B \\ &\xrightarrow[\sim]{\text{adj}} A \boxtimes B. \end{aligned}$$

Now let $A \in \mathbb{D}(\star)_0$ be an arbitrary object and consider $C = i_{\ulcorner !}(1,1)_* A \in \mathbb{D}(\square)_0$. Since $i_{\ulcorner !}$ is fully faithful (this is an easy computation; see [16, 7.1]), C is cocartesian, and by (*), this is also true of $C \boxtimes B$. Moreover $(C \boxtimes B)_{(1,0)} \cong C_{(1,0)} \otimes B \cong 0$ and, similarly, $(C \boxtimes B)_{(0,1)} \cong 0$. It follows from the following claim (**) that $\Sigma(A \otimes B) \cong \Sigma(C_{(1,1)} \otimes B)$ is isomorphic to $C_{(0,0)} \otimes B \cong \Sigma A \otimes B$, naturally in A.

(**) Let $A \in \mathbb{D}(\square)_0$ be a cocartesian object with $A_{(1,0)} \cong A_{(0,1)} \cong 0$. Then there is a canonical isomorphism $\Sigma(A_{(1,1)}) \cong A_{(0,0)}$, natural in A.

The condition that those fibers vanish implies that the counit of the adjunction

$$i_{\ulcorner}^* A \to (1,1)_* (1,1)^* i_{\ulcorner}^* A$$

is invertible (again, an easy computation, cf. [16, 8.11]). But the left hand side becomes isomorphic to A after applying $i_{\ulcorner !}$ by assumption, so we get the required isomorphism after applying $(0,0)^* i_{\ulcorner !}$.

Now let D be a distinguished triangle in $\mathbb{D}(\star)$, associated to a triangle $A \in \mathbb{D}(\boxplus)_0$. Essentially by (*), $A \boxtimes B$ is again a triangle, and essentially by (**), the distinguished triangle associated to $A \boxtimes B$ is isomorphic to $D \otimes B$. ☐

1.8. For I an object of **Cat**, throughout the chapter we fix the notation as in the following diagram where both squares are pullback squares:

$$\begin{tikzcd} 2\mathrm{tw}(I) \arrow[r, "r_2"] \arrow[d, "r_1"'] & \mathrm{tw}(I)^\circ \arrow[d, "q_2"] & \\ \mathrm{tw}(I) \arrow[r, "q_1"] & I^\circ \times I \arrow[r, "p_2"] \arrow[d, "p_1"'] & I \arrow[d, "p_I"] \\ & I^\circ \arrow[r, "p_{I^\circ}"] & \star. \end{tikzcd} \tag{Δ_I}$$

Explicitly, the objects of $2\mathrm{tw}(I)$ are pairs of arrows in I of the form

$$i \rightleftarrows i',$$

and morphisms from this object to $j \rightleftarrows j'$ are pairs of morphisms $(i \leftarrow j, i' \to j')$ rendering the following two squares commutative:

$$\begin{tikzcd} i \arrow[r] & i' \arrow[d] \\ j \arrow[u] \arrow[r] & j', \end{tikzcd} \qquad \begin{tikzcd} i & i' \arrow[l] \arrow[d] \\ j \arrow[u] & j'. \arrow[l] \end{tikzcd}$$

Note that if I lies in some diagram category then so does the whole diagram (Δ_I).

2. External hom

Fix a closed monoidal derivator $\mathbb{D}$ of type **Dia**. As explained in the introduction we would like to define an "external hom" functor which will play an essential role in the definition of the trace. It should behave with respect to the external product as does the internal hom with respect to the internal product (i. e. the monoidal structure). As a first indication of its nature, the external hom of $A \in \mathbb{D}(I)_o$ and $B \in \mathbb{D}(J)_o$ should be an object of $\mathbb{D}(I^\circ \times J)$, denoted by $\langle A, B\rangle$. Additionally, we would like the fibers of $\langle A, B\rangle$ to compute the internal hom of the fibers of A and B, because fiberwise dualizability should imply dualizability with respect to $\langle -, -\rangle$; moreover, $[A, B]$ should be expressible in terms of $\langle A, B\rangle$ in the case $I = J$. These and other desired properties of the external product are satisfied by the following construction which is due to Joseph Ayoub.

Given small categories I and J in **Dia**, we fix the following notation, all functors being the obvious ones:

$$\begin{tikzcd} \mathrm{tw}(I) \times J \arrow[r, "r"] \arrow[d, "p"'] \arrow[dr, "q"] & J \\ I^\circ \times J & I. \end{tikzcd} \tag{$\Pi_{I,J}$}$$

For any A in $\mathbb{D}(I)_o$ and B in $\mathbb{D}(J)_o$ set

$$\langle A, B\rangle := p_*[q^*A, r^*B].$$

This defines a bifunctor

$$\langle -, -\rangle : \mathbb{D}(I)^\circ \times \mathbb{D}(J) \to \mathbb{D}(I^\circ \times J),$$

whose properties we are going to list now. For the proofs the reader is referred to appendix A.

Naturality. For functors $u : I' \to I$ and $v : J' \to J$ in **Dia** there is an invertible morphism

$$\Psi : (u^\circ \times v)^*\langle A, B\rangle \xrightarrow{\sim} \langle u^*A, v^*B\rangle,$$

natural in $A \in \mathbb{D}(I)_o$, $B \in \mathbb{D}(J)_o$. Moreover, Ψ behaves well with respect to functors and natural transformations in **Dia**. In other words, $\langle -, -\rangle$ defines a 1-morphism in $\mathbf{Fun}_{\mathrm{lax}}(\mathbf{Dia}^\circ \times \mathbf{Dia}^{\circ,\circ}, \mathbf{CAT})$ from $\mathbb{D}(-)^\circ \times \mathbb{D}(-)$ to $\mathbb{D}(-^\circ \times -)$ with invertible 2-cell components (i. e. a pseudonatural transformation).

Internal hom. In the case $I = J$ there is an invertible morphism

$$\Theta : [A, B] \xrightarrow{\sim} p_{2*}q_{2*}q_2^*\langle A, B\rangle \qquad \text{(with the notation of } (\Delta_I)),$$

natural in A and $B \in \mathbb{D}(I)_o$. Moreover, for any functor $u : I' \to I$ in **Dia**, the canonical arrow $u^*[A, B] \to [u^*A, u^*B]$ is compatible with Ψ via Θ. In other words, Θ defines an invertible 2-morphism in $\mathbf{Fun}_{\mathrm{lax}}((\mathbf{Dia}_{\leq 1})^\circ, \mathbf{CAT})$ between 1-morphisms from $\mathbb{D}(-)^\circ \times \mathbb{D}(-)$ to $\mathbb{D}(-)$. Here, $\mathbf{Dia}_{\leq 1}$ is the 2-subcategory of **Dia** obtained by removing all non-identity 2-cells.

External product. Given categories $I_{(k)}$, $k = 1, \ldots, 4$, in **Dia**, $A_k \in \mathbb{D}(I_{(k)})_o$, there is a morphism

$$\Xi : \langle A_1, A_2\rangle \boxtimes \langle A_3, A_4\rangle \to \tau^*\langle A_1 \boxtimes A_3, A_2 \boxtimes A_4\rangle,$$

natural in all four arguments, where

$$\tau : I_{(1)}^\circ \times I_{(2)} \times I_{(3)}^\circ \times I_{(4)} \to I_{(1)}^\circ \times I_{(3)}^\circ \times I_{(2)} \times I_{(4)}$$

interchanges the two categories in the middle. Moreover, Ξ is compatible with Ψ and (1.2). In other words, it defines a 2-morphism in $\mathbf{Fun}_{\mathrm{lax}}(\mathbf{Dia}^\circ \times \mathbf{Dia}^{\circ,\circ} \times \mathbf{Dia}^\circ \times \mathbf{Dia}^{\circ,\circ}, \mathbf{CAT})$ between 1-morphisms from $\mathbb{D}(-)^\circ \times \mathbb{D}(-) \times \mathbb{D}(-)^\circ \times \mathbb{D}(-)$ to $\mathbb{D}(-^\circ \times - \times -^\circ \times -)$.

Adjunction. Given three categories in **Dia**, there is an invertible morphism

$$\Omega : \langle A, \langle B, C\rangle\rangle \xrightarrow{\sim} \langle A \boxtimes B, C\rangle,$$

natural in all three arguments. Moreover, Ω is compatible with Ψ and (1.2). In other words, it defines an invertible 2-morphism in $\mathbf{Fun}_{\mathrm{lax}}(\mathbf{Dia}^{\circ} \times \mathbf{Dia}^{\circ} \times \mathbf{Dia}^{\circ,\circ}, \mathbf{CAT})$ between 1-morphisms from $\mathbb{D}(-)^{\circ} \times \mathbb{D}(-)^{\circ} \times \mathbb{D}(-)$ to $\mathbb{D}(-^{\circ} \times -^{\circ} \times -)$.

Biduality. For fixed $B \in \mathbb{D}(\star)_{\mathrm{o}}$, there is a morphism

$$\Upsilon : A \to \langle\langle A, B\rangle, B\rangle,$$

natural in $A \in \mathbb{D}(I)_{\mathrm{o}}$. Moreover, Υ defines a 2-morphism in $\mathbf{Fun}_{\mathrm{lax}}(\mathbf{Dia}^{\circ,\circ}, \mathbf{CAT})$ between 1-endomorphisms of $\mathbb{D}$.

Normalization. Given $J \in \mathbf{Dia}_{\mathrm{o}}$, there is an invertible morphism

$$\Lambda : [p_J^* A, B] \xrightarrow{\sim} \langle A, B\rangle,$$

natural in $A \in \mathbb{D}(\star)_{\mathrm{o}}$ and $B \in \mathbb{D}(J)_{\mathrm{o}}$. Again, Λ is compatible with v^* for any $v : J' \to J$ in **Dia**, therefore it defines an invertible 2-morphism in $\mathbf{Fun}_{\mathrm{lax}}(\mathbf{Dia}^{\circ,\circ}, \mathbf{CAT})$ between 1-morphisms from $\mathbb{D}(\star)^{\circ} \times \mathbb{D}(-)$ to $\mathbb{D}(-)$. Moreover under this identification, all the morphisms in the statements of the previous properties reduce to the canonical morphisms in closed monoidal categories. (These morphisms are made explicit in appendix A; see p. 45.)

3. Definition of the trace

Recall that in a closed monoidal category $\mathcal{C}$, an object A is called *dualizable* (sometimes also *strongly dualizable*) if the canonical morphism

$$[A, \mathbb{1}] \otimes B \to [A, \mathbb{1} \otimes B] \tag{3.1}$$

is invertible for all $B \in \mathcal{C}_{\mathrm{o}}$, and in this case one defines a *coevaluation*

$$\mathrm{coev} : \mathbb{1} \xrightarrow{\mathrm{adj}} [A, \mathbb{1} \otimes A] \xleftarrow{\sim} [A, \mathbb{1}] \otimes A. \tag{3.2}$$

It has the characterizing property that the following diagram commutes (see [49, 1.4]):

$$\begin{tikzcd}
{[A, \mathbb{1}] \otimes A} \arrow[r, "\mathrm{ev}"] \arrow[d, "\sim"'] & \mathbb{1} \arrow[d, "\sim"] \\
{[[A, \mathbb{1}] \otimes A, \mathbb{1}]} \arrow[r, "{[\mathrm{coev}, \mathbb{1}]}"'] & {[\mathbb{1}, \mathbb{1}].}
\end{tikzcd} \tag{3.3}$$

Here the vertical morphism on the left is defined as the composition

$$[A, \mathbb{1}] \otimes A \to [A, \mathbb{1}] \otimes [[A, \mathbb{1}], \mathbb{1}] \to [A \otimes [A, \mathbb{1}], \mathbb{1} \otimes \mathbb{1}] \xrightarrow{\sim} [[A, \mathbb{1}] \otimes A, \mathbb{1}], \tag{3.4}$$

while the one on the right is

$$\mathbb{1} \xrightarrow{\mathrm{adj}} [\mathbb{1}, \mathbb{1} \otimes \mathbb{1}] \xrightarrow{\sim} [\mathbb{1}, \mathbb{1}].$$

$[A, \mathbb{1}]$ is called the *dual* of A, and is often denoted by A^*. Dualizability of A implies that the canonical morphism

$$A \to (A^*)^* \tag{3.5}$$

is invertible.

For dualizable A, the *trace map*

$$\mathrm{tr} : \mathcal{C}(A, A) \to \mathcal{C}(\mathbb{1}, \mathbb{1})$$

sends an endomorphism f to the composition

$$\mathbb{1} \xrightarrow{\text{coev}} A^* \otimes A \xrightarrow{1\otimes f} A^* \otimes A \xrightarrow{\text{ev}} \mathbb{1}.$$

More generally, [51] and [42] independently introduced a *(twisted) trace map* for any S and T in $\mathcal{C}$ (A still assumed dualizable),

$$\text{tr} : \mathcal{C}(A \otimes S, A \otimes T) \to \mathcal{C}(S, T),$$

which sends a "twisted endomorphism" f to the composition

$$S \xleftarrow{\sim} \mathbb{1} \otimes S \xrightarrow{\text{coev}\otimes 1} A^* \otimes A \otimes S \xrightarrow{1\otimes f} A^* \otimes A \otimes T \xrightarrow{\text{ev}\otimes 1} \mathbb{1} \otimes T \xrightarrow{\sim} T.$$

We will mimic this definition in our derivator setting. So fix a closed monoidal derivator $\mathbb{D}$ of type **Dia**. First of all, here is our translation of dualizability:

Definition 3.6 Let $I \in \mathbf{Dia}_0$, $A \in \mathbb{D}(I)_0$. We say that A is *fiberwise dualizable* if A_i is dualizable for all $i \in I_0$. The *dual* of A is defined to be $\langle A, \mathbb{1}_{\mathbb{D}(\star)}\rangle \in \mathbb{D}(I^\circ)_0$, also denoted by $A^\vee$.

Let I and A as in the definition, A fiberwise dualizable. Then, as was the case for dualizable objects in closed monoidal categories, the morphisms corresponding to (3.1) and (3.5) are invertible (for any $B \in \mathbb{D}(I)_0$):

$$A^\vee \boxtimes B \cong \langle A, \mathbb{1}\rangle \boxtimes [p_I^*\mathbb{1}, B] \underset{\sim}{\xrightarrow{\Lambda}} \langle A, \mathbb{1}\rangle \boxtimes \langle \mathbb{1}, B\rangle \underset{\sim}{\xrightarrow{\Xi}} \langle A \boxtimes \mathbb{1}, \mathbb{1} \boxtimes B\rangle \cong \langle A, B\rangle, \tag{3.7}$$

$$\Upsilon : A \xrightarrow{\sim} (A^\vee)^\vee. \tag{3.8}$$

This follows from the naturality and the normalization properties of the external hom. We now go about defining a coevaluation and an evaluation morphism. This will rely on the results of the previous section.

Using the relation between internal and external hom, we can consider the composition

$$\mathbb{1}_{\mathbb{D}(I)} \xrightarrow{\text{adj}} [A, \mathbb{1} \otimes A] \xrightarrow{\sim} [A, A] \underset{\sim}{\xrightarrow{\Theta}} p_{2*}q_{2*}q_2^*\langle A, A\rangle$$

and, by adjunction, we obtain

$$\text{coev} : q_{2!}\mathbb{1} \to \langle A, A\rangle \underset{\sim}{\xleftarrow{(3.7)}} A^\vee \boxtimes A. \tag{3.9}$$

Next, inspired by (3.3), we define the evaluation morphism to be simply the dual of the coevaluation morphism. For this, notice that A being fiberwise dualizable implies that also $A^\vee$ is. Hence there is an analogue of (3.4):

$$A \boxtimes A^\vee \underset{\sim}{\xrightarrow{\Upsilon}} (A^\vee)^\vee \boxtimes A^\vee \underset{\sim}{\xrightarrow{(3.7)}} \langle A^\vee, A^\vee\rangle \underset{\sim}{\xrightarrow{\Omega}} \langle A^\vee \boxtimes A, \mathbb{1}_{\mathbb{D}(\star)}\rangle. \tag{3.10}$$

Denote by $\mu : I \times I^\circ \to I^\circ \times I$ the canonical isomorphism. Then we define

$$\begin{aligned}
\text{ev} : A^\vee \boxtimes A &\xrightarrow{\sim} \mu_*(A \boxtimes A^\vee) \\
&\underset{\sim}{\xrightarrow{(3.10)}} \mu_*\langle A^\vee \boxtimes A, \mathbb{1}\rangle \\
&\xrightarrow{\langle \text{coev}, \mathbb{1}\rangle} \mu_*\langle q_{2!}\mathbb{1}, \mathbb{1}\rangle \\
&\xrightarrow{\overline{\Psi}} \mu_*(q_2)^\circ_*\langle \mathbb{1}, \mathbb{1}\rangle \\
&\xrightarrow{\sim} q_{1*}\mathbb{1}.
\end{aligned}$$

Here, $\overline{\Psi}$ is obtained by adjunction from Ψ:

$$\begin{aligned}\overline{\Psi} : \langle q_{2!}\mathbb{1}, \mathbb{1}\rangle &\xrightarrow{\text{adj}} q^{\circ}_{2*}q^{\circ *}_{2}\langle q_{2!}\mathbb{1}, \mathbb{1}\rangle \\ &\xrightarrow[\sim]{\Psi} q^{\circ}_{2*}\langle q^{*}_{2}q_{2!}\mathbb{1}, \mathbb{1}\rangle \\ &\xrightarrow{\text{adj}} q^{\circ}_{2*}\langle \mathbb{1}, \mathbb{1}\rangle.\end{aligned}$$

It follows immediately that the following diagram commutes for any $u : I' \to I$ in **Dia**:

$$\begin{tikzcd}(u \times u^{\circ})^{*}\langle q_{2!}\mathbb{1}, \mathbb{1}\rangle \arrow[r] \arrow[d, "\overline{\Psi}"'] & \langle q'_{2!}\mathrm{tw}(u)^{\circ *}\mathbb{1}, \mathbb{1}\rangle \arrow[d, "\overline{\Psi}"] \\ (u \times u^{\circ})^{*}q^{\circ}_{2*}\langle \mathbb{1}, \mathbb{1}\rangle \arrow[r] & q'^{\circ}_{2*}\langle \mathrm{tw}(u)^{\circ *}\mathbb{1}, \mathbb{1}\rangle.\end{tikzcd} \tag{3.11}$$

In the sequel we will sometimes denote by the same symbol $\overline{\Psi}$ other morphisms obtained by adjunction from Ψ in a similar way. It is hoped that this will not cause any confusion.

Finally we can put all the pieces together and define the trace:

Definition 3.12 Let $I \in \mathbf{Dia}_{\mathrm{o}}$, $A \in \mathbb{D}(I)_{\mathrm{o}}$ fiberwise dualizable, and $S, T \in \mathbb{D}(I)_{\mathrm{o}}$ arbitrary. Then we define the *(twisted) trace map*

$$\mathrm{Tr} : \mathbb{D}(I)\,(A \otimes S, A \otimes T) \to \mathbb{D}(I^{\circ} \times I)\,(q_{2!}\mathbb{1} \otimes p^{*}_{2}S, q_{1*}\mathbb{1} \otimes p^{*}_{2}T)$$

as the association which sends a twisted endomorphism f to the composition

$$\begin{tikzcd}q_{2!}\mathbb{1} \otimes p^{*}_{2}S \arrow[r, "\mathrm{coev}\otimes 1"] & (A^{\vee} \boxtimes A) \otimes p^{*}_{2}S \arrow[r, "\sim"] & A^{\vee} \boxtimes (A \otimes S) \arrow[d, "1\boxtimes f"] \\ q_{1*}\mathbb{1} \otimes p^{*}_{2}T & (A^{\vee} \boxtimes A) \otimes p^{*}_{2}T \arrow[l, "\mathrm{ev}\otimes 1"'] & A^{\vee} \boxtimes (A \otimes T) \arrow[l, "\sim"']\end{tikzcd}$$

called the *(twisted) trace of f*.

Remark 3.13 Although defined in this generality, we will be interested mainly in traces of endomorphisms twisted by "constant" objects, i. e. coming from objects in $\mathbb{D}(\star)$. In this case ($S, T \in \mathbb{D}(\star)_{\mathrm{o}}$), the trace map is an association

$$\mathbb{D}(I)\,(A \boxtimes S, A \boxtimes T) \to \mathbb{D}(I^{\circ} \times I)\,(q_{2!}\mathbb{1} \boxtimes S, q_{1*}\mathbb{1} \boxtimes T)\,.$$

Now, let g be an element of the target of this map. It induces the composite

$$q_{2!}S|_{\mathrm{tw}(I)^{\circ}} \xleftarrow{\sim} q_{2!}\mathbb{1} \boxtimes S \xrightarrow{g} q_{1*}\mathbb{1} \boxtimes T \to q_{1*}T|_{\mathrm{tw}(I)}$$

and by adjunction

$$q^{*}_{2}S|_{I^{\circ}\times I} \to q^{*}_{2}q_{1*}q^{*}_{1}T|_{I^{\circ}\times I} \xrightarrow{\sim} r_{2*}r^{*}_{1}q^{*}_{1}T|_{I^{\circ}\times I}$$

or, by another adjunction, a morphism

$$S|_{2\mathrm{tw}(I)} \to T|_{2\mathrm{tw}(I)}. \tag{3.14}$$

Applying the functor $\mathrm{dia}_{2\mathrm{tw}(I)}$ we obtain an element of

$$\mathbb{D}(\star)^{(2\mathrm{tw}(I))^{\circ}}\left(\mathrm{dia}_{2\mathrm{tw}(I)}(S|_{2\mathrm{tw}(I)}), \mathrm{dia}_{2\mathrm{tw}(I)}(T|_{2\mathrm{tw}(I)})\right) \cong \prod_{\pi_{\mathrm{o}}(2\mathrm{tw}(I))} \mathbb{D}(\star)\,(S, T)\,.$$

The component corresponding to $\gamma \in \pi_{\mathrm{o}}(2\mathrm{tw}(I))$ is called the *γ-component of g*.

Lemma 3.15 *Suppose that the following hypotheses are satisfied:*

(H1) $q_{1*}\mathbb{1}\boxtimes A\to q_{1*}(\mathbb{1}\boxtimes A)$ *is invertible for all* $A\in\mathbb{D}(\star)_0$*;*
(H2) *for each connected component* γ *of* $2\mathrm{tw}(I)$*, the functor* p_γ^* *is fully faithful.*
Then the map

$$\mathbb{D}(I^\circ\times I)(q_{2!}\mathbb{1}\boxtimes S,q_{1*}\mathbb{1}\boxtimes T)\to\prod_{\pi_0(2\mathrm{tw}(I))}\mathbb{D}(\star)(S,T)$$

defined above is a bijection. In particular, any morphism $q_{2!}\mathbb{1}\boxtimes S\to q_{1*}\mathbb{1}\boxtimes T$ *is uniquely determined by its* γ*-components,* $\gamma\in\pi_0(2\mathrm{tw}(I))$*.*

Proof. (H1) implies that the morphism $g:q_{2!}\mathbb{1}\boxtimes S\to q_{1*}\mathbb{1}\boxtimes T$ in the remark above can equivalently be described by (3.14). Moreover, starting at the bottom right corner and going once around the following square represents the identity map:

$$\begin{tikzcd}
\mathbb{D}(2\mathrm{tw}(I))(S|_{2\mathrm{tw}(I)},T|_{2\mathrm{tw}(I)}) \arrow[r, "\mathrm{dia}_{2\mathrm{tw}(I)}"] \arrow[d, "\sim"'] & \mathbb{D}(\star)^{2\mathrm{tw}(I)^\circ}(S_{\mathrm{cst}},T_{\mathrm{cst}}) \arrow[d, "\sim"] \\
\prod_{\gamma\in\pi_0(2\mathrm{tw}(I))}\mathbb{D}(\gamma)(S|_\gamma,T|_\gamma) & \prod_{\gamma\in\pi_0(2\mathrm{tw}(I))}\mathbb{D}(\star)(S,T). \arrow[l, "(p_\gamma^*)_\gamma"']
\end{tikzcd}$$

Here, the left vertical arrow is invertible by (D1). (H2) now implies that the horizontal arrow on the top is a bijection. □

In particular we see that in favorable cases (and these are the only ones we will have much to say about) the seemingly complicated twisted trace is encoded simply by a family of morphisms over the terminal category. The goal of the following section is to determine these morphisms.

4. Functoriality of the trace

Our immediate goal is to describe the components $S\to T\in\mathbb{D}(\star)_1$ associated to the trace of a (twisted) endomorphism of a fiberwise dualizable object as explained in the previous section. However, we take the opportunity to establish a more general functoriality property of the trace (Proposition 4.3). Our immediate goal will be achieved as a corollary to this result.

Throughout this section we fix a category $I\in\mathbf{Dia}_0$. An object of $2\mathrm{tw}(I)$ is a pair of arrows

$$\begin{tikzcd} i \arrow[r, shift left, "h_1"] & j \arrow[l, shift left, "h_2"] \end{tikzcd} \tag{4.1}$$

in I (cf. 1.8). There is always a morphism in $2\mathrm{tw}(I)$ from an object of the form

$$(i,h):\qquad \begin{tikzcd} i \arrow[r, shift left, "1_i"] & i \arrow[l, shift left, "h"] \end{tikzcd}$$

to (4.1), given by the pair of arrows $(1_i,h_1)$ if $h=h_2h_1$. Hence we can take some of the (i,h) as representatives for $\pi_0(2\mathrm{tw}(I))$ and it is sufficient to describe the component $S\to T$ corresponding to (i,h). This motivates the following more general functoriality statement.

Let $u:I'\to I$ be a functor, $\eta:u\to u$ a natural transformation in **Dia**; consider the basic diagram (Δ_I). Notice that this diagram is functorial in I hence there is a canonical morphism of diagrams $(\Delta_{I'})\to(\Delta_I)$ and we will use the convention that the arrows in $(\Delta_{I'})$ will be distinguished from their I-counterparts by being decorated with a prime.

Definition 4.2 Let $S, T \in \mathbb{D}(I)_\circ$. Define a pullback map

$$(u,\eta)^* : \mathbb{D}(I^\circ \times I)\,(q_{2!}\mathbb{1} \otimes p_2^* S, q_{1*}\mathbb{1} \otimes p_2^* T) \longrightarrow \mathbb{D}(I'^\circ \times I')\,(q'_{2!}\mathbb{1} \otimes p_2'^* u^* S, q'_{1*}\mathbb{1} \otimes p_2'^* u^* T)$$

by sending a morphism g to the composition

$$q'_{2!}\mathrm{tw}(u)^{\circ *}\mathbb{1} \otimes p_2'^* u^* S \to (u^\circ \times u)^*(q_{2!}\mathbb{1} \otimes p_2^* S) \xrightarrow{g} (u^\circ \times u)^*(q_{1*}\mathbb{1} \otimes p_2^* T) \xrightarrow{(1\times\eta)^*} (u^\circ \times u)^*(q_{1*}\mathbb{1} \otimes p_2^* T) \to q'_{1*}\mathrm{tw}(u)^*\mathbb{1} \otimes p_2'^* u^* T.$$

Proposition 4.3 *Let u, η, S, T as above, assume $A \in \mathbb{D}(I)_\circ$ is fiberwise dualizable. For any $f : A \otimes S \to A \otimes T$, we have*

$$(u,\eta)^*\mathrm{Tr}(f) = \mathrm{Tr}(\eta^* \circ u^* f), \tag{4.4}$$

where $\eta^* \circ u^* f$ is any of the two paths from the top left to the bottom right in the following commutative square:

$$\begin{tikzcd}
u^*A \otimes u^*S \arrow[r, "u^*f"] \arrow[d, "\eta^*_{A\otimes S}"'] & u^*A \otimes u^*T \arrow[d, "\eta^*_{A\otimes T}"] \\
u^*A \otimes u^*S \arrow[r, "u^*f"'] & u^*A \otimes u^*T.
\end{tikzcd}$$

Proof. The two outer paths in the following diagram are exactly the two sides of (4.4):

$$\begin{tikzcd}
q'_{2!}\mathrm{tw}(u)^{\circ *}\mathbb{1} \otimes p_2'^* u^* S \arrow[r] \arrow[d, "\mathrm{coev}"'] & (u^\circ \times u)^*(q_{2!}\mathbb{1} \otimes p_2^* S) \arrow[d, "\mathrm{coev}"] \\
(u^*A)^\vee \boxtimes u^*A \otimes p_2'^* u^* S \arrow[d, "1\boxtimes u^* f"'] & (u^\circ \times u)^*(A^\vee \boxtimes A \otimes p_2^* S) \arrow[l, "\Psi"'] \arrow[d, "1\boxtimes f"] \\
(u^*A)^\vee \boxtimes u^*A \otimes p_2'^* u^* T \arrow[d, "1\boxtimes\eta^*"'] & (u^\circ \times u)^*(A^\vee \boxtimes A \otimes p_2^* T) \arrow[l, "\Psi"'] \arrow[d, "1\times\eta^*"] \\
(u^*A)^\vee \boxtimes u^*A \otimes p_2'^* u^* T \arrow[d, "\sim"'] & (u^\circ \times u)^*(A^\vee \boxtimes A \otimes p_2^* T) \arrow[l, "\Psi"'] \arrow[d, "\sim"] \\
\mu'_*\langle (u^*A)^\vee \boxtimes u^*A, \mathbb{1}\rangle \otimes p_2'^* u^* T \arrow[d, "\langle \mathrm{coev}, \mathbb{1}\rangle"'] & (u^\circ \times u)^*(\mu_*\langle A^\vee \boxtimes A, \mathbb{1}\rangle \otimes p_2^* T) \arrow[l, "\Psi"'] \arrow[d, "\langle \mathrm{coev}, \mathbb{1}\rangle"] \\
\mu'_*\langle q'_{2!}\mathbb{1}, \mathbb{1}\rangle \otimes p_2'^* u^* T \arrow[d] & (u^\circ \times u)^*(\mu_*\langle q_{2!}\mathbb{1}, \mathbb{1}\rangle \otimes p_2^* T) \arrow[l, "\Psi"'] \arrow[d] \\
q'_{1*}\mathrm{tw}(u)^*\mathbb{1} \otimes p_2'^* u^* T & (u^\circ \times u)^*(q_{1*}\mathbb{1} \otimes p_2^* T). \arrow[l]
\end{tikzcd}$$

Hence it suffices to prove the commutativity of this diagram. The second and third square clearly commute, the fourth and sixth square do so by the functoriality statements in section 2 (use also (3.11)). The fifth square commutes if the first does so we are left to show commutativity of the first one.

By definition, coev is the composition

$$q_{2!}\mathbb{1} \to \langle A, A\rangle \xleftarrow{\sim} A^\vee \boxtimes A$$

and we already know that the second arrow behaves well with respect to functors in **Dia**. Thus it suffices to prove that the following diagram commutes:

$$\begin{tikzcd}
q'_{2!}q'^*_2 p'^*_2 \mathbb{1} \arrow[r] \arrow[d, "\sim"'] & q'_{2!}q'^*_2 p'^*_2 p'_{2*} q'_{2*} q'^*_2 \langle u^*A, u^*A\rangle \arrow[r, "\mathrm{adj}"] & \langle u^*A, u^*A\rangle \\
q'_{2!}q'^*_2 p'^*_2 u^* \mathbb{1} \arrow[r] \arrow[d] & q'_{2!}q'^*_2 p'^*_2 u^* p_{2*} q_{2*} q^*_2 \langle A, A\rangle \arrow[u, "\Psi"] \arrow[d] & \\
(u^\circ \times u)^* q_{2!} q^*_2 p^*_2 \mathbb{1} \arrow[r] & (u^\circ \times u)^* q_{2!} q^*_2 p^*_2 p_{2*} q_{2*} q^*_2 \langle A, A\rangle \arrow[r, "\mathrm{adj}"'] & (u^\circ \times u)^* \langle A, A\rangle. \arrow[uu, "\Psi"']
\end{tikzcd}$$

The top left square commutes by the internal hom property in section 2, the bottom left square clearly commutes, and the right rectangle is also easily seen to commute. ☐

Of course, in the Proposition we can take $u = i$ to be an object of I, and η to be the identity transformation. Denote the pullback morphism $(i, 1)^*$ by i^*.

Corollary 4.5 *Let $i \in I_0$. For any $A, S, T \in \mathbb{D}(I)_0$, A fiberwise dualizable, and for any $f : A \otimes S \to A \otimes T$, we have*

$$i^*\mathrm{Tr}(f) = \mathrm{tr}(f_i)$$

modulo the obvious identifications.

Proof. By the proposition, $i^*\mathrm{Tr}(f) = \mathrm{Tr}(i^* f)$. It remains to prove that in the case $I = \star$, the maps Tr and tr coincide. Thus assume $I = \star$ and consider the following diagram:

$$\begin{tikzcd}
1_!\mathbb{1} \otimes S \arrow[r, "\mathrm{coev}"] \arrow[d, "\sim"'] & A^\vee \otimes A \otimes S \arrow[r, "f"] & A^\vee \otimes A \otimes T \arrow[r, "\mathrm{ev}"] & 1_*\mathbb{1} \otimes T \\
\mathbb{1} \otimes S \arrow[r, "\mathrm{coev}"'] & A^* \otimes A \otimes S \arrow[r, "f"'] \arrow[u, "\sim"', "\Lambda"] & A^* \otimes A \otimes T \arrow[r, "\mathrm{ev}"'] \arrow[u, "\Lambda", "\sim"'] & \mathbb{1} \otimes T. \arrow[u, "\sim"']
\end{tikzcd}$$

The composition of the top horizontal arrows is $\mathrm{Tr}(f)$ while the composition of the bottom horizontal arrows is $\mathrm{tr}(f)$. The middle square clearly commutes. The left square commutes by the normalization property of the external hom, and commutativity of the right square can be deduced from this and (3.3). ☐

Let us come back to the situation considered at the beginning of this section. Here the proposition implies:

Corollary 4.6 *Let $A \in \mathbb{D}(I)_0$ fiberwise dualizable, $S, T \in \mathbb{D}(\star)_0$, $i \in I_0$, $h \in I(i, i)$, and $f : A \boxtimes S \to A \boxtimes T \in \mathbb{D}(I)_1$. Then, modulo the obvious identifications, the (i, h)-component of $\mathrm{Tr}(f)$ is $\mathrm{tr}(h^* \circ f_i)$.*

Proof. h defines a natural transformation $i \to i$ and we have

$$\begin{aligned}
(i, h)^*\mathrm{Tr}(f) &= \mathrm{Tr}(h^* \circ i^* f) && \text{by the proposition above,} \\
&= \mathrm{tr}(h^* \circ i^* f) && \text{by the previous corollary.}
\end{aligned}$$

We need to prove that the left hand side computes the (i, h)-component. The pair $(1_i, h)$ defines an arrow in $\mathrm{tw}(I)^\circ$ from h to 1_i. The composition of the vertical arrows on the left of the following diagram is the (i,h)-component of $\mathrm{Tr}(f)$ while the composition of the vertical arrows on the right is $(i,h)^*\mathrm{Tr}(f)$:

$$\begin{tikzcd}
(i,h)^* S|_{2\mathrm{tw}(I)} \arrow[r, equal] \arrow[d, "\mathrm{adj}"'] & h^* S|_{\mathrm{tw}(I)^\circ} \arrow[r, equal] \arrow[d, "\mathrm{adj}"'] & 1_i^* S|_{\mathrm{tw}(I)^\circ} \arrow[d, "\mathrm{adj}"] \\
(i,h)^* r_2^* q_2^* q_{2!}(\mathbb{1} \boxtimes S) \arrow[r, equal] \arrow[d, "\sim"'] & h^* q_2^* q_{2!}(\mathbb{1} \boxtimes S) \arrow[d, "\sim"'] & 1_i^* q_2^* q_{2!}(\mathbb{1} \boxtimes S) \arrow[l, "(1_i,h)^*"'] \arrow[d, "\sim"] \\
(i,h)^* r_2^* q_2^* (q_{2!}\mathbb{1} \boxtimes S) \arrow[r, equal] \arrow[d, "\mathrm{Tr}(f)"'] & h^* q_2^* (q_{2!}\mathbb{1} \boxtimes S) \arrow[d, "\mathrm{Tr}(f)"'] & 1_i^* q_2^* (q_{2!}\mathbb{1} \boxtimes S) \arrow[l, "(1_i,h)^*"'] \arrow[d, "\mathrm{Tr}(f)"] \\
(i,h)^* r_2^* q_2^* (q_{1*}\mathbb{1} \boxtimes T) \arrow[r, equal] \arrow[d] & h^* q_2^* (q_{1*}\mathbb{1} \boxtimes T) \arrow[d] & 1_i^* q_2^* (q_{1*}\mathbb{1} \boxtimes T) \arrow[l, "(1_i,h)^*"'] \arrow[d] \\
(i,h)^* r_2^* q_2^* q_{1*}(\mathbb{1} \boxtimes T) \arrow[r, equal] \arrow[d, "\sim"'] & h^* q_2^* q_{1*}(\mathbb{1} \boxtimes T) \arrow[d, "\sim"'] & 1_i^* q_2^* q_{1*}(\mathbb{1} \boxtimes T) \arrow[l, "(1_i,h)^*"'] \arrow[d, "(1_i \times h)^*"] \\
(i,h)^* r_1^* q_1^* q_{1*} T|_{\mathrm{tw}(I)} \arrow[r, equal] \arrow[d, "\mathrm{adj}"'] & 1_i^* q_1^* q_{1*} T|_{\mathrm{tw}(I)} \arrow[r, equal] \arrow[d, "\mathrm{adj}"'] & 1_i^* q_1^* q_{1*} T|_{\mathrm{tw}(I)} \arrow[d, "\mathrm{adj}"] \\
(i,h)^* T|_{2\mathrm{tw}(I)} \arrow[r, equal] & 1_i^* T|_{\mathrm{tw}(I)} \arrow[r, equal] & 1_i^* T|_{\mathrm{tw}(I)}.
\end{tikzcd}$$

The unlabeled arrows are the canonical ones; all squares clearly commute. ☐

Knowing the components of the trace we now give a better description of the indexing set $\pi_0(2\mathrm{tw}(I))$, at least for "EI-categories":

Definition 4.7 An *EI-category* I is a category whose endomorphisms are all invertible, i. e. such that for all $i \in I_0$, $G_i := I(i,i)$ is a group.

EI-categories have been of interest in studies pertaining to different fields of mathematics, especially in representation theory and algebraic topology; closest to our discussion in the sequel is their role in the study of the Euler characteristic of a category (see [21], [47]). We will see examples of EI-categories below.

Let I be an EI-category; we define the *endomorphism category* $\mathrm{end}(I)$ associated to I to be the category whose objects are endomorphisms in I and an arrow from $h \in I(i,i)$ to $k \in I(j,j)$ is a morphism $m \in I(i,j)$ such that $mh = km$. The object $h \in I(i,i)$ is sometimes denoted by (i,h). There is also a canonical functor

$$2\mathrm{tw}(I) \to \mathrm{end}(I)$$

which takes a typical object (4.1) of $2\mathrm{tw}(I)$ to its composition $h_1 h_2 \in I(j,j)$. Notice that it takes (i,h) to (i,h).

Lemma 4.8 *Let I be an EI-category, $h \in I(i,i)$, $k \in I(j,j)$. Then (i,h) and (j,k) lie in the same connected component of $2\mathrm{tw}(I)$ if and only if $h \cong k$ as objects of $\mathrm{end}(I)$. In other words, the functor defined above induces a bijection*

$$\pi_0(2\mathrm{tw}(I)) \longleftrightarrow \mathrm{end}(I)_0/\cong .$$

Proof. If $m : h \to k$ is an isomorphism in $\mathrm{end}(I)$ then (m^{-1}, m) defines a morphism in $2\mathrm{tw}(I)$ from (i, h) to (j, k).

For the converse we notice that $2\mathrm{tw}(I)$ is a groupoid. Indeed, it follows from the definition of an EI-category that in a typical object (4.1) of $2\mathrm{tw}(I)$, both h_1 and h_2 must be isomorphisms. From this, and using a similar argument, one deduces that the components of any morphism in $2\mathrm{tw}(I)$ are invertible.

Given now a morphism (m_1, m_2) from (i, h) to (j, k) in $2\mathrm{tw}(I)$, we must have $m_1 = m_2^{-1}$ and therefore m_2 defines an isomorphism from h to k in $\mathrm{end}(I)$. □

Example 4.9

(1) Let I be a preordered set considered as a category. Clearly this is an EI-category, and $\mathrm{end}(I) = I$. It follows that we have $\pi_0(2\mathrm{tw}(I)) = I_0/\cong$, the isomorphism classes of objects in I, or in other words, the (underlying set of the) poset associated to I. If the hypotheses of Lemma 3.15 are satisfied then the trace of an endomorphism f is just the family of the traces of the fibers $(\mathrm{tr}(f_i))_i$, indexed by isomorphism classes of objects in I.

(2) Let G be a group. We can consider G canonically as a category with one object, the morphisms being given by G itself, the composition being the multiplication in G. Again, this is an EI-category. Given h and k in G, an element $m \in G$ defines a morphism $m : h \to k$ if and only if it satisfies $mhm^{-1} = k$, so h and k are connected (and therefore isomorphic) in $\mathrm{end}(I)$ if and only if they are conjugate in G. It follows that $\pi_0(2\mathrm{tw}(I))$ can be identified with the set of conjugacy classes of G. If the hypotheses of Lemma 3.15 are satisfied then the trace of an endomorphism f with unique fiber $e^* f$ is just the family of traces $(\mathrm{tr}(h^* \circ e^* f))_{[h]}$, indexed by the conjugacy classes of G.

(3) Generalizing the two previous examples, for an arbitrary EI-category I, $\mathrm{end}(I)_0/\cong$ can be identified with the disjoint union of the sets C_i of conjugacy classes of the groups $G_i = I(i, i)$ for representatives i of the isomorphism classes in I, i. e.

$$\pi_0(2\mathrm{tw}(I)) \longleftrightarrow \coprod_{i \in I_0/\cong} C_i.$$

If the hypotheses of Lemma 3.15 are satisfied then the trace of an endomorphism f is just the family of traces $(\mathrm{tr}(h^* \circ f_i))_{i,[h]}$.

Remark 4.10 One can define the category $\mathrm{end}(I)$ without the hypothesis that I be an EI-category but the previous lemma does not remain true without it. However, there is the following general alternative description of $\pi_0(2\mathrm{tw}(I))$: Let $\sim$ be the equivalence relation on the set $\coprod_{i \in I_0} I(i, i)$ generated by the relation $m_1 m_2 \sim m_2 m_1$, $m_1, m_2 \in I_1$ composable. Then (i, h) and (j, k) lie in the same connected component of $2\mathrm{tw}(I)$ if and only if $h \sim k$. It follows that for arbitrary I, there is a bijection

$$\pi_0(2\mathrm{tw}(I)) \longleftrightarrow \left(\coprod_{i \in I_0} I(i, i) \right) / \sim .$$

5. The trace of the homotopy colimit

Given a closed monoidal derivator $\mathbb{D}$, a category I in the domain **Dia** of $\mathbb{D}$ and objects A of $\mathbb{D}(I)$ fiberwise dualizable, S and T of $\mathbb{D}(\star)$, we can associate to every $f : A \boxtimes S \to A \boxtimes T$ in $\mathbb{D}(I)$ its homotopy colimit $p_{I!} f : p_{I!} A \boxtimes S \to p_{I!} A \boxtimes T$ by requiring that the following square commutes:

$$\begin{tikzcd} p_{I!}(A \boxtimes S) \arrow[r, "f"] \arrow[d, "\sim", "(1.4)"'] & p_{I!}(A \boxtimes T) \arrow[d, "\sim"', "(1.4)"] \\ p_{I!}A \boxtimes S \arrow[r, "p_{I!}f"'] & p_{I!}A \boxtimes T. \end{tikzcd}$$

We will now show that, in good cases, the trace of f as defined above contains enough information to compute the trace of the homotopy colimit of f.

Definition 5.1 Given a morphism $g : q_{2!}\mathbb{1} \boxtimes S \to q_{1*}\mathbb{1} \boxtimes T$ as in Remark 3.13 (or, under the hypotheses in Lemma 3.15, the family of its γ-components, $\gamma \in \pi_0(2\mathrm{tw}(I))$), we associate to it a new morphism $\Phi(g) : S \to T$, provided that the morphism $p_{I!}p_{2*} \to p_{I^\circ *}p_{1!}$ is invertible. (This latter morphism is obtained by adjunction from the composition

$$p_{I^\circ}^* p_{I!} p_{2*} \xleftarrow{\sim} p_{1!} p_2^* p_{2*} \xrightarrow{\mathrm{adj}} p_{1!},$$

where for the first isomorphism one uses Fact 1.1.) In this case $\Phi(g)$ is defined by the requirement that the following rectangle commutes:

$$\begin{tikzcd} S \arrow[rr, "\Phi(g)"] \arrow[d, "\mathrm{adj}"'] & & T \\ p_{I^\circ *}p_{I^\circ}^* S & & p_{I!}p_I^* T \arrow[u, "\mathrm{adj}"'] \arrow[d, "\sim"', "\mathrm{adj}"] \\ p_{I^\circ *}p_{1!}q_{2!}q_2^* p_1^* p_{I^\circ}^* S \arrow[u, "\mathrm{adj}", "\sim"'] \arrow[d, "\sim"] & & p_{I!}p_{2*}q_{1*}q_1^* p_2^* p_I^* T \\ p_{I^\circ *}p_{1!}q_{2!}(\mathbb{1} \boxtimes S) \arrow[d, "\sim"] & & p_{I!}p_{2*}q_{1*}(\mathbb{1} \boxtimes T) \arrow[u, "\sim"] \\ p_{I^\circ *}p_{1!}(q_{2!}\mathbb{1} \boxtimes S) \arrow[r, "g"'] & p_{I^\circ *}p_{1!}(q_{1*}\mathbb{1} \boxtimes T) & p_{I!}p_{2*}(q_{1*}\mathbb{1} \boxtimes T). \arrow[l, "\sim"] \arrow[u] \end{tikzcd} \tag{5.2}$$

Here, the two (co)units of adjunctions going in the "wrong" direction are invertible by Lemma B.1.

Remark 5.3 Suppose that the conditions (H1) and (H2) of Lemma 3.15 are satisfied, thus $\Phi = \Phi_{S,T}$ can be identified with a map $\prod_{\pi_0(2\mathrm{tw}(I))} \mathbb{D}(\star)(S,T) \to \mathbb{D}(\star)(S,T)$. The observation is that this map is natural in both arguments, in the following sense: Given morphisms $S \to S'$ and $T \to T'$, the following diagram commutes:

$$\begin{tikzcd} \prod_{\pi_0(2\mathrm{tw}(I))} \mathbb{D}(\star)(S',T) \arrow[r, "\Phi_{S',T}"] \arrow[d] & \mathbb{D}(\star)(S',T) \arrow[d] \\ \prod_{\pi_0(2\mathrm{tw}(I))} \mathbb{D}(\star)(S,T) \arrow[r, "\Phi_{S,T}"] \arrow[d] & \mathbb{D}(\star)(S,T) \arrow[d] \\ \prod_{\pi_0(2\mathrm{tw}(I))} \mathbb{D}(\star)(S,T') \arrow[r, "\Phi_{S,T'}"] & \mathbb{D}(\star)(S,T'). \end{tikzcd}$$

This follows immediately from the definition of Φ.

Proposition 5.4 *Let $I \in \mathbf{Dia}_0$, and suppose that the following conditions are satisfied:*

(H3) the morphism $p_{I!}p_{2*} \to p_{I^\circ *}p_{1!}$ is invertible;
(H4) the morphism $p_{I^\circ *} - \otimes - \to p_{I^\circ *}(- \otimes p_{I^\circ}^* -)$ is invertible.

If $A \in \mathbb{D}(I)_0$ is fiberwise dualizable, $S, T \in \mathbb{D}(\star)_0$, and $f : A \boxtimes S \to A \boxtimes T$, then the object $p_{I!}A$ is dualizable in $\mathbb{D}(\star)$ and the following equality holds:

$$\Phi(\mathrm{Tr}(f)) = \mathrm{tr}(p_{I!}f).$$

Proof. (H4) implies that $p_{I!}$ preserves fiberwise dualizable objects. Then the proof proceeds by decomposing (5.2) into smaller pieces; since it is rather long and not very enlightening we defer it to appendix B. □

Remark 5.5 It is worth noting that the particular shape of diagram (5.2) is of no importance to us. All we will use in the sequel is that there exists a map Φ, natural in the sense of Remark 5.3, and which takes the trace of a (twisted) endomorphism to the trace of its homotopy colimit. The idea is the following: Suppose $\mathbb{D}$ is additive, and let I be a category satisfying (H1)–(H4). Then Corollary 4.6 tells us that $\mathrm{Tr}(f)$ is completely determined by the local traces $\mathrm{tr}(h^* \circ f_i)$, $(i,h) \in \pi_0(2\mathrm{tw}(I))$. If $\pi_0(2\mathrm{tw}(I))$ is finite then, by Remark 5.3, we can think of Φ as a linear map which takes the input $(\mathrm{tr}(h^* \circ f_i))_{(i,h)}$ and outputs $\sum_{(i,h)} \lambda_{(i,h)}\mathrm{tr}(h^* \circ f_i) = \mathrm{tr}(p_{I!}f)$. We will obtain a formula for the trace of the homotopy colimit by determining these coefficients $\lambda_{(i,h)}$.

Let I be a finite category. The *ζ-function on I* is defined as the association

$$\begin{aligned} \zeta_I : I_0 \times I_0 &\to \mathbb{Z} \\ (i,j) &\mapsto \#I(i,j). \end{aligned}$$

Following [47] we define an *R-coweighting on I*, R a commutative unitary ring, to be a family $(\lambda_i)_{i \in I_0}$ of elements of R such that the following equality holds for all $j \in I_0$:

$$1 = \sum_{i \in I_0} \lambda_i \zeta_I(i,j). \tag{5.6}$$

Not all finite categories possess an R-coweighting; and if one such exists it might not be unique. Preordered sets always possess an R-coweighting (and it is unique if and only if the preorder is a partial order), groups possess one if and only if their order is invertible in R (and in this case it is unique). One trivial reason why a coweighting may not be unique is the existence of isomorphic distinct objects in a category. For in this case any modification of the family $(\lambda_i)_i$ which doesn't change the sum of the coefficients λ_i for isomorphic objects leaves the right hand side of (5.6) unchanged. On the other hand, this also means that any coweighting $(\lambda_i)_i$ on I induces a coweighting $(\rho_j)_j$ on the core of I by setting $\rho_j = \sum_{i \in I_0, i \cong j} \lambda_i$. (Here, "the" *core of I* is any equivalent subcategory of I which is *skeletal*, i. e. has no distinct isomorphic objects.) Conversely, any coweighting on the core induces a coweighting on I by choosing all additional coefficients to be 0. We therefore say that I admits an *essentially unique R-coweighting* if there is a unique R-coweighting on its core. In this case we sometimes speak abusively of *the* R-coweighting, especially if the context makes it clear which core is to be chosen.

For an EI-category I we continue to denote by G_i, $i \in I_0$, the group $I(i,i)$, and by C_i the set of conjugacy classes of G_i (cf. Example 4.9). Given $h \in G_i$, we denote by $[h] \in C_i$ the conjugacy class of h in G_i.

Definition 5.7 Let I be a finite EI-category. We define its *characteristic*, denoted by $\mathrm{char}(I)$, to be the product of distinct prime factors dividing the order of the automorphism group of some object in the category, i. e.

$$\mathrm{char}(I) = \mathrm{rad}\left(\prod_{i \in I_0} \#G_i\right).$$

Lemma 5.8 (cf. [47, 1.4]) *Let I be a finite EI-category and R a commutative unitary ring. If $\mathrm{char}(I)$ is invertible in R then there is an essentially unique R-coweighting on $\mathrm{end}(I)$. It is given as follows:*

Choose a core $J \subset \mathrm{end}(I)$ of objects $\{(i,h)\}$. Then

$$\lambda_{(j,k)} = \sum_{(i,h) \in J_0} \sum_{n \geq 0} (-1)^n \sum \frac{\#[h_0]}{\#G_{i_0}} \cdots \frac{\#[h_n]}{\#G_{i_n}},$$

where the last sum is over all non-degenerate paths

$$(i,h) = (i_0,h_0) \to (i_1,h_1) \to \cdots \to (i_n,h_n) = (j,k)$$

from (i,h) to (j,k) in J (i. e. the (i_l,h_l) are pairwise distinct, or, equivalently, the i_l are pairwise distinct, or, also equivalently, none of the arrows is the identity).

Proof. The data $(\zeta_J(h,k))_{h,k \in J_0}$ can be identified in an obvious way with a square matrix ζ_J with coefficients in $\mathbb{Z}$. For the first claim in the lemma, it suffices to prove that ζ_J is an invertible matrix in R, for then

$$\begin{aligned} \begin{bmatrix} \cdots & \lambda_{(i,h)} & \cdots \end{bmatrix} &= \begin{bmatrix} \cdots & \lambda_{(i,h)} & \cdots \end{bmatrix} (\zeta_J \zeta_J^{-1}) \\ &= \left(\begin{bmatrix} \cdots & \lambda_{(i,h)} & \cdots \end{bmatrix} \zeta_J \right) \zeta_J^{-1} \\ &= \begin{bmatrix} \cdots & 1 & \cdots \end{bmatrix} \zeta_J^{-1}. \end{aligned}$$

For any $(i,h) \in J_0$, the endomorphism monoid is $G_{(i,h)} = C_{G_i}(h)$, the centralizer of h, hence J is also a finite EI-category. This implies that we can find an object $(i,h) \in J_0$ which has no incoming arrows from other objects. Proceeding inductively we can thus choose an ordering of J_0 such that the matrix ζ_J is upper triangular. Consequently, $\det(\zeta_J) = \prod_{(i,h) \in J_0} \#C_{G_i}(h)$ is invertible in R by assumption.

The proof in [47, 1.4] goes through word for word to establish the formula given in the lemma (the relation between "Möbius inversion" and coweighting is given in [47, p. 28]). ☐

Example 5.9

(1) Let I be a finite skeletal category with no non-identity endomorphisms (e.g. a partially ordered set). Then for any ring R there is a unique R-coweighting on $I = \mathrm{end}(I)$ given by (cf. [47, 1.5])

$$\lambda_j = \sum_{i \in I_0} \sum_{n \geq 0} (-1)^n \#\{\text{non-degenerate paths of length } n \text{ from } i \text{ to } j\}$$

for any $j \in I_0$.

(2) Let $I = G$ be a finite group. By Example 4.9, the objects of the core of $\mathrm{end}(G)$ can be identified with the conjugacy classes of G. For a $\mathbb{Z}[1/\#G]$-algebra R, the R-coweighting on $\mathrm{end}(G)$ is given by

$$\lambda_{[k]} = \frac{\#[k]}{\#G}$$

for any conjugacy class $[k]$ of G.

Example 5.10 Let us go back to the situation considered in the introduction: Let $\ulcorner$ be the category of (0.8). It follows from the first example above that for any ring R, the unique R-coweighting on $\ulcorner = \mathrm{end}(\ulcorner)$ is given by

$$\begin{tikzcd} -1 & 1 \arrow[l] \\ 1 \arrow[u] & \end{tikzcd}$$

and one notices that these are precisely the coefficients in the formula for the trace of the homotopy colimit (0.9). This is an instance of the following theorem.

Theorem 5.11 *Let $\mathbb{D}$ be a closed monoidal stable derivator of type* **Dia***, let I be a finite EI-category in* **Dia** *and suppose that* $\mathrm{char}(I)$ *is invertible in $R_{\mathbb{D}}$. If $S, T \in \mathbb{D}(\star)_0$, $f : A \boxtimes S \to A \boxtimes T \in \mathbb{D}(I)_1$, with $A \in \mathbb{D}(I)_0$ fiberwise dualizable, then the object $p_{I!}A$ is dualizable in $\mathbb{D}(\star)$, and we have*

$$\mathrm{tr}(p_{I!}f) = \sum_{\substack{i \in I_0/\cong \\ [h] \in C_i}} \lambda_{(i,h)} \mathrm{tr}(h^* \circ f_i)$$

where $(\lambda_{(i,h)})_{(i,h)}$ is the $R_{\mathbb{D}}$-coweighting on $\mathrm{end}(I)$.

We will prove the theorem under the additional assumption that all of the hypotheses (H1)–(H4) are satisfied. In the next section we will show that they in fact automatically hold (Proposition 6.4).

Proof. We view $\pi_0(2\mathrm{tw}(I))$ as the set of pairs (i, h) where i runs through a full set of representatives for the isomorphism classes of objects of I, and h runs through a full set of representatives for the conjugacy classes of G_i (use Example 4.9).

Lemma 3.15 tells us that we may consider Φ as a group homomorphism

$$\prod_{\pi_0(2\mathrm{tw}(I))} \mathbb{D}(\star)(S, T) \to \mathbb{D}(\star)(S, T).$$

We first assume $S = T$, set $R = \mathbb{D}(\star)(S, S)$. In this case, Remark 5.3 tells us that Φ is both left and right R-linear hence there exist $\lambda_{(i,h)} \in Z(R)$, the center of R, such that for every $g = (g_{(i,h)})_{(i,h)}$ in the domain,

$$\Phi(g) = \sum_{(i,h) \in \pi_0(2\mathrm{tw}(I))} \lambda_{(i,h)} g_{(i,h)}.$$

In particular, if $g = \mathrm{Tr}(f)$ we get

$$\begin{aligned} \mathrm{tr}(p_{I!}f) &= \Phi(\mathrm{Tr}(f)) && \text{by Proposition 5.4,} \\ &= \sum_{(i,h)} \lambda_{(i,h)} \mathrm{tr}(h^* \circ f_i) && \text{by Corollary 4.6.} \end{aligned} \tag{5.12}$$

Now, fix $(j, k) \in \pi_0(2\mathrm{tw}(I))$. Below we will define a specific endomorphism f satisfying

$$\mathrm{tr}(p_{I!}f) = 1_S \tag{5.13}$$

and

$$\mathrm{tr}(h^* \circ f_i) = \zeta_{\mathrm{end}(I)}(h, k) \tag{5.14}$$

for any $(i, h) \in \pi_0(2\mathrm{tw}(I))$. Letting $(j, k) \in \pi_0(2\mathrm{tw}(I))$ vary, (5.12) thus says that the $\lambda_{(i,h)}$ define a $Z(R)$-coweighting on the core of $\mathrm{end}(I)$ and by Lemma 5.8 this is unique (by assumption, $\mathrm{char}(I)$ is invertible in $R_{\mathbb{D}}$ but then it must also be invertible in $Z(R)$). It must therefore be (the image of) the unique $R_{\mathbb{D}}$-coweighting on the core of $\mathrm{end}(I)$ and this would complete the proof of the theorem in the case $S = T$.

Before we come to the construction of f, let us explain how the general case (i. e. when not necessarily $S = T$) can be deduced. Set $U = S \oplus T$ and denote by $\iota : S \to U$ and $\pi : U \to T$ the canonical inclusion and projection, respectively. By Remark 5.3, the following diagram commutes:

$$\begin{tikzcd}
\prod_{\pi_0(2\mathrm{tw}(I))} \mathbb{D}(\star)(U,U) \arrow[r, "\Phi_{U,U}"] \arrow[d, "\iota^*"'] & \mathbb{D}(\star)(U,U) \arrow[d, "\iota^*"] \\
\prod_{\pi_0(2\mathrm{tw}(I))} \mathbb{D}(\star)(S,U) \arrow[r, "\Phi_{S,U}"] \arrow[d, "\pi_*"'] & \mathbb{D}(\star)(S,U) \arrow[d, "\pi_*"] \\
\prod_{\pi_0(2\mathrm{tw}(I))} \mathbb{D}(\star)(S,T) \arrow[r, "\Phi_{S,T}"] & \mathbb{D}(\star)(S,T).
\end{tikzcd}$$

Given a family $(g_{(i,h)})_{(i,h)}$ in the bottom left, there is a canonical lift $(\widetilde{g_{(i,h)}})_{(i,h)}$ in the top left, similarly for the right hand side. In particular, given S, T, A, f as in the statement of the theorem,

$$\begin{aligned}
\mathrm{tr}(p_{I!}f) &= \Phi_{S,T}(\mathrm{Tr}(f)) && \text{by the proposition,}\\
&= \Phi_{S,T}(\pi_*\iota^*(\widetilde{\mathrm{Tr}(f)_{(i,h)}})_{(i,h)}) \\
&= \pi_*\iota^*\Phi_{U,U}((\widetilde{\mathrm{Tr}(f)_{(i,h)}})_{(i,h)}) \\
&= \pi_*\iota^*\textstyle\sum_{(i,h)}\lambda_{(i,h)}\mathrm{tr}(\widetilde{h^*\circ f_i}) && \text{by the previous argument,}\\
&= \textstyle\sum_{(i,h)}\pi\lambda_{(i,h)}\mathrm{tr}(\widetilde{h^*\circ f_i})\iota \\
&= \textstyle\sum_{(i,h)}\lambda_{(i,h)}\pi\mathrm{tr}(\widetilde{h^*\circ f_i})\iota \\
&= \textstyle\sum_{(i,h)}\lambda_{(i,h)}\mathrm{tr}(h^*\circ f_i).
\end{aligned}$$

This completes the argument in the general case.

Now we come to the construction of the endomorphism f mentioned above. We will freely use the fact that for any finite group $G \in \mathbf{Dia}_0$ whose order is invertible in $R_{\mathbb{D}}$ (such as G_i for all $i \in I_0$ by assumption), the underlying diagram functor

$$\mathrm{dia}_G : \mathbb{D}(G) \to \mathbf{CAT}(G^\circ, \mathbb{D}(\star))$$

is fully faithful. We postpone the proof of this to appendix C.

Fix $(j,k) \in \pi_0(2\mathrm{tw}(I))$. Denote by $e_j : \star \to G_j$ the unique functor; by (Dia2), this is a functor in **Dia**. Then $e_{j!}S$ is the right regular representation of G_j° associated to S (for more details, see appendix C); we denote the action by $r_{(-)}$. Left translation by k, l_k, defines a G_j°-endomorphism of $e_{j!}S$. By transitivity of the action,

$$p_{G_j!}l_k : S = p_{G_j!}e_{j!}S = e_{j!}S/G_j^\circ \to e_{j!}S/G_j^\circ = p_{G_j!}e_{j!}S = S$$

is just the identity.

Let $\overline{j} : G_j \to I$ be the fully faithful inclusion pointing j and set $f = \overline{j}_! l_k$. To be completely precise, we should set $A = j_!\mathbb{1}$, and f to be the endomorphism of $A \boxtimes S$ induced by $\overline{j}_! l_k$ via the canonical isomorphism

$$j_!\mathbb{1} \boxtimes S \xleftarrow[\sim]{(1.4)} j_!(\mathbb{1} \boxtimes S) \xrightarrow{\sim} j_!S \xrightarrow{\sim} \overline{j}_! e_{j!}S.$$

However, for the sake of clarity, we will continue to use this identification implicitly.

Then we have

$$\begin{aligned}\mathrm{tr}(p_{I!}\bar{j}_! l_k) &= \mathrm{tr}(p_{G_j!} l_k) \\ &= \mathrm{tr}(1_S) && \text{as seen above,} \\ &= 1_S\end{aligned}$$

i. e. (5.13) holds.

For (5.14) we must understand $h^* \circ i^* \bar{j}_! l_k$. Write $\mathrm{S}(m)$ for the stabilizer subgroup of $m \in I(i,j)$ in G_j and consider the following comma square in **Dia**

$$\begin{tikzcd} \coprod_m \mathrm{S}(m) \arrow[r, "w"] \arrow[d, "p"'] \arrow[dr, phantom, "\eta \nearrow"] & G_j \arrow[d, "\bar{j}"] \\ \star \arrow[r, "i"'] & I \end{tikzcd}$$

where the disjoint union is indexed by a full set of representatives for the G_j-orbits of $I(i,j)$, w is the canonical inclusion on each component, and η is m on the component of m. Under the identification $i^*\bar{j}_! \cong p_! w^*$ (by (D4)),

$$i^*\bar{j}_! e_{j!} S \cong p_! w^* e_{j!} S \cong \bigoplus_m \big(e_{j!} S/\mathrm{S}(m)\big) \cong \bigoplus_m \bigoplus_{G_j/\mathrm{S}(m)} S,$$

and $i^*\bar{j}_! l_k$ corresponds to the morphism which takes the $g\mathrm{S}(m)$-summand identically to the $k^{-1}g\mathrm{S}(m)$-summand. It follows that under the identification $i^*\bar{j}_! e_{j!} S \cong i^* j_! S \cong \oplus_{I(i,j)} S$ (again by (D4)), it corresponds to the morphism which takes the m-summand identically to the $k^{-1}m$-summand.

Writing out explicitly the Beck-Chevalley transformation above we obtain the horizontal arrows in the following diagram:

$$\begin{tikzcd} \oplus_{I(i,j)} \arrow[r, "\mathrm{adj}"] \arrow[d, "m \mapsto mh"'] & \oplus_{I(i,j)} j^* j_! \arrow[r, "(m^*)_m"] \arrow[d, "m \mapsto mh"] & \oplus_{I(i,j)} i^* j_! \arrow[r, "\Sigma"] & i^* j_! \arrow[d, "h^*"] \\ \oplus_{I(i,j)} \arrow[r, "\mathrm{adj}"] & \oplus_{I(i,j)} j^* j_! \arrow[r, "(m^*)_m"] & \oplus_{I(i,j)} i^* j_! \arrow[r, "\Sigma"] & i^* j_!. \end{tikzcd}$$

Obviously, the diagram is commutative. In total we get that $h^* \circ i^* \bar{j}_! l_k$ corresponds to the morphism which takes the m-summand identically to the $k^{-1}mh$-summand. It follows that the trace of this composition is equal to

$$\begin{aligned}\mathrm{tr}(h^* \circ i^* \bar{j}_! l_k) &= \#\{m \in I(i,j) \mid k^{-1}mh = m\} \\ &= \#\mathrm{end}(I)(h,k) \\ &= \zeta_{\mathrm{end}(I)}(h,k).\end{aligned}$$

□

6. $\mathbb{Q}$-linearity and stability

Let $\mathbb{D}$ be a monoidal stable derivator. In this section we will show that for any finite EI-category $I \in \mathbf{Dia}_{\mathrm{o}}$, if $\mathrm{char}(I)$ is invertible in $R_{\mathbb{D}}$ then all hypotheses (H1)–(H4) automatically hold. The main tool used in the proof is Lemma 6.1 below, in essence suggested to me by Joseph Ayoub, where it is shown how invertibility of $\mathrm{char}(I)$ in $R_{\mathbb{D}}$ and $\mathbb{D}$ being stable imply the existence of nice generators for $\mathbb{D}(I)$. In fact, this is the only place in the chapter where the triangulated structure plays any role.

Recall that a subcategory of a triangulated category is called *thick* if it is a triangulated subcategory and closed under direct factors. If $\mathcal{T}$ is a triangulated category and $S \subset \mathcal{T}_0$ a family of objects we denote by

$$\langle S \rangle \qquad (\text{resp. } \langle S \rangle_s)$$

the triangulated (resp. thick) subcategory generated by S.

Let I be an EI-category. If $i \in I_0$ is an object we denote by G_i its automorphism group $I(i,i)$, and by $\bar{i} : G_i \to I$ the fully faithful embedding of the "point" i into I. This is to distinguish it from the inclusion $i : \star \to I$.

Lemma 6.1 *Let $\mathbb{D}$ be a stable derivator, and let $I \in$ **Dia**$_0$ be a finite EI-category. Then we have the following equality:*

$$\mathbb{D}(I) = \langle \bar{i}_! A \mid i \in I_0, A \in \mathbb{D}(G_i)_0 \rangle.$$

Suppose that for all $i \in I_0$, the canonical functor $e_i : \star \to G_i$ induces a faithful functor $e_i^* : \mathbb{D}(G_i) \to \mathbb{D}(\star)$. Then we also have the following equality:

$$\mathbb{D}(I) = \langle i_! A \mid i \in I_0, A \in \mathbb{D}(\star)_0 \rangle_s.$$

All these statements remain true if we replace $(-)_!$ by $(-)_*$ everywhere.

Remark 6.2 We will prove in appendix C that if n is invertible in $R_{\mathbb{D}}$ then $e : \star \to G$ induces a faithful functor $e^* : \mathbb{D}(G) \to \mathbb{D}(\star)$ for any group $G \in$ **Dia**$_0$ of order n. In particular, if $\operatorname{char}(I)$ is invertible in $R_{\mathbb{D}}$ then the second equality in Lemma 6.1 holds.

Proof of Lemma 6.1. Note that since $I \in$ **Dia**$_0$ so is G_i, $i \in I_0$, by (Dia2). Therefore, the statement of the lemma at least makes sense.

The first equality is proved by induction on the number n of objects in I. Clearly, we may assume I to be skeletal. If $n = 1$, the claim is obviously true. If $n > 1$ we find an object $i \in I_0$ which is maximal in the sense that the implication $I(i,j) \neq \varnothing \Rightarrow i = j$ holds. For any $B \in \mathbb{D}(I)_0$, consider the morphism

$$\mathrm{adj} : \bar{i}_! \bar{i}^* B \to B$$

and let C be the cone. One checks easily that $i^*\mathrm{adj}$ is an isomorphism hence $i^* C = 0$ which implies that C is of the form $u_! B'$, some $B' \in \mathbb{D}(U)_0$ where $u : U \hookrightarrow I$ is the open embedding of the full subcategory of objects different from i in I (see [16, 8.11]). By induction, $B' \in \langle \bar{j}_! A \mid j \in U_0, A \in \mathbb{D}(G_j)_0 \rangle$, hence it suffices to prove

$$u_! \langle \bar{j}_! A \mid j \in U_0, A \in \mathbb{D}(G_j)_0 \rangle \subset \langle \bar{j}_! A \mid j \in I_0, A \in \mathbb{D}(G_j)_0 \rangle.$$

But this follows from the fact that $u_!$ is a triangulated functor and $u_! \bar{j}_! = \bar{j}_!$.

For the second equality, it will follow from the first as soon as we prove, for each $i \in I_0$,

$$\mathbb{D}(G_i) = \langle e_{i!} A \mid A \in \mathbb{D}(\star)_0 \rangle_s. \tag{6.3}$$

So let $B \in \mathbb{D}(G_i)_0$ and consider the counit $e_{i!} e_i^* B \to B$. By assumption this is an epimorphism. But in a triangulated category every epimorphism is complemented, i. e.

$$e_{i!} e_i^* B \cong B \oplus B',$$

some $B' \in \mathbb{D}(G_i)_0$. This proves (6.3) and hence the second equality.

The last claim of the lemma can be established by dualizing the whole proof. □

Proposition 6.4 *Let $\mathbb{D}$ be a monoidal stable derivator, let I be a finite EI-category in* **Dia**, *and suppose that* $\mathrm{char}(I)$ *is invertible in* $R_{\mathbb{D}}$. *Then all hypotheses (H1)–(H4) are satisfied.*

PROOF.

(1) For (H1) we will prove more generally that

$$q_{1*}A \otimes T|_{I^\circ \times I} \to q_{1*}(A \otimes T|_{\mathrm{tw}(I)}) \tag{6.5}$$

is invertible for all $T \in \mathbb{D}(\star)_0$, $A \in \mathbb{D}(\mathrm{tw}(I))_0$. Also we will only need that I has finite hom-sets.

Fix $i, j \in I_0$ and consider the following pullback square:

$$\begin{tikzcd} I(i,j) \arrow[r, "r_{i,j}"] \arrow[d, "p_{I(i,j)}"'] & \mathrm{tw}(I) \arrow[d, "q_1"] \\ \star \arrow[r, "(i,j)"'] & I^\circ \times I. \end{tikzcd}$$

Since q_1 is an opfibration, Fact 1.1 tells us that the first vertical morphisms on the left and on the right in the following diagram are invertible:

$$\begin{tikzcd} (i,j)^*(q_{1*}A \otimes T|_{I^\circ \times I}) \arrow[r] \arrow[d, "\sim"'] & (i,j)^* q_{1*}(A \otimes T|_{\mathrm{tw}(I)}) \arrow[d, "\sim"] \\ p_{I(i,j)*} r_{i,j}^* A \otimes T \arrow[r] \arrow[d, "\sim"'] & p_{I(i,j)*} r_{i,j}^*(A \otimes T|_{\mathrm{tw}(I)}) \arrow[d, "\sim"] \\ \prod_{h \in I(i,j)} h^* r_{i,j}^* A \otimes T \arrow[r] & \prod_{h \in I(i,j)} (h^* r_{i,j}^* A \otimes T). \end{tikzcd}$$

Here, $h : \star \to I(i,j)$ is the functor defined by the object h of the discrete category $I(i,j)$, and the axiom (D1) is used for the second vertical morphisms on the left and on the right. Clearly both squares commute. Moreover, the bottom horizontal arrow is invertible since $I(i,j)$ is finite and the internal product in $\mathbb{D}(\star)$ is additive (Lemma 1.10). Therefore also the top horizontal arrow is invertible which implies (by (D2), and letting i and j vary) that (6.5) is.

(2) For (H2), let γ be a connected component of $2\mathrm{tw}(I)$. Since I is a finite EI-category, γ is equivalent to a finite group G whose order divides $\mathrm{char}(I)$. As explained in Remark 6.2, this implies that e_G^* is faithful ($e_G : \star \to G$). Since p_G^* is a section of e_G^*, it follows that p_G^* is fully faithful.

(3) (H3) states that $p_{I!}p_{2*} \to p_{I^\circ *}p_{1!}$ is invertible. Since also $I^\circ \times I$ is a finite EI-category and since $\mathrm{char}(I^\circ \times I) = \mathrm{char}(I)$ we may prove this on objects in the image of $(1_{I^\circ} \times i)_!$, where $i \in I_0$, by the previous lemma. Consider the following square:

$$\begin{tikzcd} p_{I!}p_{2*}(1_{I^\circ} \times i)_! \arrow[r] & p_{I^\circ *}p_{1!}(1_{I^\circ} \times i)_! \arrow[d, "\sim"] \\ p_{I!}i_!p_{I^\circ *} \arrow[u] \arrow[r, "\sim"'] & p_{I^\circ *}. \end{tikzcd}$$

It is easy to see that it commutes hence it suffices to prove invertibility of the left vertical arrow.

For this we use (D2), so fix $j \in I_0$ an object. Then

$$\begin{aligned} j^* i_! p_{I^\circ *} &\cong \oplus_{I(j,i)} p_{I^\circ *}, \\ j^* p_{2*}(1_{I^\circ} \times i)_! &\cong p_{I^\circ *}(1_{I^\circ} \times j)^*(1_{I^\circ} \times i)_! \\ &\cong p_{I^\circ *} \oplus_{I(j,i)} . \end{aligned}$$

The claim follows since $p_{I^\circ *}$ is additive. (It is easy to see that this identification is compatible with the vertical arrow above.)

(4) Since also I° is a finite EI-category and $\operatorname{char}(I^\circ) = \operatorname{char}(I)$, we may replace I by I°. (H4) then is the statement that

$$p_{I*} A \otimes B \to p_{I*}(A \otimes p_I^* B)$$

is invertible, and by Lemma 6.1 we may assume $A = i_* C$, some $C \in \mathbb{D}(\star)_0$ and $i \in I_0$ (here we use that $- \otimes B$ and $- \otimes p_I^* B$ both take distinguished triangles to distinguished triangles, by Lemma 1.10).

Clearly, the following square commutes:

$$\begin{tikzcd} p_{I*} i_* C \otimes B \arrow[r] \arrow[d, "\sim"'] & p_{I*}(i_* C \otimes p_I^* B) \arrow[d] \\ C \otimes B & p_{I*} i_*(C \otimes B), \arrow[l, "\sim"] \end{tikzcd}$$

hence it suffices to prove invertible the vertical arrow on the right. Again we use (D2), so let $j \in I_0$ an object. Then:

$$\begin{aligned} j^* i_* C \otimes j^* p_I^* B &\cong \oplus_{I(i,j)} C \otimes B, \\ j^* i_*(C \otimes B) &\cong \oplus_{I(i,j)} (C \otimes B). \end{aligned}$$

Again, the claim follows from the additivity of the functor $- \otimes B$.

□

A. Properties of the external hom

In this section we want to give proofs for the properties of the external hom listed in section 2. We take them up one by one. Throughout the section we fix a closed monoidal derivator $\mathbb{D}$ of type **Dia**.

Naturality. Given $u : I' \to I$ and $v : J' \to J$ in **Dia** there is an induced morphism of diagrams $(\Pi_{I',J'}) \to (\Pi_{I,J})$ and we distinguish the morphisms in the former from their counterparts in the latter by decorating them with a prime. We deduce a morphism

$$\begin{aligned} \Psi^{u,v}_{A,B} : (u^\circ \times v)^* \langle A, B \rangle &=\!= (u^\circ \times v)^* p_*[q^* A, r^* B] \\ &\to p'_*(\operatorname{tw}(u) \times v)^*[q^* A, r^* B] \\ &\to p'_*[(\operatorname{tw}(u) \times v)^* q^* A, (\operatorname{tw}(u) \times v)^* r^* B] \\ &=\!= \langle u^* A, v^* B \rangle. \end{aligned}$$

Clearly, this morphism is natural in A and B, moreover it behaves well with respect to identities and composition of functors as well as natural transformations in $\mathbf{Dia}^\circ \times \mathbf{Dia}^{\circ,\circ}$ so that we have defined a lax natural transformation. The following proposition thus concludes the proof of the naturality property.

Proposition A.1 *For u,v and A, B as above the morphism $\Psi^{u,v}_{A,B}$ is invertible.*

Proof. We proceed in several steps.

(1) Let $i \in I'_0$, $j \in J'_0$. It suffices to prove that $(i,j)^*$ applied to the morphism $\Psi^{u,v}_{A,B}$ is invertible. But this means that it suffices to prove that $\Psi^{ui,vj}_{-,-}$ and $\Psi^{i,j}_{-,-}$ are invertible; in other words we may assume $I' = J' = \star$, $u = i \in I_0$, $v = j \in J_0$.

(2) We factor $(i,j) : \star \xrightarrow{i} I \xrightarrow{1_I \times j} I \times J$, and first deal with $\Psi^{1_I, j}_{-,-}$. In this case, the square

$$\begin{tikzcd}
\mathrm{tw}(I) \arrow[r, "1_{\mathrm{tw}(I)} \times j"] \arrow[d, "p'"'] & \mathrm{tw}(I) \times J \arrow[d, "p"] \\
I^\circ \arrow[r, "1_{I^\circ} \times j"'] & I^\circ \times J
\end{tikzcd}$$

is a pullback square, and p an opfibration, therefore the first arrow in the definition of Ψ is invertible (Fact 1.1). For the second arrow in the definition, it suffices to prove invertible

$$(1_{\mathrm{tw}(I)} \times j)^*[(1_{\mathrm{tw}(I)} \times p_J)^* -, -] \to [-, (1_{\mathrm{tw}(I)} \times j)^* -].$$

By passing to $\mathbb{D}_{\mathrm{tw}(I)}$ we may thus assume $I = \star$ and prove instead invertible

$$j^*[p_J^* -, -] \to [-, j^* -].$$

By adjunction, this corresponds to the morphism

$$(1_\star \times j)_!(- \boxtimes -) \xrightarrow{(1.4)} - \boxtimes j_! -$$

which we know to be invertible.

(3) Thus from now on we may assume $J = \star$. Factor $i : \star \xrightarrow{1_i} i\backslash I \xrightarrow{t} I$. Exactly the same argument as in the previous step shows that the first arrow in the definition of $\Psi^{t,1_\star}$ is invertible. Moreover, $\mathrm{tw}(t)$ is a fibration hence, by Lemma 1.5, also the second arrow in the definition of $\Psi^{t,1_\star}$ is invertible.

(4) From now on, we may assume that I has initial object i and we need to prove $\Psi^{i,1_\star}$ invertible. Consider the following diagram:

$$\begin{tikzcd}
i^* p_*[q^* -, p^*_{\mathrm{tw}(I)} -] \arrow[r] \arrow[dr, "\sim"'] & 1_i^*[q^* -, p^*_{\mathrm{tw}(I)} -] \arrow[r] & {[1_i^* q^* -, 1_i^* p^*_{\mathrm{tw}(I)} -]} \arrow[r, equal] & {[i^* -, -]} \\
 & p_{\mathrm{tw}(I)*}[q^* -, p^*_{\mathrm{tw}(I)} -] \arrow[u] & {[p_{\mathrm{tw}(I)!} q^* -, -]} \arrow[l, "\sim"] \arrow[u] & {[p_{I!} -, -].} \arrow[l, "\sim"] \arrow[u, "\sim"']
\end{tikzcd}$$

The composition of the top horizontal arrows is nothing but $\Psi^{i,1_\star}$. The triangle on the left arises from the Beck-Chevalley transformations associated to the squares

$$\begin{tikzcd}
\star \arrow[r, "1_i"] \arrow[d, equal] & \mathrm{tw}(I) \arrow[r, equal] \arrow[d, "p_{\mathrm{tw}(I)}"'] \arrow[dr, phantom, "\swarrow"] & \mathrm{tw}(I) \arrow[d, "p"] \\
\star \arrow[r, equal] & \star \arrow[r, "i"'] & I^\circ.
\end{tikzcd}$$

It follows that the triangle commutes and the slanted morphism is invertible by (D4). The first bottom horizontal arrow is invertible by Lemma 1.5, the second one arises from the counit $q_! q^* \to 1$ which is invertible by Lemma B.1. The middle vertical arrow is induced by the "dual" of the left vertical arrow, $1_i^* \xrightarrow{\mathrm{adj}} 1_i^* p^*_{\mathrm{tw}(I)} p_{\mathrm{tw}(I)!} = p_{\mathrm{tw}(I)!}$. The commutativity of the left square is therefore immediate, as is the commutativity of the right square (the right vertical arrow is induced by the canonical identification $i^* \cong p_{I!}$ as i is an initial object of I).

$\square$

Internal hom. We now want to show that in case $I = J \in \mathbf{Dia}_0$, internal hom can be expressed in terms of external hom. Consider the following category $3I$: Objects are two composable arrows in I and morphisms from the top to the bottom are of the form:

$$\begin{tikzcd} i_2 \arrow[r] \arrow[d] & i_1 \arrow[r] & i_0 \arrow[d] \\ j_2 \arrow[r] & j_1 \arrow[r] \arrow[u] & j_0. \end{tikzcd}$$

We have canonical functors $t_k : 3I \to I, k = 0, 2$. Moreover, there are functors $p' : 3I \to \mathrm{tw}(I)^\circ$ and $q' : 3I \to \mathrm{tw}(I) \times I$, the first one forgetting the 0-component, the second one mapping the two components 0 and 1 to $\mathrm{tw}(I)$ and component 2 to I. It is easy to see that one gets a pullback square:

$$\begin{tikzcd} 3I \arrow[r, "p'"] \arrow[d, "q'"'] & \mathrm{tw}(I)^\circ \arrow[d, "q_2"] \\ \mathrm{tw}(I) \times I \arrow[r, "p"'] & I^\circ \times I. \end{tikzcd}$$

Notice that there is a canonical natural transformation $t_2 \to t_0$ and hence one can define the following morphism:

$$\begin{aligned} \Theta^I_{A,B} : [A, B] &\xrightarrow{\mathrm{adj}} [t_{2!} t_2^* A, B] \\ &\longrightarrow [t_{2!} t_0^* A, B] \\ &\longrightarrow t_{2*}[t_0^* A, t_2^* B] \\ &\xleftarrow{\sim} p_{2*} q_{2*} p'_*[q'^* q^* A, q'^* r^* B] \\ &\xleftarrow{\sim} p_{2*} q_{2*} p'_* q'^*[q^* A, r^* B] \\ &\xleftarrow{\sim} p_{2*} q_{2*} q_2^* p_*[q^* A, r^* B]. \end{aligned} \tag{A.2}$$

Here the last isomorphism is due to Fact 1.1 and q_2 being a fibration. Therefore also q' is a fibration and Lemma 1.5 gives us the second to last isomorphism.

Again, $\Theta^I_{A,B}$ is clearly natural in A and B and one checks easily (if tediously) that the following diagram commutes for any $u : I' \to I$ in $\mathbf{Dia}_1$:

$$\begin{tikzcd} u^*[A, B] \arrow[r, "\Theta^I"] \arrow[d] & u^* p_{2*} q_{2*} q_2^* \langle A, B \rangle \arrow[r, dotted] & p'_{2*} q'_{2*} \mathrm{tw}(u)^{\circ *} q_2^* \langle A, B \rangle \arrow[d, equal] \\ {[u^* A, u^* B]} \arrow[r, "\Theta^{I'}"'] & p'_{2*} q'_{2*} q_2'^* \langle u^* A, u^* B \rangle & p'_{2*} q'_{2*} q_2'^* (u^\circ \times u)^* \langle A, B \rangle. \arrow[l, dotted, "\Psi"'] \end{tikzcd}$$

It follows that if we take the composition of the dotted arrows in the diagram as components of the 2-cells for the lax natural transformation $p_{2*} q_{2*} q_2^* \langle -, - \rangle$, then Θ defines a modification as claimed in section 2. It now remains to prove that it is invertible.

Proposition A.3 *$\Theta^I_{A,B}$ is invertible for all I, A and B as above.*

Proof. It is easy to see that t_2 is a fibration. Hence it follows from Lemma 1.5 that the third arrow in (A.2) is invertible, and it now suffices to prove that

$$t_{2!}t_0^* \to t_{2!}t_2^* \xrightarrow{\text{adj}} 1 \tag{A.4}$$

is invertible. Let $i \in I_0$ be an arbitrary object. We will show that i^* applied to (A.4) is invertible which is enough for the claim by (D2).

Consider the following two diagrams:

$$\begin{tikzcd} 3I_i \arrow[r, "w"] \arrow[d, "p_{3I_i}"'] & 3I \arrow[d, "t_2"] \\ \star \arrow[r, "i"'] & I, \end{tikzcd} \qquad \begin{tikzcd}[column sep=large] 3I_i \arrow[rr, "w"] \arrow[d, "u"'] & & 3I \arrow[d, bend right, "t_2"'] \arrow[d, bend left, "t_0"] \arrow[d, phantom, "\Rightarrow"] \\ i\backslash I \arrow[rr, bend left=15, "ip_{i\backslash I}"] \arrow[rr, bend right=15, "v"'] \arrow[rr, phantom, "\Downarrow"] & & I. \end{tikzcd}$$

The first one is a pullback square, in the second one u is defined by $u(i \to i_1 \to i_0) = i \to i_0$, while $v(i \to i_0) = i_0$ and $ip_{i\backslash I} \to v$ is the canonical natural transformation. This second diagram is commutative in the sense that $ip_{i\backslash I}u \to vu$ is equal to $t_2 w \to t_0 w$. Consequently the second inner square on the left of the following diagram commutes:

$$\begin{tikzcd} i^*t_{2!}t_0^* \arrow[r] & i^*t_{2!}t_2^* \arrow[r, "\text{adj}"] & i^* \\ p_{3I_i!}w^*t_0^* \arrow[r] \arrow[u, "\sim"] \arrow[d, "\sim"'] & p_{3I_i!}w^*t_2^* \arrow[r, equal] \arrow[u, "\sim"] \arrow[d, "\sim"'] & p_{3I_i!}p_{3I_i}^*i^* \arrow[u, "\text{adj}"'] \arrow[dl, "\sim"] \\ p_{i\backslash I!}u_!u^*v^* \arrow[r] \arrow[d, "\text{adj}"'] & p_{i\backslash I!}u_!u^*p_{i\backslash I}^*i^* \arrow[d, "\text{adj}"'] & \\ p_{i\backslash I!}v^* \arrow[r, "\sim"'] & p_{i\backslash I!}p_{i\backslash I}^*i^*. \arrow[uuur, bend right=40, "\sim", "\text{adj}"'] & \end{tikzcd}$$

The rest is clearly commutative. Moreover, the top row is the fiber of (A.4) over i. The isomorphism of functors $p_{i\backslash I!} \cong 1_i^*$ (1_i being the initial object of $i\backslash I$) implies that the bottom horizontal as well as the bent arrow induced by the counit of the adjunction $p_{i\backslash I!} \dashv p_{i\backslash I}^*$ are invertible, hence it suffices to prove $u_!u^* \to 1$ an isomorphism. But this is true since u admits a fully faithful right adjoint

$$\begin{aligned} i\backslash I &\longrightarrow 3I_i \\ (i \to j) &\longmapsto (i \xrightarrow{1_i} i \to j). \end{aligned}$$

$\square$

External product. Recall that for any closed monoidal category there is a canonical morphism

$$[A_1, A_2] \otimes [A_3, A_4] \to [A_1 \otimes A_3, A_2 \otimes A_4] \tag{A.5}$$

defined by adjunction as follows:

$$([A_1, A_2] \otimes [A_3, A_4]) \otimes (A_1 \otimes A_3) \xrightarrow{\sim} ([A_1, A_2] \otimes A_1) \otimes ([A_3, A_4] \otimes A_3) \xrightarrow{\mathrm{ev}\otimes\mathrm{ev}} A_2 \otimes A_4.$$

From this we deduce for $A_1, A_3 \in \mathbb{D}(I)_0$, $A_2, A_4 \in \mathbb{D}(J)_0$ $(I, J \in \mathbf{Dia}_0)$:

$$\begin{aligned} \langle A_1, A_2\rangle \otimes \langle A_3, A_4\rangle &=\!= p_*[q^*A_1, r^*A_2] \otimes p_*[q^*A_3, r^*A_4] \\ &\longrightarrow p_*([q^*A_1, r^*A_2] \otimes [q^*A_3, r^*A_4]) \\ &\xrightarrow{(A.5)} p_*([q^*A_1 \otimes q^*A_3, r^*A_2 \otimes r^*A_4]) \\ &\xrightarrow{\sim} \langle A_1 \otimes A_3, A_2 \otimes A_4\rangle. \end{aligned} \tag{A.6}$$

Now, fix categories $I_{(k)}$, $k = 1, \ldots 4$ in **Dia** and objects $A_k \in \mathbb{D}(I_{(k)})_0$. Set $K = I_{(1)}^\circ \times I_{(2)} \times I_{(3)}^\circ \times I_{(4)}$. We can now finally define the morphism Ξ:

$$\begin{aligned} \Xi^{I_{(1)}, I_{(2)}, I_{(3)}, I_{(4)}}_{A_1, A_2, A_3, A_4} : \quad & \langle A_1, A_2\rangle \boxtimes \langle A_3, A_4\rangle = \langle A_1, A_2\rangle|_K \otimes \langle A_3, A_4\rangle|_K \\ & \underset{\sim}{\xrightarrow{\Psi}} \tau^*\langle A_1|_{I_{(1)}\times I_{(3)}}, A_2|_{I_{(2)}\times I_{(4)}}\rangle \otimes \tau^*\langle A_3|_{I_{(1)}\times I_{(3)}}, A_4|_{I_{(2)}\times I_{(4)}}\rangle \\ & \underset{\sim}{\leftarrow} \tau^*\left(\langle A_1|_{I_{(1)}\times I_{(3)}}, A_2|_{I_{(2)}\times I_{(4)}}\rangle \otimes \langle A_3|_{I_{(1)}\times I_{(3)}}, A_4|_{I_{(2)}\times I_{(4)}}\rangle\right) \\ & \xrightarrow{(A.6)} \tau^*\langle A_1 \boxtimes A_3, A_2 \boxtimes A_4\rangle. \end{aligned} \tag{A.7}$$

Clearly, $\Xi^{I_{(1)}, I_{(2)}, I_{(3)}, I_{(4)}}$ is a natural transformation. To conclude the proof of the external product property it remains to verify the following lemma.

Lemma A.8 *Let $u_k : I'_{(k)} \to I_{(k)}$, $k = 1, \ldots, 4$. Then the following diagram commutes:*

$$\begin{tikzcd} (u_1^\circ \times u_3^\circ \times u_2 \times u_4)^*(\langle A_1, A_2\rangle \boxtimes \langle A_3, A_4\rangle) \arrow[r, "\Xi"] \arrow[d, "\Psi"', "\sim"] & (u_1^\circ \times u_3^\circ \times u_2 \times u_4)^*\tau^*\langle A_1 \boxtimes A_3, A_2 \boxtimes A_4\rangle \arrow[d, "\sim"', "\Psi"] \\ \langle u_1^*A_1, u_2^*A_2\rangle \boxtimes \langle u_3^*A_3, u_4^*A_4\rangle \arrow[r, "\Xi"'] & \tau^*\langle u_1^*A_1 \boxtimes u_3^*A_3, u_2^*A_2 \boxtimes u_4^*A_4\rangle. \end{tikzcd}$$

Proof. By decomposing the horizontal arrows according to their definition in (A.7) one immediately reduces to showing that (A.6) behaves well with respect to the functors u_k; in other words one reduces to showing that for $A_1, A_3 \in \mathbb{D}(I)_0$, $A_2, A_4 \in \mathbb{D}(J)_0$ and functors $u : I' \to I$, $v : J' \to J$, the following diagram commutes:

$$\begin{tikzcd} (u^\circ \times v)^*(\langle A_1, A_2\rangle \otimes \langle A_3, A_4\rangle) \arrow[r, "(A.6)"] \arrow[d, "\Psi"', "\sim"] & (u^\circ \times v)^*\langle A_1 \otimes A_3, A_2 \otimes A_4\rangle \arrow[d, "\sim"', "\Psi"] \\ \langle u^*A_1, v^*A_2\rangle \otimes \langle u^*A_3, v^*A_4\rangle \arrow[r, "(A.6)"'] & \langle u^*A_1 \otimes u^*A_3, v^*A_2 \otimes v^*A_4\rangle. \end{tikzcd}$$

Since the unit and counit of the adjunction $p^* \dashv p_*$ behave well with respect to pulling back along $u^\circ \times v$ and $\mathrm{tw}(u) \times v$ one reduces further to showing that (A.5) is functorial in this sense which is clear. $\square$

Adjunction. Fix three categories I, J, K in **Dia**, and objects $A \in \mathbb{D}(I)_o, B \in \mathbb{D}(J)_o, C \in \mathbb{D}(K)_o$. Fix also the following notation:

$$\begin{tikzcd}
J & I \times J \arrow[l] \arrow[r] & I \\
\mathrm{tw}(J) \times K \arrow[u, "q"] \arrow[d, "r"] \arrow[dd, bend right, "p"'] & \mathrm{tw}(I \times J) \times K \arrow[l, "\beta"'] \arrow[r, "\alpha"] \arrow[u, "q''"] \arrow[d, "p''"] \arrow[dl, "r''"] & \mathrm{tw}(I) \times J^\circ \times K \arrow[u, "q'"'] \arrow[dl, "p'"] \arrow[ddll, bend left, "r'"] \\
K & I^\circ \times J^\circ \times K & \\
J^\circ \times K. & &
\end{tikzcd}$$

Then the morphism in the statement of the adjunction property is given by:

$$\begin{aligned}
\Omega^{I,J,K}_{A,B,C} : p'_*[q'^*A, r'^*p_*[q^*B, r^*C]] &\xrightarrow{\sim} p'_*[q'^*A, \alpha_*\beta^*[q^*B, r^*C]] \\
&\xrightarrow{\sim} p'_*\alpha_*[\alpha^*q'^*A, [\beta^*q^*B, \beta^*r^*C]] \\
&\xrightarrow{\sim} p'_*\alpha_*[\alpha^*q'^*A \otimes \beta^*q^*B, \beta^*r^*C] \\
&\xrightarrow{\sim} p''_*[q''^*(A|_{I\times J} \otimes B|_{I\times J}), r''^*C].
\end{aligned}$$

It is clear that this morphism is natural in the three arguments. Moreover, as above it is straightforward to check that it behaves well with respect to functors $u : I' \to I$, $v : J' \to J$, $w : K' \to K$.

Biduality. Fix $B \in \mathbb{D}(\star)_o$, $I \in \mathbf{Dia}_o$ and $A \in \mathbb{D}(I)_o$. We also fix the following notation:

$$\begin{tikzcd}
 & \star & \\
\mathrm{tw}(I^\circ) \arrow[ur, "\overline{r}"] \arrow[rr, "\mu"] \arrow[d, "\overline{q}"'] \arrow[drr, "\overline{p}"' near end] & & \mathrm{tw}(I) \arrow[ul, "r"'] \arrow[d, "q"] \arrow[dll, "p" near end] \\
I^\circ & & I.
\end{tikzcd}$$

Here, μ is the isomorphism of categories taking $j \to i$ in I° to $i \to j$ in I. We then define the morphism mentioned in the statement of the biduality property,

$$\Upsilon^I_A : A \to \langle\langle A, B\rangle, B\rangle, \tag{A.9}$$

by adjunction as follows:

$$\begin{aligned}
\overline{p}^*A \otimes \overline{q}^*p_*[q^*A, r^*B] &=\!=\!= \overline{p}^*A \otimes \mu^*p^*p_*[q^*A, r^*B] \\
&\xrightarrow{\text{adj}} \overline{p}^*A \otimes \mu^*[q^*A, r^*B] \\
&\longrightarrow \overline{p}^*A \otimes [\overline{p}^*A, \overline{r}^*B] \\
&\xrightarrow{\text{ev}} \overline{r}^*B.
\end{aligned}$$

This is clearly natural in A. If $u : I' \to I$ is a functor in **Dia** we define a morphism

$$u^*\langle\langle A, B\rangle, B\rangle \underset{\sim}{\xrightarrow{\Psi}} \langle u^{\circ *}\langle A, B\rangle, B\rangle \underset{\sim}{\xleftarrow{\Psi}} \langle\langle u^*A, B\rangle, B\rangle.$$

As we know by the naturality property, this morphism is invertible, natural in A, and behaves well with respect to identity and composition of functors as well as natural transformations in **Dia**. Therefore we have defined a pseudonatural transformation $\langle\langle -, B\rangle, B\rangle$. To check that (A.9) defines a modification of pseudonatural transformations as claimed in section 2 it suffices to prove the following lemma.

Lemma A.10 *With the notation above the following diagram commutes:*

$$\begin{tikzcd}
u^*A \arrow[r, "\Upsilon"] \arrow[d, "\Upsilon"'] & u^*\langle\langle A, B\rangle, B\rangle \arrow[d, "\Psi", "\sim"'] \\
\langle\langle u^*A, B\rangle, B\rangle \arrow[r, "\sim", "\Psi"'] & \langle u^{\circ *}\langle A, B\rangle, B\rangle.
\end{tikzcd}$$

Proof. Using adjunction, the square can be equivalently written as the outer rectangle of the following diagram:

$$\begin{tikzcd}
\overline{p}'^* u^* A \otimes \overline{q}'^* u^{\circ *} p_*[q^*A, r^*B] \arrow[r] \arrow[d] & \mathrm{tw}(u^\circ)^*(\overline{p}^* A \otimes \overline{q}^* p_*[q^*A, r^*B]) \arrow[d] \arrow[r, dash] & \cdots \\
\overline{p}'^* u^* A \otimes [\overline{p}'^* u^* A, \overline{r}'^* B] \arrow[d, "\mathrm{ev}"'] & \mathrm{tw}(u^\circ)^*(\overline{p}^* A \otimes [\overline{p}^* A, \overline{r}^* B]) \arrow[d, "\mathrm{ev}"'] & \\
\overline{r}'^* B \arrow[r, equal] & \mathrm{tw}(u^\circ)^* \overline{r}^* B \arrow[r, equal] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{tw}(u^\circ)^*(\overline{p}^*\langle\langle A, B\rangle, B\rangle \otimes \overline{q}^*\langle A, B\rangle) \arrow[d] & \overline{p}'^* u^*\langle\langle A, B\rangle, B\rangle \otimes \overline{q}'^* u^{\circ *}\langle A, B\rangle \arrow[l, "\sim"'] \arrow[d] \\
& \mathrm{tw}(u^\circ)^*([\overline{q}^*\langle A, B\rangle, \overline{r}^* B] \otimes \overline{q}^*\langle A, B\rangle) \arrow[d, "\mathrm{ev}"'] & [\overline{q}'^* u^{\circ *}\langle A, B\rangle, \overline{r}'^* B] \otimes \overline{q}'^* u^{\circ *}\langle A, B\rangle \arrow[d, "\mathrm{ev}"] \\
\cdots \arrow[r, equal] & \mathrm{tw}(u^\circ)^* \overline{r}^* B \arrow[r, equal] & \overline{r}'^* B.
\end{tikzcd}$$

All three parts are easily seen to commute. ☐

Normalization. Given $J \in \mathbf{Dia}_0$, $A \in \mathbb{D}(\star)_0$ and $B \in \mathbb{D}(J)_0$, the morphism $\Lambda^J_{A,B}$ is the canonical identification induced by the strict functoriality of $\mathbb{D}$:

$$[p_J^* A, B] \xrightarrow{\sim} 1_{J*}[p_J^* A, B] = \langle A, B\rangle.$$

Clearly, this is natural in A and B, and behaves well with respect to functors $v : J' \to J$. The last claim in section 2 about Λ explicitly amounts to the following:

- for $A, B \in \mathbb{D}(\star)_0$, Θ is the canonical composition

$$[A, B] \underset{\sim}{\to} 1_*[A, B] \underset{\sim}{\to} 1_* 1_*[A, B] \underset{\sim}{\xrightarrow{\Lambda}} 1_* 1_*\langle A, B\rangle$$

  where 1 is the unique endofunctor of the terminal category $\star$;

- for $A, C \in \mathbb{D}(\star)_0$, $B \in \mathbb{D}(I)_0$, $D \in \mathbb{D}(J)_0$, Ξ fits into the commutative diagram:

$$\begin{tikzcd}
\langle A,B\rangle \boxtimes \langle C,D\rangle \arrow[rr, "\Xi"] & & \langle A\boxtimes C, B\boxtimes D\rangle \\
{[p_I^*A,B]|_{I\times J} \otimes [p_J^*C,D]|_{I\times J}} \arrow[u, "\Lambda", "\sim"'] \arrow[d, "\sim"] & & {[p_{I\times J}^*(A\otimes C), B|_{I\times J}\otimes D|_{I\times J}]} \arrow[u, "\sim", "\Lambda"'] \arrow[d, "\sim"] \\
{[p_{I\times J}^*A, B|_{I\times J}] \otimes [p_{I\times J}^*C, D|_{I\times J}]} \arrow[rr, "(A.5)"] & & {[p_{I\times J}^*A\otimes p_{I\times J}^*C, B|_{I\times J}\otimes D|_{I\times J}].}
\end{tikzcd}$$

- for $A, B \in \mathbb{D}(\star)_0$ and $C \in \mathbb{D}(J)_0$, Ω fits into the commutative diagram:

$$\begin{tikzcd}
\langle A, \langle B,C\rangle\rangle \arrow[r, "\Omega", "\sim"'] & \langle A\otimes B, C\rangle \\
{[p_J^*A, [p_J^*B, C]]} \arrow[u, "\Lambda", "\sim"'] \arrow[dr, "\sim"'] & {[p_J^*(A\otimes B), C]} \arrow[u, "\sim", "\Lambda"'] \arrow[d, "\sim"] \\
 & {[p_J^*A\otimes p_J^*B, C].}
\end{tikzcd}$$

- for $A, B \in \mathbb{D}(\star)_0$, Υ is identified with the morphism $A \to [[A,B],B]$ which by adjunction corresponds to $\mathrm{ev}: A\otimes [A,B] \to B$.

All these statements follow easily from the constructions in this section.

B. The external trace and homotopy colimits

In this section the proof of Proposition 5.4 will be given. Throughout we fix a closed monoidal derivator $\mathbb{D}$ of type **Dia**. We start with a preliminary result, already needed to define the association Φ on page 31.

Lemma B.1 *Let $I \in \mathbf{Dia}_0$. Then the following three morphisms are invertible:*

(1) $p_{1!}q_{2!}q_2^*p_1^* \to 1$ *(counit of adjunction),*
(2) $1 \to p_{2*}q_{1*}q_1^*p_2^*$ *(unit of adjunction),*
(3) $\overline{\Psi} : [p_{I!}A, B] \to p_{I^\circ *}\langle A, B\rangle$ *for* $A \in \mathbb{D}(I)_0$, $B \in \mathbb{D}(\star)_0$.

Proof. For the first morphism, fix $i \in I_0$ and consider the following pullback square:

$$\begin{tikzcd}
\mathrm{tw}(I)_i^\circ \arrow[r] \arrow[d, "p_i"'] & \mathrm{tw}(I)^\circ \arrow[d, "p_1q_2"] \\
\star \arrow[r, "i"'] & I^\circ.
\end{tikzcd}$$

Since q_2 and p_1 are both fibrations so is their composition and by Fact 1.1 the Beck-Chevalley transformation corresponding to the square above is invertible. It follows that for the counit $p_{1!}q_{2!}q_2^*p_1^* \to 1$ to be invertible it is necessary and sufficient that $p_{i!}p_i^* \to 1$ is (for all $i \in I_0$, by (D2)). This is equivalent to $1 \to p_{i*}p_i^*$ being invertible, and this is true since $\mathrm{tw}(I)_i^\circ = I/i$ and thus $p_{i*} = 1_i^*$. The second morphism in the statement of the Lemma is treated in the same way.

For the last morphism, we consider the following factorization:

$$
\begin{tikzcd}
{[p_{I!}A, B]} \arrow[r, "\mathrm{adj}"] \arrow[d, "\sim"] & p_{I^\circ *} p_{I^\circ}^* [p_{I!}A, B] \arrow[r, "\Psi\circ\Lambda"] \arrow[d, "\sim"] & p_{I^\circ *}\langle p_I^* p_{I!}A, B\rangle \arrow[d, "\mathrm{adj}"] \\
p_{I*}[A, p_I^* B] \arrow[r, "\mathrm{adj}"] \arrow[d, "\Theta"', "\sim"] & p_{I^\circ *} p_{I^\circ}^* p_{I*}[A, p_I^* B] \arrow[d, "\Theta"', "\sim"] & p_{I^\circ *}\langle A, B\rangle \arrow[dd, "\mathrm{adj}"] \\
p_{I*}p_{2*}q_{2*}q_2^*\langle A, p_I^* B\rangle \arrow[r, "\mathrm{adj}"] & p_{I^\circ *} p_{I^\circ}^* p_{I*}p_{2*}q_{2*}q_2^*\langle A, p_I^* B\rangle & \\
p_{I^\circ *}p_{1*}q_{2*}q_2^*p_1^*\langle A, B\rangle \arrow[u, "\Psi", "\sim"'] & p_{I^\circ *} p_{I^\circ}^* p_{I^\circ *}p_{1*}q_{2*}q_2^*p_1^*\langle A, B\rangle \arrow[l, "\mathrm{adj}"'] \arrow[u, "\Psi", "\sim"'] & p_{I^\circ *}p_{1*}q_{2*}q_2^*p_1^*\langle A, B\rangle . \arrow[l, "\mathrm{adj}"']
\end{tikzcd}
$$

Notice that all the vertical arrows on the left are invertible (the first one by Lemma 1.5, the second and third by the results of section 2) as is the vertical arrow on the bottom right by part 1 of the lemma. And the composition of the horizontal arrows at the bottom is the identity so we only need to prove commutativity of the diagram.

This is clear for the left half of the diagram while the right half may be decomposed as follows:

$$
\begin{tikzcd}
p_{I^\circ}^*[p_{I!}A, B] \arrow[r, "\Lambda", "\sim"'] \arrow[d, "\mathrm{adj}"'] & p_{I^\circ}^*\langle p_{I!}A, B\rangle \arrow[r, dash, "\Psi"] \arrow[d, "\mathrm{adj}"] & \cdots \\
p_{I^\circ}^* p_{I*} p_I^*[p_{I!}A, B] \arrow[r, "\Lambda", "\sim"'] \arrow[d] & p_{I^\circ}^* p_{I*} p_I^*\langle p_{I!}A, B\rangle \arrow[r, dash, "\mathrm{adj}"] \arrow[d, phantom, "\textcircled{1}"] & \cdots \\
p_{I^\circ}^* p_{I*}[p_I^* p_{I!}A, p_I^* B] \arrow[r, "\Theta", "\sim"'] \arrow[d, "\mathrm{adj}"'] & p_{I^\circ}^* p_{I*}p_{2*}q_{2*}q_2^*\langle p_I^* p_{I!}A, p_I^* B\rangle \arrow[r, dash, "\sim"] \arrow[d, "\mathrm{adj}"'] & \cdots \\
p_{I^\circ}^* p_{I*}[A, p_I^* B] \arrow[r, "\Theta", "\sim"'] & p_{I^\circ}^* p_{I*}p_{2*}q_{2*}q_2^*\langle A, p_I^* B\rangle \arrow[r, dash, "\sim"] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdots \arrow[r] & \langle p_I^* p_{I!}A, B\rangle \arrow[r, "\mathrm{adj}"] & \langle A, B\rangle \arrow[d, "\mathrm{adj}"] \\
\cdots \arrow[r] & p_{I^\circ}^* p_{I*}p_{2*}q_{2*}q_2^*(p_I^\circ \times p_I)^*\langle p_{I!}A, B\rangle \arrow[d, "\sim"', "\Psi"] & p_{1*}q_{2*}q_2^*p_1^*\langle A, B\rangle \arrow[dd, "\Psi"] \\
\cdots \arrow[r] & p_{I^\circ}^* p_{I^\circ *}p_{1*}q_{2*}q_2^*\langle p_I^* p_{I!}A, p_I^* B\rangle \arrow[d, "\mathrm{adj}"] & \\
\cdots \arrow[r] & p_{I^\circ}^* p_{I^\circ *}p_{1*}q_{2*}q_2^*\langle A, p_I^* B\rangle \arrow[r, "\mathrm{adj}"] & p_{1*}q_{2*}q_2^*\langle A, p_I^* B\rangle .
\end{tikzcd}
$$

Everything except possibly ① clearly commutes; and ① does so by the internal hom property in section 2. □

From now on we take the assumptions of Proposition 5.4 to be satisfied. First we prove:

Lemma B.2 *$p_{I!}A$ is dualizable.*

Proof. We are given an object B in $\mathbb{D}(\star)$ and we need to show that the top arrow in the following diagram is invertible:

$$\begin{tikzcd}
{[p_{I!}A, \mathbb{1}] \otimes B} \arrow[r] \arrow[d, "\overline{\Psi}"', "\sim"] & {[p_{I!}A, \mathbb{1} \otimes B]} \arrow[d, "\sim"', "\overline{\Psi}"] \\
p_{I^\circ *}\langle A, \mathbb{1}\rangle \otimes B \arrow[d, "\sim"] & p_{I^\circ *}\langle A, \mathbb{1} \otimes B\rangle \arrow[d, equal] \\
p_{I^\circ *}(\langle A, \mathbb{1}\rangle \boxtimes B) \arrow[r, "\sim", "\Xi"'] & p_{I^\circ *}\langle A, \mathbb{1} \boxtimes B\rangle.
\end{tikzcd}$$

The two arrows labeled $\overline{\Psi}$ are invertible by the previous lemma, as is the vertical arrow on the bottom left by hypothesis (H4). Given $i \in I_o$, the fiber over i of the morphism $\Xi : \langle A, \mathbb{1}\rangle \boxtimes B \to \langle A, \mathbb{1} \boxtimes B\rangle$ corresponds to the morphism $[i^*A, \mathbb{1}] \otimes B \to [i^*A, \mathbb{1} \otimes B]$ by the external product and normalization properties in section 2. The latter morphism is invertible since A is fiberwise dualizable hence also the bottom horizontal arrow in the diagram is invertible (by (D2)). It now suffices to prove its commutativity which we leave as an easy exercise. □

To prove commutativity of the diagram (5.2) with $g = \mathrm{Tr}(f)$ and the top horizontal arrow replaced by $\mathrm{Tr}(p_{I!}f)$ we decompose $\mathrm{Tr}(f)$ into coevaluation, the morphism induced by f and evaluation, and similarly for $\mathrm{Tr}(p_{I!}f)$. Schematically:

$$\begin{tikzcd}
S \arrow[r, "\mathrm{coev}"] \arrow[d] & (p_{I!}A)^* \otimes p_{I!}A \otimes S \arrow[r, "p_{I!}f"] \arrow[d] & \cdots \\
p_{I^\circ *}p_{1!}(q_{2!}\mathbb{1} \otimes S|_{I^\circ \times I}) \arrow[r, "\mathrm{coev}"'] & p_{I^\circ *}p_{1!}(A^\vee \boxtimes A \otimes S|_{I^\circ \times I}) \arrow[r, "f"'] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & (p_{I!}A)^* \otimes p_{I!}A \otimes T \arrow[r, "\mathrm{ev}"] & T \\
\cdots \arrow[r] & p_{I^\circ *}p_{1!}(A^\vee \boxtimes A \otimes T|_{I^\circ \times I}) \arrow[r, "\sim"', "\mathrm{oev}"'] \arrow[u] & p_{I!}p_{2*}(q_{1*}\mathbb{1} \otimes T|_{I^\circ \times I}). \arrow[u]
\end{tikzcd} \tag{B.3}$$

The vertical morphisms in the middle will be described below but we can already say here that they will be easily seen to make the square in the middle commute. Now the fact that we have isomorphisms

$$p_{I^\circ *}(- \otimes p_{I^\circ}^* -) \cong p_{I^\circ *} - \otimes -, \qquad p_{1!}(- \otimes p_1^* -) \cong p_{1!} - \otimes -$$

allows us to neglect the twisting:

Lemma B.4 *We may assume $S = T = \mathbb{1}$.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
\mathbb{1} \otimes S \arrow[r, "\mathrm{coev}"] \arrow[d] & (p_{I!}A)^* \otimes p_{I!}A \otimes S \arrow[d] \\
p_{I^\circ *}p_{1!}q_{2!}\mathbb{1} \otimes S \arrow[r, "\mathrm{coev}"] \arrow[d, "\sim"'] & p_{I^\circ *}p_{1!}(A^\vee \boxtimes A) \otimes S \arrow[d, "\sim"] \\
p_{I^\circ *}p_{1!}(q_{2!}\mathbb{1} \otimes p_2^* p_I^* S) \arrow[r, "\mathrm{coev}"] & p_{I^\circ *}p_{1!}(A^\vee \boxtimes A \otimes p_2^* p_I^* S).
\end{tikzcd}$$

It is easy to check that the composition of the two vertical morphisms on the left equals the left vertical morphism in (B.3). Moreover the bottom square clearly commutes thus we are left to prove the commutativity of the top square but this does not depend on S. A similar argument shows that we may assume $T = \mathbb{1}$. □

Lemma B.5 *The left square in (B.3) commutes.*

Proof. By the previous lemma we may assume $S = \mathbb{1}$. Again, we factor the coevaluation morphisms on the top and bottom into two parts as in (3.2) and (3.9) respectively. This decomposes the left square in (B.3) into two parts which we consider separately.

By adjunction, the first one may be expanded as follows (the arrows labeled with a small Greek letter will be defined below):

$$
\begin{tikzcd}
p_{I^\circ}^*\mathbb{1} \arrow[r, "\mathrm{adj}"] & p_{I^\circ}^*[p_{I!}A, p_{I!}A] \arrow[r, dash, "\alpha"] & \cdots \\
(p_1q_2)_!(p_1q_2)^*\mathbb{1} \arrow[u, "\mathrm{adj}", "\sim"'] \arrow[r, "\mathrm{adj}"] \arrow[d, "\sim"'] & (p_1q_2)_!(p_1q_2)^*p_{I^\circ}^*[p_{I!}A, p_{I!}A] \arrow[u, "\sim", "\mathrm{adj}"'] \arrow[r, dash, "\alpha"] \arrow[d, equal] \arrow[dr, phantom, "①"] & \cdots \\
(p_1q_2)_!(p_2q_2)^*\mathbb{1} \arrow[r, "\mathrm{adj}"] \arrow[d, equal] & (p_1q_2)_!(p_2q_2)^*p_I^*[p_{I!}A, p_{I!}A] \arrow[r, dash, "\beta"] & \cdots \\
(p_1q_2)_!(p_2q_2)^*\mathbb{1} \arrow[rr, dash, "\Theta\circ\mathrm{adj}"] & & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdots \arrow[r] & \langle p_I^*p_{I!}A, p_{I!}A\rangle \arrow[r, "\mathrm{adj}"] & \langle A, p_{I!}A\rangle \\
\cdots \arrow[r] & (p_1q_2)_!(p_1q_2)^*\langle p_I^*p_{I!}A, p_{I!}A\rangle \arrow[u, "\sim", "\mathrm{adj}"'] & \\
\cdots \arrow[r] & (p_1q_2)_!(p_2q_2)^*(p_2q_2)_*q_2^*\langle p_I^*p_{I!}A, p_I^*p_{I!}A\rangle \arrow[u, "\gamma"'] \arrow[d, "\mathrm{adj}"'] & p_{1!}p_1^*\langle A, p_{I!}A\rangle \arrow[uu, "\mathrm{adj}"'] \arrow[d, "\sim", "\Psi"'] \\
& (p_1q_2)_!(p_2q_2)^*(p_2q_2)_*q_2^*\langle A, p_I^*p_{I!}A\rangle \arrow[r, "\mathrm{adj}"] & p_{1!}\langle A, p_I^*p_{I!}A\rangle \\
\cdots \arrow[r] & (p_1q_2)_!(p_2q_2)^*(p_2q_2)_*q_2^*\langle A, A\rangle \arrow[u, "\mathrm{adj}"'] \arrow[r, "\mathrm{adj}"'] & p_{1!}\langle A, A\rangle, \arrow[u, "\mathrm{adj}"']
\end{tikzcd}
$$

and the second one as follows:

$$
\begin{tikzcd}
p_{I^\circ}^*[p_{I!}A, p_{I!}A] \arrow[d, "\mathrm{adj}\circ\alpha"'] & p_{I^\circ}^*([p_{I!}A, \mathbb{1}] \otimes p_{I!}A) \arrow[l, "\sim"'] \arrow[d, "\delta"] \arrow[dl, phantom, "②"] \\
\langle A, p_{I!}A\rangle & \langle A, \mathbb{1}\rangle \otimes p_{I^\circ}^*p_{I!}A \arrow[l, "\sim"', "(3.7)"] \arrow[dl, phantom, "③"] \\
p_{1!}\langle A, A\rangle \arrow[u, "\overline{\Psi}", "\sim"'] & p_{1!}(p_1^*\langle A, \mathbb{1}\rangle \otimes p_2^*A). \arrow[l, "\sim"', "(3.7)"] \arrow[u, "\sim"']
\end{tikzcd}
$$

Notice first that these two diagrams indeed “glue” together. Thus it suffices to show commutativity of the rectangles marked with a number (the other ones are easily seen to commute).

(1) may be expanded as follows (set $B = p_{I!}A$):

$$\begin{tikzcd}
q_2^* p_1^* p_{I^\circ}^*[B,B] \arrow[r, "\Lambda", "\sim"'] \arrow[d, equal] & q_2^* p_1^* p_{I^\circ}^*\langle B,B\rangle \arrow[r, "\Psi", "\sim"'] & q_2^* p_1^*\langle p_I^* B, B\rangle \arrow[d, "\Psi", "\sim"'] \\
q_2^* p_2^* p_I^*[B,B] \arrow[r, "\Lambda", "\sim"'] \arrow[d, equal] & q_2^* p_2^* p_I^*\langle B,B\rangle \arrow[r, "\Psi", "\sim"'] & q_2^*\langle p_I^* B, p_I^* B\rangle \\
q_2^* p_2^* p_I^*[B,B] \arrow[r, "\sim"'] & q_2^* p_2^*[p_I^* B, p_I^* B] \arrow[r, "\Theta", "\sim"'] & q_2^* p_2^* p_{2*} q_{2*} q_2^*\langle p_I^* B, p_I^* B\rangle . \arrow[u, "\mathrm{adj}"']
\end{tikzcd}$$

The top rectangle commutes by the naturality property, the bottom rectangle by the internal hom property of section 2.

For (2) consider the following decomposition (by adjunction again):

$$\begin{tikzcd}
{[p_{I!}A, p_{I!}A]} \arrow[d, "\Lambda\circ\mathrm{adj}"'] & {[p_{I!}A, \mathbb{1}] \otimes p_{I!}A} \arrow[l, "\sim"'] \arrow[r, equal] \arrow[d] & {[p_{I!}A, \mathbb{1}] \otimes p_{I!}A} \arrow[d, "\Lambda\circ\mathrm{adj}"] \\
p_{I^\circ *} p_{I^\circ}^*\langle p_{I!}A, p_{I!}A\rangle \arrow[dd, "\Psi"', "\sim"] & p_{I^\circ *} p_{I^\circ}^*(\langle p_{I!}A, \mathbb{1}\rangle \otimes p_{I!}A) \arrow[l, "\sim", "(3.7)"'] \arrow[dr, "\sim"'] & p_{I^\circ *} p_{I^\circ}^*\langle p_{I!}A, \mathbb{1}\rangle \otimes p_{I!}A \arrow[d, "\sim"] \\
 & & p_{I^\circ *}(p_{I^\circ}^*\langle p_{I!}A, \mathbb{1}\rangle \otimes p_{I^\circ}^* p_{I!}A) \arrow[d, "\Psi", "\sim"'] \\
p_{I^\circ *}\langle p_I^* p_{I!}A, p_{I!}A\rangle \arrow[d, "\mathrm{adj}"'] & & p_{I^\circ *}(\langle p_I^* p_{I!}A, \mathbb{1}\rangle \otimes p_{I^\circ}^* p_{I!}A) \arrow[ll, "\sim", "(3.7)"'] \arrow[d, "\mathrm{adj}"] \\
p_{I^\circ *}\langle A, p_{I!}A\rangle & & p_{I^\circ *}(\langle A, \mathbb{1}\rangle \otimes p_{I^\circ}^* p_{I!}A) . \arrow[ll, "\sim", "(3.7)"']
\end{tikzcd}$$

The top left square commutes by the normalization property, the pentagon in the middle by the external product and normalization properties of section 2. The rest is clearly commutative. (One also needs here Lemma B.1 to ensure that the morphism corresponding to δ under adjunction is invertible.)

Next, we may decompose (3) by adjunction as follows:

$$\begin{tikzcd}
p_1^*\langle A, p_{I!}A\rangle \arrow[d, "\Psi"', "\sim"] & p_1^*(\langle A, \mathbb{1}\rangle \boxtimes p_{I!}A) \arrow[l, "(3.7)"', "\sim"] \arrow[d, "\sim"] \\
\langle A, p_I^* p_{I!}A\rangle & \langle A, \mathbb{1}\rangle \boxtimes p_I^* p_{I!}A \arrow[l, "(3.7)"', "\sim"] \\
\langle A, A\rangle \arrow[u, "\mathrm{adj}"] & \langle A, \mathbb{1}\rangle \boxtimes A . \arrow[l, "(3.7)"', "\sim"] \arrow[u, "\mathrm{adj}"']
\end{tikzcd}$$

Both squares commute by the external product property in section 2. ☐

The following lemma completes the proof of Proposition 5.4.

Lemma B.6 *The right square in (B.3) commutes.*

Proof. Again, we may assume $T = \mathbb{1}$ by the lemma above. First, (3.3) lets us replace the evaluation morphism on the top by the following composition (the arrows labeled with a small Greek letter will be defined below):

$$\begin{tikzcd}
(p_{I!}A)^* \otimes p_{I!}A \arrow[r, "\sim"'] \arrow[d, "\Lambda"', "\sim"] \arrow[dr, phantom, "4"] & (p_{I!}A)^* \otimes (p_{I!}A)^{**} \arrow[r] \arrow[d, "\overline{\Psi}\circ\Lambda"', "\sim"] \arrow[dr, phantom, "6"] & ((p_{I!}A)^* \otimes p_{I!}A)^* \arrow[r, "\mathrm{coev}"] \arrow[dr, phantom, "7"] & \mathbb{1} \\
p_{I^\circ *}A^\vee \otimes p_{I!}A \arrow[r, "\Upsilon", "\sim"'] \arrow[dr, phantom, "5"] & p_{I!}A^{\vee\vee} \otimes p_{I^\circ *}A^\vee & p_{I!}(A^\vee \boxtimes p_{I!}A)^\vee \arrow[u, "\theta"', "\sim"] \arrow[d, "\overline{\Psi}"', "\sim"] & \\
p_{I^\circ *}p_{1!}(A^\vee \boxtimes A) \arrow[r, "\sim"'] \arrow[u, "\varepsilon"', "\sim"] & p_{I!}p_{2*}\mu_*(A^{\vee\vee} \boxtimes A^\vee) \arrow[r, "\Xi"'] \arrow[u, "\eta"', "\sim"] & p_{I!}p_{2*}\mu_*(A^\vee \boxtimes A)^\vee \arrow[r, "\mathrm{coev}"'] & p_{I!}p_{2*}\mu_*\langle q_{2!}\mathbb{1}, \mathbb{1}\rangle. \arrow[uu]
\end{tikzcd}$$

The commutativity of 4 can be checked on each tensor factor separately; only one of them is possibly non-obvious:

$$\begin{tikzcd}
A \arrow[r, "\mathrm{adj}"] \arrow[d, "\Upsilon"', "\sim"] & p_I^* p_{I!}A \arrow[r, "\Upsilon", "\sim"'] \arrow[d, "\Upsilon"', "\sim"] & p_I^*\langle\langle p_{I!}A, \mathbb{1}\rangle, \mathbb{1}\rangle \arrow[d, "\sim"', "\Psi"] \\
\langle\langle A, \mathbb{1}\rangle, \mathbb{1}\rangle \arrow[r, "\mathrm{adj}"] \arrow[d, "\mathrm{adj}"'] & \langle\langle p_I^* p_{I!}A, \mathbb{1}\rangle, \mathbb{1}\rangle \arrow[r, "\Psi", "\sim"'] & \langle p_{I^\circ}^*\langle p_{I!}A, \mathbb{1}\rangle, \mathbb{1}\rangle \\
\langle p_{I^\circ}^* p_{I^\circ *}\langle A, \mathbb{1}\rangle, \mathbb{1}\rangle \arrow[urr, "\overline{\Psi}"] & & p_I^*\langle p_{I^\circ *}\langle A, \mathbb{1}\rangle, \mathbb{1}\rangle. \arrow[ll, "\sim", "\Psi"'] \arrow[uu, bend right=60, "\overline{\Psi}"']
\end{tikzcd}$$

The two squares in the top row commute by the biduality property of section 2 while the rest is clearly commutative.

5 may be decomposed as follows:

$$\begin{tikzcd}
p_{I^\circ *}A^\vee \otimes p_{I!}A \arrow[r, equal] \arrow[d] & p_{I^\circ *}A^\vee \otimes p_{I!}A \arrow[r] & p_{I!}A \otimes p_{I^\circ *}A^\vee \arrow[r, "\Upsilon"] & p_{I!}A^{\vee\vee} \otimes p_{I^\circ *}A^\vee \\
p_{I^\circ *}(A^\vee \otimes p_{I^\circ}^* p_{I!}A) & p_{I!}(p_I^* p_{I^\circ *}A^\vee \otimes A) \arrow[r] \arrow[u] \arrow[d] & p_{I!}(A \otimes p_I^* p_{I^\circ *}A^\vee) \arrow[r, "\Upsilon"] \arrow[u] \arrow[d] & p_{I!}(A^{\vee\vee} \otimes p_I^* p_{I^\circ *}A^\vee) \arrow[u] \arrow[d] \\
p_{I^\circ *}(A^\vee \otimes p_{1!}p_2^* A) \arrow[u] & p_{I!}(p_{2*}p_1^* A^\vee \otimes A) \arrow[r] \arrow[d] & p_{I!}(A \otimes p'_{1*}p'^*_2 A^\vee) \arrow[r, "\Upsilon"] \arrow[d] & p_{I!}(A^{\vee\vee} \otimes p'_{1*}p'^*_2 A^\vee) \arrow[d] \\
p_{I^\circ *}p_{1!}(A^\vee \boxtimes A) \arrow[u] & p_{I!}p_{2*}(A^\vee \boxtimes A) \arrow[l] \arrow[r] & p_{I!}p_{2*}\mu_*(A \boxtimes A^\vee) \arrow[r, "\Upsilon"] & p_{I!}p_{2*}\mu_*(A^{\vee\vee} \boxtimes A^\vee).
\end{tikzcd}$$

Here, p'_1 and p'_2 are the projections onto the factors of $I \times I^\circ$ and all arrows are invertible. All rectangles of this diagram are easily seen to commute (for the leftmost one may use [2, 2.1.105]).

Next we turn to 6. In the decomposition of it (use the normalization property of section 2 for the top horizontal arrow),

$$
\begin{tikzcd}
(p_{I!}A)^{\vee\vee} \boxtimes (p_{I!}A)^{\vee} \arrow[r, "\Xi"] & ((p_{I!}A)^{\vee} \boxtimes p_{I!}A)^{\vee} \\
(p_{I^\circ *}A^{\vee})^{\vee} \boxtimes (p_{I!}A)^{\vee} \arrow[r, "\Xi"] \arrow[u, "\overline{\Psi}"] & (p_{I^\circ *}A^{\vee} \boxtimes p_{I!}A)^{\vee} \arrow[u, "\overline{\Psi}"'] \\
p_{I!}(A^{\vee\vee}) \boxtimes (p_{I!}A)^{\vee} \arrow[u, "\overline{\Psi}"] & (p_{I^\circ *}(A^{\vee} \boxtimes p_{I!}A))^{\vee} \arrow[u] \\
p_{I!}(A^{\vee\vee} \boxtimes (p_{I!}A)^{\vee}) \arrow[r, "\Xi"] \arrow[u] \arrow[d] & p_{I!}(A^{\vee} \boxtimes p_{I!}A)^{\vee} \arrow[u, "\overline{\Psi}"'] \arrow[d] \\
p_{I!}(A^{\vee\vee} \boxtimes p_{I^\circ *}A^{\vee}) \arrow[d] & p_{I!}(A^{\vee} \otimes p_{1!}p_2^{*}A)^{\vee} \arrow[d] \\
p_{I!}(A^{\vee\vee} \otimes p'_{1*}p'^{*}_{2}A^{\vee}) \arrow[d] & p_{I!}(p_{1!}(A^{\vee} \boxtimes A))^{\vee} \arrow[d, "\overline{\Psi}"] \\
p_{I!}p_{2*}\mu_{*}(A^{\vee\vee} \boxtimes A^{\vee}) \arrow[r, "\Xi"] & p_{I!}p_{2*}\mu_{*}(A^{\vee} \boxtimes A)^{\vee},
\end{tikzcd}
$$

everything commutes by the external product property of section 2 (and adjunction). All vertical arrows are invertible.

It remains to prove the commutativity of 7. In the diagram

$$
\begin{tikzcd}
\langle (p_{I!}A)^{*} \otimes p_{I!}A, \mathbb{1} \rangle \arrow[r, "\mathrm{coev}"] & \langle \mathbb{1}, \mathbb{1} \rangle \arrow[r] & \mathbb{1} \\
\langle p_{I^\circ *}p_{1!}(A^{\vee} \boxtimes A), \mathbb{1} \rangle \arrow[r, "\mathrm{coev}"] \arrow[u] \arrow[d, "\overline{\Psi}"'] & \langle p_{I^\circ *}p_{1!}q_{2!}\mathbb{1}, \mathbb{1} \rangle \arrow[u] \arrow[d, "\overline{\Psi}"'] & p_{I!}p_{2*}q_{1*}q_1^{*}p_2^{*}p_I^{*}\langle \mathbb{1}, \mathbb{1} \rangle \arrow[ul] \arrow[d, "\Psi", "\sim"'] \\
p_{I!}p_{2*}\mu_{*}\langle A^{\vee} \boxtimes A, \mathbb{1} \rangle \arrow[r, "\mathrm{coev}"] & p_{I!}p_{2*}\mu_{*}\langle q_{2!}\mathbb{1}, \mathbb{1} \rangle \arrow[r, "\overline{\Psi}"] & p_{I!}p_{2*}q_{1*}\langle \mathbb{1}, \mathbb{1} \rangle,
\end{tikzcd}
$$

the top left square is simply $\langle -, \mathbb{1} \rangle$ applied to the left square in (B.3). It follows that this square is commutative. Moreover it is easy to see that the composition of the left vertical arrows is the same as of the ones in 7. Thus this diagram is a decomposition of 7. The rest of the diagram clearly commutes. □

C. $\mathbb{D}(G)$ for a finite group G

The question, given a category I, whether I-diagrams and morphisms of such in the homotopy categories can be lifted (and if so whether uniquely) to the homotopy categories of I-diagrams has always been of interest (see e. g. [28, chapitre IV] or [32, p. 2]). The goal of this last section is to give a proof for the (well-known) answer in the case of I a finite group.

Proposition C.1 *Let $\mathbb{D}$ be an additive derivator of type* **Dia***, let G be a finite group in* **Dia** *and assume that $\#G$ is invertible in $R_{\mathbb{D}}$. Then the canonical functor*

$$\mathrm{dia}_G : \mathbb{D}(G) \to \mathbf{CAT}(G^\circ, \mathbb{D}(\star))$$

is fully faithful. If, in addition, $\mathbb{D}(G)$ is pseudo-abelian then the functor is an equivalence of categories.

Remark C.2 Suppose that $\mathbb{D}$ is stable and that **Dia** contains countable discrete categories. In this case $\mathbb{D}(G)$ has countable direct sums, and it follows from [58, 1.6.8] that $\mathbb{D}(G)$ is pseudo-abelian.

Proof of Proposition C.1. We need to understand the two adjunctions $e_! \dashv e^*$ and $e^* \dashv e_*$ where $e : \star \to G$ is the unique functor.

Consider the following comma square where η on the component corresponding to $g \in G$ is g:

$$\begin{tikzcd} \coprod_G \star \arrow[r, "p"] \arrow[d, "p"'] \arrow[dr, phantom, "\nearrow\eta"] & \star \arrow[d, "e"] \\ \star \arrow[r, "e"'] & G, \end{tikzcd}$$

By (D4), the two compositions

$$p_! p^* \xrightarrow{\mathrm{adj}} p_! p^* e^* e_! \xrightarrow{\eta^*} p_! p^* e^* e_! \xrightarrow{\mathrm{adj}} e^* e_!,$$

$$p_* p^* \xleftarrow[\mathrm{adj}]{} p_* p^* e^* e_* \xleftarrow[\eta^*]{} p_* p^* e^* e_* \xleftarrow[\mathrm{adj}]{} e^* e_*$$

are invertible, yielding identifications

$$e^* e_! \cong \textstyle\coprod_G, \qquad\qquad e^* e_* \cong \textstyle\prod_G,$$

and therefore a canonical morphism $e^* e_! \to e^* e_*$ which is invertible if G is finite.

Under these identifications the (contravariant) action of G on $e^* e_!$ (obtained by applying dia_G to $e_!$) is given by right translation, and on $e^* e_*$ by left translation. Indeed, let $A \in \mathbb{D}(\star)_0$ be an arbitrary object and set $B = e^* e_! A$, fix also $g \in G$. Then the following diagram commutes where $r_g((x_h)_h) = (x_h)_{hg}$:

$$\begin{tikzcd} \coprod_{h\in G} A \arrow[r, "\mathrm{adj}"] \arrow[d, "r_g"'] & \coprod_{h\in G} B \arrow[r, "\coprod_h h^*"] \arrow[d, "r_g"'] & \coprod_{h\in G} B \arrow[r, "\Sigma"] & B \arrow[d, "g^*"] \\ \coprod_{h\in G} A \arrow[r, "\mathrm{adj}"'] & \coprod_{h\in G} B \arrow[r, "\coprod_h h^*"'] & \coprod_{h\in G} B \arrow[r, "\Sigma"'] & B. \end{tikzcd}$$

Thus the claim in the case of $e^* e_!$; the case of $e^* e_*$ is proved in a similar way.

Next, we would like to describe the units and counits of the adjunctions. We first deal with the unit of $e_! \dashv e^*$. Let $i : \star \to \coprod_G \star$ be the inclusion of the component corresponding to 1_G.

$$\begin{tikzcd}
p_! p^* \arrow[r, "\mathrm{adj}"] & p_! p^* e^* e_! \arrow[r, "\eta^*"] & p_! p^* e^* e_! \arrow[dd, "\mathrm{adj}"] \\
p_! i_! i^* p^* \arrow[u, "\mathrm{adj}"] \arrow[r, "\mathrm{adj}"] \arrow[d, "\sim"'] & p_! i_! i^* p^* e^* e_! \arrow[u, "\mathrm{adj}"'] \arrow[ur, "\mathrm{adj}"] \arrow[dr, "\sim"] & \\
1 \arrow[rr, "\mathrm{adj}"'] & & e^* e_! .
\end{tikzcd}$$

The diagram clearly commutes and hence the unit $1 \to e^* e_!$ is given by the inclusion of the unit component into $\coprod_G$. Similarly, the counit $e^* e_* \to 1$ is the projection onto the component corresponding to 1_G.

Next, we want to describe the other two (co)units (at least after applying e^*). For this consider the composition of the unit and the counit of the adjunction,

$$e^* \to \textstyle\coprod_G e^* \to e^*,$$

which we know to be the identity. By the description of the first morphism above we see that the 1_G-component of the second morphism has to be the identity. But this second morphism is also G-equivariant so the description of the G-action above implies that the morphism is the action of g on the g-component for any $g \in G$. Similarly, the counit $e^* \to \prod_G e^*$ is given by the action of g on the g-component.

We now have enough information to describe the composition

$$\xi : e_! \xrightarrow{\mathrm{adj}} e_* e^* e_! \to e_* e^* e_* \xrightarrow{\mathrm{adj}} e_*$$

after applying e^*. Indeed, it can then be identified with the following one:

$$\begin{array}{ccccccc}
\coprod_G & \longrightarrow & \prod_G \coprod_G & \longrightarrow & \prod_G \prod_G & \longrightarrow & \prod_G \\
(x_h)_h & \longmapsto & ((x_{hg^{-1}})_h)_g & \longmapsto & ((x_{hg^{-1}})_h)_g & \longmapsto & (x_{g^{-1}})_g .
\end{array}$$

Since this morphism is invertible and e^* conservative (by (D2)), also ξ is invertible, and it thus makes sense to consider the composition

$$1 \to e_* e^* \xrightarrow{\xi^{-1}} e_! e^* \to 1. \tag{C.3}$$

After applying e^* it can be identified with

$$\begin{array}{ccccccc}
e^* & \longrightarrow & \prod_G e^* & \longrightarrow & \coprod_G e^* & \longrightarrow & e^* \\
x & \longmapsto & (g^* x)_g & \longmapsto & ((g^{-1})^* x)_g & \longmapsto & \sum_{g \in G} g^* (g^{-1})^* x = \#G \cdot x .
\end{array}$$

If $\#G$ is invertible in $R_{\mathbb{D}}$ then this morphism and (again, by (D2)) also (C.3) is invertible, in particular there is, for every $B \in \mathbb{D}(G)_0$, a factorization of the identity morphism of B: $B \to e_* e^* B \to B$. For any $A \in \mathbb{D}(G)_0$, this factorization in turn induces the horizontal arrows in the following commutative diagram ($\mathcal{C} = \mathbf{CAT}(G^\circ, \mathbb{D}(\star))$, $d = \mathrm{dia}_G$):

$$\begin{tikzcd}
\mathbb{D}(G)(A, B) \arrow[r] \arrow[d, "d"] & \mathbb{D}(G)(A, e_* e^* B) \arrow[r] \arrow[d, "d"] & \mathbb{D}(G)(A, B) \arrow[d, "d"] \\
\mathcal{C}(dA, dB) \arrow[r] & \mathcal{C}(dA, de_* e^* B) \arrow[r] & \mathcal{C}(dA, dB).
\end{tikzcd}$$

The first top horizontal arrow is injective hence if the middle vertical arrow is injective then so is the left vertical one. Similarly, the second bottom horizontal arrow is surjective hence if the middle vertical arrow is surjective then so is the right vertical one. Consequently, to prove fully faithfulness of dia_G it suffices to prove bijective the middle vertical arrow (for all A and B). Now, the source of this map can be identified with $\mathbb{D}(\star)(e^*A, e^*B)$ by adjunction, while the target is the set of G°-morphisms in $\mathbb{D}(\star)$ from e^*A to the left regular representation associated to e^*B — which is also $\mathbb{D}(\star)(e^*A, e^*B)$.

It remains to show essential surjectivity of dia_G. Given an object $A \in \mathbb{D}(\star)_o$ with a G°-action ρ, consider the two morphisms

$$\begin{array}{ccc} A \xrightarrow{\ \alpha\ } \prod_G A & \text{and} & \prod_G A \xrightarrow{\qquad \beta \qquad} A \\ x \longmapsto (\rho(g)x)_g & & (x_g)_g \longmapsto \frac{1}{\#G}\sum_{g\in G}\rho(g^{-1})x_g. \end{array}$$

They give rise to a G°-equivariant decomposition of the identity on A:

$$1_A : A \xrightarrow{\alpha} \mathrm{dia}_G(e_*A) \xrightarrow{\beta} A.$$

By fullness of dia_G proved above, there exists $p \in \mathbb{D}(G)(e_*A, e_*A)$ with $\mathrm{dia}_G(p) = \alpha\beta$. By faithfulness also proved above, the equality

$$\mathrm{dia}_G(p^2) = \mathrm{dia}_G(p)^2 = (\alpha\beta)^2 = \alpha\beta = \mathrm{dia}_G(p)$$

implies that p is a projector, and therefore if $\mathbb{D}(G)$ is pseudo-abelian then there is a decomposition

$$e_*A = \ker(p) \oplus \mathrm{Im}(p).$$

Let $\alpha' : \mathrm{Im}(p) \to e_*A$ be the inclusion, and $\beta' : e_*A \to \mathrm{Im}(p)$ the projection. Then

$$\begin{aligned}(\mathrm{dia}_G(\beta')\alpha)(\beta\mathrm{dia}_G(\alpha')) &= \mathrm{dia}_G(\beta')\mathrm{dia}_G(p)\mathrm{dia}_G(\alpha') \\ &= \mathrm{dia}_G(\beta' p \alpha') \\ &= \mathrm{dia}_G(1_{\mathrm{Im}(p)}) \\ &= 1_{\mathrm{dia}_G(\mathrm{Im}(p))},\end{aligned}$$

and

$$\begin{aligned}(\beta\mathrm{dia}_G(\alpha'))(\mathrm{dia}_G(\beta')\alpha) &= \beta\mathrm{dia}_G(\alpha'\beta')\alpha \\ &= \beta\mathrm{dia}_G(p)\alpha \\ &= \beta\alpha\beta\alpha \\ &= 1_A.\end{aligned}$$

We conclude that $A \cong \mathrm{dia}_G(\mathrm{Im}(p))$. □

III

HOMOTOPY THEORY OF DG SHEAVES

Let $(\mathcal{C}, \tau)$ be a small Grothendieck site. Our goal in this chapter is to describe in detail one specific homotopy theoretic model for the unbounded derived category of τ-sheaves on $\mathcal{C}$. This description will be used in the following chapter in our study of motives. Although the model is well-known, there were several facts we needed in our application but were not able to find in the literature, which is why we decided to include the present chapter. To be useful in other contexts as well, we place ourselves in a general setting, in particular we try to make as few assumptions as possible regarding the site $(\mathcal{C}, \tau)$.

Let us quickly give the definition of the model. Start with the category of presheaves of unbounded complexes on $\mathcal{C}$ and declare weak equivalences and fibrations to be objectwise quasi-isomorphisms and epimorphisms, respectively. This yields the *projective model structure*. The *τ-local model structure* arises from it by a left Bousfield localization with respect to τ-local weak equivalences, i. e. morphisms inducing isomorphisms on all homology τ-sheaves. The resulting model category is our model for the derived category of τ-sheaves.

In §2 we recall the basic properties of the model category and describe the cofibrations. As an application we construct in §3 an explicit cofibrant replacement functor which resolves any presheaf of complexes by representables. The main theorem of §4 states that the τ-fibrant objects are precisely those presheaves satisfying descent with respect to τ-hypercovers. The analogous statement for simplicial presheaves is well-known, and our strategy is to reduce to this case via the Dold-Kan correspondence. We use the same strategy to prove a generalization of the Verdier hypercover theorem, expressing the hypercohomology of a complex of sheaves in terms of hypercovers only. We also describe some modifications to our model and deduce some useful consequences from the main theorem. In the final section 5 we prove that the Godement resolution defines a fibrant replacement functor for our model.

We would like to remark that the model described in this chapter is not quite arbitrary but has a very satisfying universal property. To describe it, recall the easy fact from category theory that for a small category $\mathcal{C}$, the category of presheaves on $\mathcal{C}$ is its *universal (or free) cocompletion*. This means that any functor from $\mathcal{C}$ into a cocomplete category factors via a cocontinuous functor through the Yoneda embedding $\mathcal{C} \to \mathbf{PSh}(\mathcal{C})$ in an essentially unique way. This basic idea finds repercussions in the following two results:

- For a small dg category $\mathcal{D}$, the category of dg modules $[\mathcal{D}^{\mathrm{op}}, \mathbf{Cpl}]$ is its *universal dg cocompletion*.

- In [19], Dugger proves that any functor from $\mathcal{C}$ into a model category factors via a left Quillen functor through the category of simplicial presheaves on $\mathcal{C}$ with the projective model structure, in an essentially unique way. In other words, this is the *universal model category* associated to $\mathcal{C}$.

Combining these two examples we naturally arrive at the following guess: $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cpl}]$ with the projective model structure is the *universal model dg category* associated to the small category $\mathcal{C}$. We couldn't resist prepending a section (§1) in order to explain this result (in fact, a more general version where chain complexes are replaced by quite arbitrary enriching categories).

Such a statement invites us to conceive of $\mathcal{C}$ as *generating* the dg category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Cpl}]$, while the Bousfield localization yielding the local model structure plays the role of imposing *relations*. Namely, the localization stipulates that any object in $\mathcal{C}$ may be homotopically decomposed into the pieces of any cover. In a very precise sense then (cf. Corollary 4.15) our model for the derived category of τ-sheaves is the universal τ-local model dg category associated to $\mathcal{C}$.

Contents

1. Universal enriched model categories

This section is very much inspired by Dugger's [19] where he proves the existence of a universal model category associated to a small category. Our goal is to establish an analogue of this result in the enriched setting.

"Monoidal" is an abbreviation for "unital monoidal"; the monoidal structure is always denoted by $\otimes$, the unit by $\mathbb{1}$. Fix a bicomplete closed symmetric monoidal category $\mathcal{V}$. We are first going to recall some basics in $\mathcal{V}$-enriched category theory, and for this we follow the terminology in [46].

1.1. Free enriched cocompletion. Let $\mathcal{C}$ and $\mathcal{M}$ be $\mathcal{V}$-categories and assume that $\mathcal{C}$ is small. Recall ([46, §2]) that there is a $\mathcal{V}$-functor category $[\mathcal{C}, \mathcal{M}]$ whose underlying category is just the category of $\mathcal{V}$-functors $\mathcal{C} \to \mathcal{M}$ together with $\mathcal{V}$-natural transformations. Given such a $\mathcal{V}$-functor $\gamma : \mathcal{C} \to \mathcal{M}$ consider the $\mathcal{V}$-functor $\gamma_* : \mathcal{M} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ which takes m to $\mathcal{M}(\gamma(\bullet), m)$. In particular, if $\mathcal{C} = \mathcal{M}$ and γ the identity then γ_* is the Yoneda embedding $y : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$. As in the classical case, the Yoneda embedding provides the free cocompletion as we are now going to explain (see [46, Thm. 4.51]).

Recall that a $\mathcal{V}$-category $\mathcal{M}$ is cocomplete if it has all small indexed colimits (sometimes also called weighted colimits). In practice, the functors $\mathcal{M}(\bullet, m)_0 : \mathcal{M}_0^{\mathrm{op}} \to \mathcal{V}_0$ often preserve limits (for example, if $\mathcal{M}$ is cotensored or if $\mathcal{V}$ is conservative). In this case cocompleteness is equivalent to $\mathcal{M}$ being tensored and the underlying category being cocomplete in the ordinary sense. The first condition means that there exists a $\mathcal{V}$-bifunctor (called the tensor)

$$\bullet \odot \bullet : \mathcal{V} \otimes \mathcal{M} \to \mathcal{M}$$

together with, for each $v \in \mathcal{V}$ and each $m \in \mathcal{M}$, $\mathcal{V}$-natural isomorphisms

$$\mathcal{M}(v \odot m, \bullet) \cong \mathcal{V}(v, \mathcal{M}(m, \bullet)).$$

Accordingly, a $\mathcal{V}$-functor is cocontinuous if and only if it commutes with tensors and the underlying functor is cocontinuous. Dually one defines complete $\mathcal{V}$-categories and continuous $\mathcal{V}$-functors.

An example of a cocomplete $\mathcal{V}$-category is $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ for a small $\mathcal{V}$-category $\mathcal{C}$. From now on, we denote it by $\mathbf{U}_{\mathcal{V}}\mathcal{C}$. The tensor of $v \in \mathcal{V}$ and $f \in \mathbf{U}_{\mathcal{V}}\mathcal{C}$ is given by

$$v \odot f = v_{\mathrm{cst}} \otimes f,$$

where v_{cst} denotes the constant presheaf with value v, and $\otimes$ denotes the objectwise tensor product in $\mathcal{V}$.

Fact 1.1 *Let $\gamma : \mathcal{C} \to \mathcal{M}$ be a $\mathcal{V}$-functor and assume that $\mathcal{C}$ is small and $\mathcal{M}$ is cocomplete.*

(1) There is a $\mathcal{V}$-adjunction

$$(\gamma^*, \gamma_*) : \mathbf{U}_{\mathcal{V}}\mathcal{C} \to \mathcal{M},$$

where $\gamma^*(f)$ is given by the tensor product of f and γ, $f \odot_{\mathcal{C}} \gamma$.

(2) The association $\gamma \mapsto \gamma^*$ induces an equivalence of $\mathcal{V}$-categories

$$[\mathcal{C}, \mathcal{M}] \simeq [\mathbf{U}_{\mathcal{V}}\mathcal{C}, \mathcal{M}]_{\mathrm{coc}}$$

where $(\bullet)_{\mathrm{coc}}$ picks out the cocontinuous $\mathcal{V}$-functors.

(3) There is a canonical isomorphism $\gamma^* y \cong \gamma$.

The $\mathcal{V}$-functor γ^* is called the left $\mathcal{V}$-Kan extension *of γ along the Yoneda embedding.*

Here, the tensor product of the two $\mathcal{V}$-functors f and γ is the coend $\int^{c \in \mathcal{C}} f(c) \odot \gamma(c)$. Notice that part of the statement is the existence of $[\mathbf{U}_{\mathcal{V}}\mathcal{C}, \mathcal{M}]^{\mathrm{coc}}$ as a $\mathcal{V}$-category (this is not clear since $\mathbf{U}_{\mathcal{V}}\mathcal{C}$ is not necessarily small).

If $\beta : \mathcal{D} \to \mathcal{C}$ is a $\mathcal{V}$-functor between small $\mathcal{V}$-categories, we denote $(y\beta)^*$ by β^* if no confusion is likely to arise. With this abuse of notation, there is a canonical isomorphism $(\gamma\beta)^* \cong \gamma^*\beta^*$. Similarly, if $\delta : \mathcal{M} \to \mathcal{N}$ is a cocontinous functor into another cocomplete $\mathcal{V}$-category $\mathcal{N}$, then $(\delta\gamma)^* \cong \delta\gamma^*$.

Assume now that $\mathcal{C}$ is a (symmetric) monoidal $\mathcal{V}$-category (this is the canonical translation of a (symmetric) monoidal structure to the enriched context; or see [17, p. 2f]). $\mathbf{U}_{\mathcal{V}}\mathcal{C}$ inherits a (symmetric) monoidal structure called the (Day) convolution product ([17, Thm. 3.3 and 4.1]). Explicitly, the monoidal product of two presheaves f and g is given by

$$f \otimes g = \int^{c, c'} f(c) \otimes g(c') \otimes \mathcal{C}(\bullet, c \otimes c'),$$

and the unit by $y(\mathbb{1}) = \mathcal{C}(\bullet, \mathbb{1})$. It is clear that the Yoneda embedding $y : \mathcal{C} \to \mathbf{U}_{\mathcal{V}}\mathcal{C}$ is (symmetric) monoidal.

Lemma 1.2 *In the setting of Fact 1.1, assume in addition that γ is (lax) (symmetric) monoidal, and that the monoidal product in $\mathcal{M}$ commutes with indexed colimits. Then:*

(1) γ^* is (lax) (symmetric) monoidal.
(2) The canonical isomorphism $\gamma^* y \cong \gamma$ is monoidal.
(3) The association $\gamma \mapsto \gamma^*$ induces an equivalence of ordinary categories

$$\mathcal{V}\text{–Fun}_{\otimes}(\mathcal{C}, \mathcal{M}) \simeq \mathcal{V}\text{–Fun}_{\mathrm{coc},\otimes}(\mathbf{U}_{\mathcal{V}}\mathcal{C}, \mathcal{M})$$

of (lax) (symmetric) monoidal $\mathcal{V}$-functors.

Proof. Let $f, g \in \mathbf{U}_{\mathcal{V}}\mathcal{C}$. The (lax) monoidal structure on γ^* is defined as follows:

$$\begin{aligned}\Big(\int f \odot \gamma\Big) \otimes \Big(\int g \odot \gamma\Big) &\cong \int^{c,d} (f(c) \otimes g(d)) \odot (\gamma(c) \otimes \gamma(d)) \\ &\to \int^{c,d} (f(c) \otimes g(d)) \odot (\gamma(c \otimes d)) \\ &\cong \int^{e} \Big(\int^{c,d} f(c) \otimes g(d) \otimes \mathcal{C}(e, c \otimes d)\Big) \odot \gamma(e) \\ &\cong \int (f \otimes g) \odot \gamma\end{aligned}$$

and

$$\mathbb{1} \to \gamma(\mathbb{1}) \cong \gamma^*(\mathbb{1}).$$

We leave the details to the reader. □

In this sense, if $\mathcal{C}$ is (symmetric) monoidal then $\mathbf{U}_{\mathcal{V}}\mathcal{C}$ is the free *(symmetric) monoidal* $\mathcal{V}$-cocompletion. Notice also that the "pseudo-functoriality" mentioned above, to wit $(\gamma\beta)^* \cong \gamma^*\beta^*$ and $(\delta\gamma)^* \cong \delta\gamma^*$, is compatible with monoidal structures.

1.2. Enriched model categories. We now discuss the interplay between basic enriched category theory as above and Quillen model structures. From now on we assume that the underlying category $\mathcal{V}_0$ is a symmetric monoidal model category in the sense of [36, Def. 4.2.6]. We also assume that this model structure is cofibrantly generated.

Fix a small ordinary category $\mathcal{C}$ and set $\mathcal{C}[\mathcal{V}]$ to be the associated free $\mathcal{V}$-category. It has the same objects as $\mathcal{C}$ and the $\mathcal{V}$-structure is given by

$$\mathcal{C}[\mathcal{V}](c, c') = \coprod_{\mathcal{C}(c,c')} \mathbb{1}$$

with an obvious composition. By definition, giving a $\mathcal{V}$-functor $\mathcal{C}[\mathcal{V}] \to \mathcal{M}$ into a $\mathcal{V}$-model category $\mathcal{M}$ is the same as giving an (ordinary) functor $\mathcal{C} \to \mathcal{M}_0$. In the sequel, we will often write abusively $\mathcal{C} \to \mathcal{M}$, sometimes thinking of the datum as a $\mathcal{V}$-functor, sometimes as an ordinary functor. We are positive that this will not lead to any confusion.

Thus the general small $\mathcal{V}$-category $\mathcal{C}$ in §1.1 will now always be of this special form. We impose this restriction because it simplifies most of the statements and proofs drastically, and because it is all we will need later on.

Since the underlying category of $\mathbf{U}_{\mathcal{V}}\mathcal{C} := \mathbf{U}_{\mathcal{V}}\mathcal{C}[\mathcal{V}]$ is just the category of presheaves on $\mathcal{C}$ with values in $\mathcal{V}$, the following result is well-known.

Fact 1.3 *(1) $(\mathbf{U}_{\mathcal{V}}\mathcal{C})_0$ admits a cofibrantly generated model structure with weak equivalences and fibrations defined objectwise.*

(2) *If $\mathcal{V}_0$ is left (resp. right) proper then so is $(\mathbf{U}_{\mathcal{V}}\mathcal{C})_0$.*
(3) *If $\mathcal{V}_0$ is combinatorial (resp. tractable, cellular) then so is $(\mathbf{U}_{\mathcal{V}}\mathcal{C})_0$.*
(4) *If $\mathcal{V}_0$ is stable then so is $(\mathbf{U}_{\mathcal{V}}\mathcal{C})_0$.*

This is called the *projective model structure*. If not mentioned otherwise, we will consider $\mathbf{U}_{\mathcal{V}}\mathcal{C}$ as endowed with the projective model structure from now on.

Proof. See [35, Thm. 11.6.1] for the first statement, [35, Thm. 13.1.14] for the second, [35, Pro. 12.1.5] and [9, Thm. 2.14] for the third. The last statement is obvious. □

Definition 1.4 Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{V}$-categories with model structures on their underlying categories. A $\mathcal{V}$-adjunction $(\delta, \varepsilon) : \mathcal{M} \to \mathcal{N}$ is called a *Quillen $\mathcal{V}$-adjunction* if the underlying adjunction $(\delta_0, \varepsilon_0) : \mathcal{M}_0 \to \mathcal{N}_0$ is a Quillen adjunction. In that case δ is called a *left*, ε a *right Quillen $\mathcal{V}$-functor*.

We now come back to the situation of Fact 1.1. The question we should like to answer is: When is (γ^*, γ_*) a Quillen $\mathcal{V}$-adjunction?

Lemma 1.5 *Assume that $\mathcal{M}_0$ is endowed with a model structure. The following conditions are equivalent:*
(1) *(γ^*, γ_*) is a Quillen $\mathcal{V}$-adjunction.*
(2) *For each $c \in \mathcal{C}$, $\mathcal{M}_0(\gamma(c), \bullet)$ is a right Quillen functor.*
(3) *For each $c \in \mathcal{C}$, $\bullet \odot \gamma(c)$ is a left Quillen functor.*

Proof. The equivalence between the last two conditions is clear. The equivalence between the first two conditions follows from the description of γ_* given above and the fact that we imposed the projective model structure on $(\mathbf{U}_{\mathcal{V}}\mathcal{C})_0$. □

In particular, these equivalent conditions are satisfied if the image of γ consists of cofibrant objects, and the tensor on $\mathcal{M}$ is a "Quillen $\mathcal{V}$-adjunction of two variables", i. e. a $\mathcal{V}$-adjunction of two variables such that the underlying data form a Quillen adjunction of two variables in the sense of [36, Def. 4.2.1].

Definition 1.6 A *model $\mathcal{V}$-category* is a bicomplete $\mathcal{V}$-category $\mathcal{M}$ together with a model structure on $\mathcal{M}_0$ such that

- the tensor is a Quillen $\mathcal{V}$-adjunction of two variables;
- for any cofibrant object $m \in \mathcal{M}$, $\mathbb{1}_c \odot m \to \mathbb{1} \odot m$ is a weak equivalence, for a cofibrant replacement $\mathbb{1}_c \to \mathbb{1}$.

A *(symmetric) monoidal model $\mathcal{V}$-category* is a model $\mathcal{V}$-category $\mathcal{M}$ together with a Quillen $\mathcal{V}$-adjunction of two variables $\otimes : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ with a unit, and associativity (and symmetry) constraints satisfying the usual axioms.

These are equivalent to the definitions in [36, Def. 4.2.18, 4.2.20]. Also, it is a straightforward generalization of the notion of a simplicial model category.

Example 1.7
(1) If $\mathcal{V}$ is the category of simplicial sets with the standard model structure then we recover the notion of a simplicial model category.
(2) Our main example will be obtained by taking $\mathcal{V}$ to be the category of (unbounded) chain complexes of Λ-modules, Λ a (commutative unital) ring, with the projective model structure and the usual tensor product. A model $\mathcal{V}$-category will be called a model dg category. See §§2.

Fact 1.8 $\mathbf{U}_{\mathcal{V}}\mathcal{C}$ *is a model* $\mathcal{V}$*-category. Moreover, if* $\mathcal{C}$ *is (symmetric) monoidal and the unit in* $\mathcal{V}$ *cofibrant, then* $\mathbf{U}_{\mathcal{V}}\mathcal{C}$ *is a (symmetric) monoidal model* $\mathcal{V}$*-category for the Day convolution product.*

Proof. The first statement is straightforward to check. The second is [40, Pro. 2.2.15]. ☐

Notice that if $\mathcal{C}$ is cartesian monoidal then the Day convolution product coincides with the objectwise monoidal product on $\mathbf{U}_{\mathcal{V}}\mathcal{C}$.

1.3. Statement and proof. Our goal is to establish $y : \mathcal{C} \to \mathbf{U}_{\mathcal{V}}\mathcal{C}$ (really, $\mathcal{C}[\mathcal{V}] \to \mathbf{U}_{\mathcal{V}}\mathcal{C}$) as the universal functor into a model $\mathcal{V}$-category. But first, we need to make precise what we mean by the universality in the statement. For this fix a model $\mathcal{V}$-category $\mathcal{M}$ and a functor $\gamma : \mathcal{C} \to \mathcal{M}$. Define a *factorization of* γ *through* y to be a pair (L, η) where $L : \mathbf{U}_{\mathcal{V}}\mathcal{C} \to \mathcal{M}$ is a left Quillen $\mathcal{V}$-functor, and $\eta : Ly \to \gamma$ a natural transformation which is objectwise a weak equivalence. A morphism of such factorizations $(L, \eta) \to (L', \eta')$ is a natural transformation $L \to L'$ compatible with η and η'. This clearly defines a category $\mathrm{Fact}(\gamma, y)$.

Proposition 1.9 *Assume that the unit in* $\mathcal{V}$ *is cofibrant. For any* γ*, the category* $\mathrm{Fact}(\gamma, y)$ *is contractible.*

Notice that in a homotopical context it is unreasonable to expect the category of choices to be a groupoid (“uniqueness up to unique isomorphism”) and contractibility is usually the right thing to ask of this category.

Let $\mathrm{CofRep}(\gamma)$ be the category of cofibrant replacements of γ. Its objects are functors $\gamma' : \mathcal{C} \to \mathcal{M}$ together with a natural transformation $\gamma' \to \gamma$ which is objectwise a weak equivalence and such that the image of γ' is cofibrant. The morphisms are the obvious ones.

Lemma 1.10 *Assume that the unit in* $\mathcal{V}$ *is cofibrant. There is a canonical equivalence of categories* $\mathrm{Fact}(\gamma, y) \simeq \mathrm{CofRep}(\gamma)$.

Proof. We give functors in both directions. That these are quasi-inverses to each other will then be seen to follow from the $\mathcal{V}$-equivalence of categories in Fact 1.1.

- Given $\gamma' \to \gamma$ on the right hand side, define $L = (\gamma')^*$ and choose the natural transformation $(\gamma')^* y \cong \gamma' \to \gamma$. Functoriality follows from the functoriality statement in Fact 1.1.
- Given $(L, Ly \to \gamma)$ on the left hand side, $Ly \to \gamma$ defines a cofibrant replacement since L is a left Quillen $\mathcal{V}$-functor and the image of y is cofibrant. Functoriality is obvious. ☐

Proof of Proposition 1.9. By the previous lemma, we need to show contractibility of $\mathrm{CofRep}(\gamma)$. Fix a cofibrant replacement functor F for the model structure on $\mathcal{M}_0$. Composing with γ we obtain an object $(F\gamma, F\gamma \to \gamma)$ of $\mathrm{CofRep}(\gamma)$. Given any other object $(\gamma', \gamma' \to \gamma)$, functoriality of F yields a commutative square

$$\begin{tikzcd} F\gamma' \arrow[r] \arrow[d] & F\gamma \arrow[d] \\ \gamma' \arrow[r] & \gamma \end{tikzcd}$$

and thus a zig-zag $\gamma' \leftarrow F\gamma' \to F\gamma$ in $\mathrm{CofRep}(\gamma)$. Moreover, this zig-zag is natural in γ' hence this construction provides a zig-zag of homotopies between the identity functor on $\mathrm{CofRep}(\gamma)$ and the constant functor $(F\gamma, F\gamma \to \gamma)$. ☐

For the reader's convenience we reformulate our main result.

Corollary 1.11 *Let $\mathcal{C}$ be a small category, and $\mathcal{V}$ a cofibrantly generated symmetric monoidal model category whose unit is cofibrant. There exists a functor $y : \mathcal{C} \to \mathbf{U}_{\mathcal{V}}\mathcal{C}$ into a model $\mathcal{V}$-category, universal in the sense that for any solid diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "y"] \arrow[dr, "\gamma"'] & \mathbf{U}_{\mathcal{V}}\mathcal{C} \arrow[d, dotted, "L"] \\ & \mathcal{M} \end{tikzcd}$$

with $\mathcal{M}$ a model $\mathcal{V}$-category, there exists a left Quillen $\mathcal{V}$-functor L as indicated by the dotted arrow, unique up to a contractible choice, making the diagram commutative up to a weak equivalence $Ly \to \gamma$.

Remark 1.12 One can dualize the discussion of this section in order to obtain universal model $\mathcal{V}$-categories for *right* Quillen $\mathcal{V}$-functors, as in [19, §4]. Unsurprisingly, one finds that this universal model $\mathcal{V}$-category associated to $\mathcal{C}$ is given by $[\mathcal{C}, \mathcal{V}]^{\mathrm{op}}$ with the opposite of the projective model structure. This can also be deduced from Corollary 1.11 applied to $\mathcal{C}^{\mathrm{op}}$.

2. Universal model dg categories

We now specialize to the case of dg categories. Fix a commutative unital ring Λ, denote by $\mathbf{Mod}(\Lambda)$ the category of Λ-modules, and by $\mathbf{Cpl}(\Lambda)$ the category of unbounded chain complexes of Λ-modules. Our conventions for chain complexes are homological, i. e. the differentials decrease the indices, and the shift operator satisfies $(A[p])_n = A_{p+n}$. The subobject of n-cycles (resp. n-boundaries) of A is denoted by $Z_n A$ (resp. $B_n A$). As usual, the nth homology is denoted by $\mathrm{H}_n A = Z_n A / B_n A$.

$\mathbf{Cpl}(\Lambda)$ has a tensor product, defined by

$$(A \otimes B)_n = \oplus_{p+q=n} A_p \otimes B_q$$

with the Koszul sign convention for the differential. It also admits the "projective model structure" for which the weak equivalences are the quasi-isomorphisms, and the fibrations the epimorphisms (i. e. the degreewise surjections). In that way, $\mathbf{Cpl}(\Lambda)$ becomes a symmetric monoidal model category. In this section we always take $\mathcal{V}$ to be $\mathbf{Cpl}(\Lambda)$. The universal model category underlying a model dg category $(\mathbf{U}_{\mathrm{dg}}\mathcal{C})_{\mathrm{o}}$ will now be denoted by $\mathbf{U}\mathcal{C}$. The complex of morphisms from K to K' in $\mathbf{U}_{\mathrm{dg}}\mathcal{C}$ is denoted by $\underline{\mathrm{hom}}_{\mathrm{dg}}(K, K') \in \mathbf{Cpl}(\Lambda)$. Recall that it is given explicitly by $\mathrm{Tot}^{\Pi}(\mathrm{hom}_{\mathbf{PSh}(\mathcal{C},\Lambda)}(K_p, K'_q))_{p,q}$.

Our main goal in this section is to better understand the model structure on $\mathbf{U}\mathcal{C}$ (defined in Fact 1.3). In the last part we will also discuss a specific instance of a left dg Kan extension used in the following chapter.

2.1. Basic properties of the model category $\mathbf{U}\mathcal{C}$.

By Fact 1.8 we know that $\mathbf{U}_{\mathrm{dg}}\mathcal{C}$ is a model dg category, and a (symmetric) monoidal model dg category if $\mathcal{C}$ is (symmetric) monoidal. It follows from Fact 1.3 that the model category $\mathbf{U}\mathcal{C}$ is about as nice as it can get.

Corollary 2.1 $\mathbf{U}\mathcal{C}$ *is a*

(1) proper,
(2) stable,
(3) tractable (in particular combinatorial),
(4) cellular,

model category.

We will now describe explicitly sets of generating (trivial) cofibrations.

Definition 2.2 Let, for any presheaf F, S^nF be the complex of presheaves which has F in degree n and is 0 otherwise, and let D^nF be the complex of presheaves which has F in degree n and $n-1$, is 0 otherwise, and whose nontrivial differential is given by the identity on F. There exists a canonical morphism $S^{n-1}F \to D^nF$. Let I be the set of morphisms $S^{n-1}\Lambda(c) \to D^n\Lambda(c)$ for all $c \in \mathcal{C}$ and let J be the set of maps $0 \to D^n\Lambda(c)$.

Notice that there are adjunctions

$$(S^n, Z_n) \text{ and } (D^n, (\bullet)_n) : \mathbf{PSh}(\mathcal{C}, \Lambda) \to \mathbf{U}\mathcal{C}.$$

The same arguments as in [36, Pro. 2.3.4, 2.3.5] then establish the following result.

Fact 2.3 *A morphism in $\mathbf{U}\mathcal{C}$ is a fibration (resp. trivial fibration) if and only if it has the right lifting property with respect to J (resp. I).*

We will use another set of generating cofibrations later on.

Definition 2.4 Given a presheaf F of Λ-modules, let Δ^nF be the complex which has F in degree n and $F \oplus F$ in degree $n-1$, and zero otherwise, and whose only non-zero differential is given by $\mathrm{id} \times (-\mathrm{id}) : F \to F \oplus F$. Define also $\partial\Delta^nF$ to be the complex which has $F \oplus F$ in degree $n-1$ and 0 otherwise. Let I' be the set of morphisms $\partial\Delta^n\Lambda(c) \to \Delta^n\Lambda(c)$ which is the identity in degree n, for all $n \in \mathbb{Z}$ and $c \in \mathcal{C}$.

Lemma 2.5 *A morphism in $\mathbf{U}\mathcal{C}$ is a trivial fibration if and only if it has the right lifting property with respect to I'.*

Proof. Morphisms in I' are cofibrations by Fact 2.11. Conversely we will exhibit any morphism in I as a retract of some morphism in I'. Thus fix $c \in \mathcal{C}$ and $n \in \mathbb{Z}$, and consider the following diagram:

$$\begin{tikzcd}
S^n\Lambda(c) \arrow[r, "\mathrm{id}\times(-\mathrm{id})"] \arrow[d] & \partial\Delta^{n+1}\Lambda(c) \arrow[r, "(\mathrm{id},0)"] \arrow[d] & S^n\Lambda(c) \arrow[d] \\
D^{n+1}\Lambda(c) \arrow[r, "r"] & \Delta^{n+1}\Lambda(c) \arrow[r, "s"] & D^{n+1}\Lambda(c)
\end{tikzcd}$$

Here, r in degree n is $\mathrm{id} \times (-\mathrm{id})$ and in degree $n+1$ is id, while s in degree n is the first projection and in degree $n+1$ the identity. It is easy to see that the diagram commutes and the compositions of each row are the identity morphism. ☐

2.2. Projective cofibrations.

Since the fibrations and weak equivalences are given explicitly in $\mathbf{U}\mathcal{C}$ our goal is to better understand the cofibrations. They are called projective cofibrations. The discussion runs parallel to the description of projective cofibrations for the category of chain complexes (i. e. the case of $\mathcal{C}$ the terminal category), in [36, §2.3].

Lemma 2.6 *If $f : K \to K' \in \mathbf{U}\mathcal{C}$ is a trivial fibration then f induces a surjective morphism $f : Z_nK \to Z_nK'$ for all $n \in \mathbb{Z}$.*

Proof. Since f is degreewise surjective, it induces a surjective morphism on the boundaries $B_nK \to B_nK'$. Now consider the morphism of exact sequences:

$$\begin{tikzcd}
B_nK \arrow[r] \arrow[d] & Z_nK \arrow[r] \arrow[d] & \mathrm{H}_nK \arrow[d] \\
B_nK' \arrow[r] & Z_nK' \arrow[r] & \mathrm{H}_nK'
\end{tikzcd}$$

The first and last vertical arrows are surjective, hence the middle one is too. □

Definition 2.7 A presheaf of Λ-modules $F \in \mathbf{PSh}(\mathcal{C}, \Lambda)$ is called *projective* if

$$\hom_{\mathbf{PSh}(\mathcal{C},\Lambda)}(F, \bullet) : \mathbf{PSh}(\mathcal{C}, \Lambda) \to \mathbf{Mod}(\Lambda)$$

is exact.

Example 2.8 For any $c \in \mathcal{C}$ the representable presheaf $\Lambda(c)$ is projective. Direct sums of projectives are projective.

Lemma 2.9 *For any projective presheaf $F \in \mathbf{PSh}(\mathcal{C}, \Lambda)$, the complex $S^{\circ}F$ is projective cofibrant.*

Proof. We have to prove that for any trivial fibration $f : K \to K' \in \mathbf{U}\mathcal{C}$, the induced morphism

$$\hom_{\mathbf{U}\mathcal{C}}(S^{\circ}F, K) \to \hom_{\mathbf{U}\mathcal{C}}(S^{\circ}F, K')$$

is surjective. But for any complex $L \in \mathbf{U}\mathcal{C}$, we have

$$\hom_{\mathbf{U}\mathcal{C}}(S^{\circ}F, L) = \hom_{\mathbf{Mod}(\Lambda)}(F, Z_{\circ}L).$$

Now the result follows from Lem. 2.6. □

Fact 2.10 *Let $K \in \mathbf{U}\mathcal{C}$. If K is projective cofibrant then each K_n is a projective presheaf. As a partial converse, if K is bounded below and each K_i is projective then K is projective cofibrant.*

Proof. The proof of [36, Lemma 2.3.6] applies. □

Fact 2.11 *A map $f : K \to K' \in \mathbf{U}\mathcal{C}$ is a projective cofibration if and only if f is a degreewise split injection and the cokernel of f is projective cofibrant.*

Proof. The proof of [36, Pro. 2.3.9] applies. □

Corollary 2.12 *Let $K = \varinjlim_{n\in\mathbb{N}} K^{(n)} \in \mathbf{U}\mathcal{C}$, such that $K^{(n)}$ is projective cofibrant and bounded below for each n, and such that the transition morphisms $K^{(n)} \to K^{(n+1)}$ are degreewise split injective. Then K is projective cofibrant.*

Proof. We use the fact that K is a sequential colimit of projective cofibrant objects with transition morphisms which are split injective in each degree hence the cokernel has projective objects in each degree. This implies together with boundedness and the previous lemma that the transition morphisms are projective cofibrations. Hence K is projective cofibrant. □

Independently of monoidal structures on $\mathcal{C}$, we can always define an objectwise tensor product on presheaves. The following lemma gives a necessary and sufficient condition for this product to be a Quillen bifunctor.

Lemma 2.13 *$\mathbf{U}\mathcal{C}$ is a symmetric monoidal model category for the objectwise tensor product if and only if for any pair of objects $c, d \in \mathcal{C}$, the presheaf of Λ-modules $\Lambda(c) \otimes \Lambda(d)$ is projective.*

Proof. Since representables are cofibrant (Fact 2.10) the condition is clearly necessary. For the converse, it suffices to prove the pushout-product $i \square j$ a (trivial) cofibration if i and j are generating cofibrations (and one of them a generating trivial cofibration). By Fact 2.3, i and j are of the form $i' \odot \Lambda(c)$ and $j' \odot \Lambda(d)$ for cofibrations i', j' of $\mathbf{Cpl}(\Lambda)$ (one of which is acylic), $c, d \in \mathcal{C}$. $i \square j$ can then be identified with $(i' \square j') \odot (\Lambda(c) \otimes \Lambda(d))$. $i' \square j'$ is a (trivial) cofibration since $\mathbf{Cpl}(\Lambda)$ is a symmetric monoidal model category. If $\Lambda(c) \otimes \Lambda(d)$ is projective then Lemma 2.9 together with Fact 1.8 yields what we want. □

2.3. Dold-Kan correspondence. Fix an abelian category $\mathcal{A}$. We start by recalling some basic constructions relating simplicial objects and connective chain complexes in $\mathcal{A}$.

Given a simplicial object $a_\bullet$ in $\mathcal{A}$, one can associate to it a connective chain complex (called the Moore complex, and usually still denoted by $a_\bullet$) which is a_n in degree n and whose differentials are given by

$$\sum_{i=0}^{n}(-1)^i d_i : a_n \to a_{n-1}.$$

This clearly defines a functor $\Delta^{\mathrm{op}}\mathcal{A} \to \mathbf{Cpl}_{\geq 0}(\mathcal{A})$. Since every object in $\Delta^{\mathrm{op}}\mathcal{A}$ is canonically split, we get a second functor $N : \Delta^{\mathrm{op}}\mathcal{A} \to \mathbf{Cpl}_{\geq 0}(\mathcal{A})$, which associates to $a_\bullet$ the normalized chain complex:

$$N(a_\bullet)_n = \bigcap_{i=0}^{n-1}\ker(d_i : a_n \to a_{n-1}), \qquad (-1)^n d_n : N(a_\bullet)_n \to N(a_\bullet)_{n-1}.$$

Clearly, there is a canonical embedding $N(a_\bullet) \subset a_\bullet$ but more is true:

Fact 2.14 *(1) The inclusion $N(a_\bullet) \to a_\bullet$ is a natural chain homotopy equivalence.*
(2) There is a functorial splitting $a_\bullet = N(a_\bullet) \oplus N'(a_\bullet)$ and N' is an acyclic functor.
(3) N is an equivalence of categories with quasi-inverse Γ.
(4) For any $n \in \mathbb{N}$, there is a natural isomorphism $\pi_n\Gamma \cong \mathrm{H}_n$.

In particular, we obtain a sequence of adjunctions

$$\Delta^{\mathrm{op}}\mathrm{Set} \underset{}{\overset{\Lambda}{\rightleftarrows}} \Delta^{\mathrm{op}}\mathbf{Mod}(\Lambda) \underset{\Gamma}{\overset{N}{\rightleftarrows}} \mathbf{Cpl}_{\geq 0}(\Lambda) \underset{\tau_{\geq 0}}{\rightleftarrows} \mathbf{Cpl}(\Lambda)\ , \tag{2.15}$$

where the first is the “free-forgetful” adjunction, and the last is the obvious adjunction between connective and unbounded chain complexes involving the good truncation functor $\tau_{\geq 0}$. Endow the category of simplicial sets with the Bousfield-Kan model structure for which cofibrations are levelwise injections and weak equivalences are weak homotopy equivalences, i. e. isomorphisms on the homotopy groups. By transfer along the forgetful functor this induces a model structure on simplicial Λ-modules, for which the Dold-Kan correspondence becomes a Quillen equivalence with the projective model structure on $\mathbf{Cpl}_{\geq 0}(\Lambda)$ (i. e. weak equivalences are quasi-isomorphisms, fibrations are surjections in positive degrees). It is clear that the last adjunction in (2.15) is Quillen as well.

Proposition 2.16 *The sequence in (2.15) induces a Quillen adjunction*

$$(N\Lambda, \Gamma\tau_{\geq 0}) : \Delta^{\mathrm{op}}\,\mathbf{PSh}(\mathcal{C}) \to \mathbf{U}\mathcal{C}.$$

Here both categories are equipped with the projective model structure.

Proof. Consider presheaves on $\mathcal{C}$ with values in the different categories appearing in (2.15). There is an induced sequence of adjunctions between these presheaf categories, similar to (2.15). If we endow each of them with the projective model structure, then each of the right adjoint preserves (trivial) fibrations by our discussion above. $\square$

Lemma 2.17 *Let $K \in \mathbf{U}\mathcal{C}$ be cofibrant, $K' \in \mathbf{U}\mathcal{C}$ arbitrary. Then*

$$\Gamma\tau_{\geq 0}\,\underline{\mathrm{hom}}_{\mathrm{dg}}(K, K')$$

is a (left) homotopy function complex from K to K' (in the sense of [35, Def. 17.1.1]).

Proof. Since K is cofibrant, the functor $\bullet \odot K : \mathbf{Cpl}(\Lambda) \to \mathbf{U}\mathcal{C}$ is left Quillen, with right adjoint $\underline{\mathrm{hom}}_{\mathrm{dg}}(K, \bullet)$. We know that $\Delta^\bullet$ is a (the "standard") cosimplicial resolution of the terminal object in simplicial sets. By [35, Pro. 17.4.16], the left homotopy function complex from K to K' is then given by

$$\underline{\mathrm{hom}}_{\mathrm{dg}}(N\Lambda(\Delta^\bullet) \odot K, K') \cong \Delta^{\mathrm{op}}\mathrm{Set}(\Delta^\bullet, \Gamma\tau_{\geq 0}\, \underline{\mathrm{hom}}_{\mathrm{dg}}(K, K')) \cong \Gamma\tau_{\geq 0}\, \underline{\mathrm{hom}}_{\mathrm{dg}}(K, K').$$

□

Corollary 2.18 *Let $K, K' \in \mathbf{U}\mathcal{C}$ and assume that K is cofibrant. For any $n \in \mathbb{Z}$, there is a natural isomorphism*

$$\mathrm{hom}_{\mathbf{Ho}(\mathbf{U}\mathcal{C})}(K, K'[n]) \cong \mathrm{H}_n\, \underline{\mathrm{hom}}_{\mathrm{dg}}(K, K'). \tag{2.19}$$

Proof. By [35, Pro. 17.7.1], $\pi_0\Gamma\tau_{\geq 0}\, \underline{\mathrm{hom}}_{\mathrm{dg}}(K, K'[n])$ is naturally isomorphic to the set of homotopy classes from K to $K'[n]$ which is equal to the left hand side of (2.19), by general properties of model categories. But

$$\begin{aligned}\pi_0\Gamma\tau_{\geq 0}\, \underline{\mathrm{hom}}_{\mathrm{dg}}(K, K'[n]) &\cong \mathrm{H}_0\, \underline{\mathrm{hom}}_{\mathrm{dg}}(K, K'[n]) \\ &\cong \mathrm{H}_n\, \underline{\mathrm{hom}}_{\mathrm{dg}}(K, K').\end{aligned}$$

□

Lemma 2.20 *Let $K \in \Delta^{\mathrm{op}}\mathbf{U}\mathcal{C}$ be a simplicial object in $\mathbf{U}\mathcal{C}$. Then the homotopy colimit $\mathrm{L}\,\mathrm{colim}_{\Delta^{\mathrm{op}}} K$ is given by*

$$\mathrm{Tot}^{\oplus}(K) \simeq \mathrm{Tot}^{\oplus}(NK).$$

Proof. The category $\mathbf{U}\mathcal{C}$ together with the class of quasi-isomorphisms and the functor $\mathrm{Tot}^{\oplus} : \Delta^{\mathrm{op}}\mathbf{U}\mathcal{C} \to \mathbf{U}\mathcal{C}$ defines a "simplicial descent category" in the sense of [63, 62], see [62, §5.2]. The result for the first object now follows from [63, Thm. 5.1.i]. Since the Moore complex and the normalized complexes are homotopy equivalent (see Fact 2.14), the result for the second object follows from this (or see [62, Rem. 5.2.3]). □

The Moore and normalized complexes also induce functors from cosimplicial objects to coconnective chain complexes.

Lemma 2.21 *Let $K \in \Delta\mathbf{U}\mathcal{C}$ be a cosimplicial object in $\mathbf{U}\mathcal{C}$. Then the homotopy limit $\mathrm{R}\lim_{\Delta} K$ is given by*

$$\mathrm{Tot}^{\Pi}(K) \simeq \mathrm{Tot}^{\Pi}(NK).$$

Proof. This can be deduced from the proof of the previous lemma by passing to the opposite categories. □

Finally, we will often use the following result in chapter IV.

Lemma 2.22 *The derived category $\mathbf{Ho}(\mathbf{U}\mathcal{C})$ is compactly generated by the representable objects.*

Proof. If $\mathrm{hom}_{\mathbf{Ho}(\mathbf{U}\mathcal{C})}(\Lambda(c), K[n]) = 0$ for every $c \in \mathcal{C}$ and $n \in \mathbb{Z}$ then this means by Lemma 2.18 that K is objectwise acyclic and hence the zero object in the derived category.

Moreover, given a set $(K^{(i)})_{i\in I}$ of objects in $\mathbf{U}\mathcal{C}$ and $c \in \mathcal{C}$, the canonical morphism

$$\bigoplus_i \mathrm{hom}_{\mathbf{Ho}(\mathbf{U}\mathcal{C})}(\Lambda(c), K^{(i)}) \to \mathrm{hom}_{\mathbf{Ho}(\mathbf{U}\mathcal{C})}(\Lambda(c), \bigoplus_i K^{(i)})$$

is identified, again by Lemma 2.18, with

$$\bigoplus_i \mathrm{H}_0 K^{(i)}(c) \to \mathrm{H}_0 \bigoplus_i K^{(i)}(c),$$

which is invertible, thus the representable objects are also compact. □

2.4. An example of a left dg Kan extension. We would now like to give a more explicit description of the left dg Kan extension in a specific situation arising in chapter IV. The setup is as follows: Let $\mathcal{C}$ be a small ordinary category, and $\mathcal{B}$ a cocomplete Λ-linear category which is tensored over $\mathbf{Mod}(\Lambda)$. Finally, we are given a functor $\gamma : \mathcal{C} \to \mathbf{Cpl}(\mathcal{B})$.

First, notice that $\mathbf{Cpl}(\mathcal{B})$ is canonically a dg category, and the tensors on $\mathcal{B}$ induce a tensor operation of $\mathbf{Cpl}(\Lambda)$ on $\mathbf{Cpl}(\mathcal{B})$, by

$$(K \odot B)_n = \oplus_{p+q=n} K_p \odot B_q$$

with the usual differentials.

Notice that by considering a presheaf of Λ-modules as concentrated in degree 0, we can consider the restriction of γ^* to $\mathbf{PSh}(\mathcal{C}, \Lambda)$, still denoted by γ^*. The following lemma gives an alternative characterization of (the underlying functor of) such a left dg Kan extension.

Lemma 2.23

(1) γ^* is the composition

$$\mathbf{U}\mathcal{C} \xrightarrow{\mathbf{Cpl}(\gamma^*)} \mathbf{Cpl}(\mathbf{Cpl}(\mathcal{B})) \xrightarrow{\mathrm{Tot}^{\oplus}} \mathbf{Cpl}(\mathcal{B}). \tag{2.24}$$

(2) Conversely, γ^* is characterized (up to natural isomorphism) by:
(a) γ^* admits a factorization as in (2.24).
(b) γ^* is cocontinuous.
(c) $\gamma^* \circ \Lambda(\bullet) \cong \gamma$ naturally.

Proof.

(1) This follows easily from our definition of the tensor operation on $\mathbf{Cpl}(\mathcal{B})$ together with the fact that colimits in $\mathbf{Cpl}(\mathcal{B})$ are computed degreewise.

(2) We know that γ^* satisfies the three properties in the statement. Conversely, let us prove that they characterize a functor G completely (in terms of γ). By the first property we reduce to prove it for a presheaf K concentrated in degree 0. Then:

$$\begin{aligned} G(K) &\cong G\Big(\int^c K(c)_{\mathrm{cst}} \otimes \Lambda(c)\Big) && \text{by the Yoneda lemma} \\ &\cong \int^c G(K(c)_{\mathrm{cst}} \otimes \Lambda(c)) && \text{by cocontinuity.} \end{aligned}$$

We are thus reduced to show

$$G(K_{\mathrm{cst}} \otimes \Lambda(c)) \cong K \odot \gamma(c),$$

naturally in modules K and objects $c \in C$. For this we can take a functorial exact sequence

$$\oplus_{I_2} \Lambda \xrightarrow{\alpha} \oplus_{I_1} \Lambda \to K \to 0$$

of Λ-modules, by which we easily reduce to K free using the cocontinuity of G. Again by cocontinuity we further reduce to $K = \Lambda$ and then our contention follows from the third property. ☐

3. Cofibrant replacement

Our goal in this section is to resolve functorially any presheaf of complexes by a cofibrant object made up of representables. It is clear how to resolve a single presheaf of Λ-modules, and it is also not difficult to extend this to bounded below complexes of presheaves (essentially due to Fact 2.10). As the example in [36, 2.3.7] shows, not every complex of representables is cofibrant hence naively extending the procedure to the unbounded case might apriori run into problems. However, we will show that such problems do not occur.

3.1. Preliminaries from homological algebra. Recall the following basic facts in homological algebra.

Lemma 3.1 *Let $\mathcal{A}$ be a Grothendieck abelian category, and let $D, D' : \mathbb{N} \to \mathbf{Cpl}(\mathcal{A})$ be two diagrams of complexes in $\mathcal{A}$ ($\mathbb{N}$ considered as an ordered set). If $g : D \to D'$ is a morphism of diagrams of complexes which is objectwise a quasi-isomorphism, then also $\varinjlim g$ is a quasi-isomorphism.*

Proof. The sequence

$$0 \to \oplus_{n\geq 0} D_n \to \oplus_{n\geq 0} D_n \to \varinjlim D \to 0$$

is exact, where the second arrow on D_n is defined to be $\mathrm{id} - D_{n\to n+1}$. Indeed, the only non-trivial part is exactness on the left, and for this one notices that the analogous map

$$\oplus_{n=0}^{m} A_n \to \oplus_{n=0}^{m+1} A_n$$

is a mono and hence stays so after taking the limit over m because $\mathcal{A}$ satisfies (AB5).

g then induces a morphism of short exact sequences of complexes and hence a morphism of distinguished triangles in the derived category (which exists because $\mathcal{A}$ is a Grothendieck category). It is then clear that the two vertical arrows $\oplus_n g_n$ are isomorphisms in the derived category hence so is the third vertical arrow, $\varinjlim g$. □

Lemma 3.2 *Let $\mathcal{A}$ be an abelian Grothendieck category and let C, C' be two bounded below bicomplexes (i.e. $C_{\bullet,q} = 0$ for all $q \ll 0$) in $\mathcal{A}$, and let $f : C \to C'$ be a morphism of bicomplexes. If $f_{\bullet,q} : C_{\bullet,q} \to C'_{\bullet,q}$ is a quasi-isomorphism of complexes for all q, then* $\mathrm{Tot}^{\oplus}(f)$ *is a quasi-isomorphism.*

Proof. Without loss of generality, $C_{\bullet,q} = 0$ for all negative q. Let $C(n) = C_{\bullet,\leq n}$, $n \geq 0$, be the stupid truncation. In other words, $C(n)$ is the subbicomplex of C satisfying

$$C(n)_{p,q} := \begin{cases} C_{p,q} & : q \leq n \\ 0 & : q > n; \end{cases}$$

similarly for C' and f. We claim that $\mathrm{Tot}^{\oplus}(f(n))$ is a quasi-isomorphism for all n. This is proved by induction on n. For $n = 0$ it is true because of our assumption on f. For the induction step we use the short exact sequence

$$0 \to \mathrm{Tot}^{\oplus}(C(n-1)) \to \mathrm{Tot}^{\oplus}(C(n)) \to C_{\bullet,n}[-n] \to 0$$

of complexes in $\mathcal{A}$. f gives rise to a morphism of short exact sequences, where the induction hypothesis for $n-1$ together with our assumption on f show that the outer two arrows are quasi-isomorphisms. By the 5-lemma also the middle one, i. e. $\mathrm{Tot}^{\oplus}(f(n))$, is a quasi-isomorphism.

Now apply the previous lemma to $D_n = \mathrm{Tot}^{\oplus}(C(n))$, $D'_n = \mathrm{Tot}^{\oplus}(C'(n))$, and $g_n = \mathrm{Tot}^{\oplus}(f(n))$ to get the result. (One uses here that $\mathrm{Tot}^{\oplus}$ preserves colimits.) □

3.2. Construction and proof. Consider the functor category $\mathbf{PSh}(\mathcal{C}, \Lambda)$. It is a Grothendieck abelian category. We call an object of $\mathbf{PSh}(\mathcal{C}, \Lambda)$ *semi-representable* if it is a small coproduct of representables. An *SR-resolution* of an object $K \in \mathbf{PSh}(\mathcal{C}, \Lambda)$ is a complex $K_\bullet$ of semi-representables in $\mathbf{PSh}(\mathcal{C}, \Lambda)$ together with a quasi-isomorphism of complexes $K_\bullet \to S^\circ K$. Similarly one defines SR-resolutions for complexes in $\mathbf{PSh}(\mathcal{C}, \Lambda)$. Note that a bounded below *SR-resolution* is a cofibrant replacement by Fact 2.10.

Lemma 3.3 *Objects in* $\mathbf{PSh}(\mathcal{C},\Lambda)$ *possess a functorial SR-resolution; more precisely there exists a functor*

$$P : \mathbf{PSh}(\mathcal{C},\Lambda) \to \mathbf{U}\mathcal{C}$$

together with a natural transformation $P \to S^0$ *satisfying:*

- *the components of* $P \to S^0$ *are all bounded below SR-resolutions;*
- P *maps the zero morphism to the zero morphism;*
- P *takes injective morphisms to degreewise split injective morphisms.*

Proof. Let K be an arbitrary object of $\mathbf{PSh}(\mathcal{C},\Lambda)$. There is a canonical epimorphism

$$K_0 := \bigoplus_{K(c)\setminus 0} \Lambda(c) \to \operatorname*{colim}_{K(c)} \Lambda(c) \xrightarrow{\sim} K.$$

Taking the kernel and repeating this construction we get a complex $K_\bullet$ together with a quasi-isomorphism $K_\bullet \to S^0 K$.

Given $f : K \to K'$ and $x \in K(c)\setminus 0$ such that $f(x) = 0$, the component $\Lambda(c)$ corresponding to x is mapped to 0, otherwise it maps identically to $\Lambda(c)$ corresponding to $f(x)$. It is easily checked that this induces a morphism $\ker(K_0 \to K) \to \ker(K'_0 \to K')$ hence repeating we obtain $P(f) : P(K) \to P(K')$. Functoriality is clear.

If f is injective then by this description $f_0 : K_0 \to K'_0$ is split injective, and the induced morphism $\ker(K_0 \to K) \to \ker(K'_0 \to K')$ is injective. Repeating this argument, we see that the induced morphism $P(f)$ is degreewise split injective. ☐

Proposition 3.4 *There exists an endofunctor* $Q : \mathbf{U}\mathcal{C} \to \mathbf{U}\mathcal{C}$ *together with a natural transformation* $Q \to \mathrm{id}$ *satisfying:*

- *the components of* $Q \to \mathrm{id}$ *are trivial fibrations;*
- *the image of* Q *consists of projective cofibrant complexes of semi-representables.*

In particular, Q *is a cofibrant replacement functor.*

Proof. Apply the functor P of the previous lemma in each degree, obtaining an SR-resolution $P(K_n)$ of K_n for each $n \in \mathbb{Z}$. We get maps $P(K_n) \to P(K_{n-1})$ of complexes which in total define a bicomplex $P(K) := P_\bullet(K_\bullet)$ (since P takes 0 to 0) together with a map of bicomplexes $P(K) \to K$, the latter concentrated in horizontal degree 0. Taking the total complexes yields a morphism

$$Q(K) := \mathrm{Tot}^\oplus(P_\bullet(K_\bullet)) \to \mathrm{Tot}^\oplus(K_\bullet) = K. \tag{3.5}$$

Functoriality follows from functoriality in the previous lemma as well as functoriality of $\mathrm{Tot}^\oplus$. It remains to prove that (3.5) is a quasi-isomorphism with projective cofibrant domain.

For this let $\tau_{\geq n}K$ ($n \in \mathbb{Z}$) be the subcomplex of K satisfying

$$(\tau_{\geq n}K)_q = \begin{cases} K_q & : q > n \\ Z_n K & : q = n \\ 0 & : q < n. \end{cases}$$

Note that there are canonical morphisms $\tau_{\geq n}K \to \tau_{\geq n-1}K$ and the canonical morphism $\varinjlim_{n\in\mathbb{N}} \tau_{\geq -n}K \to K$ is an isomorphism. But also $\varinjlim_{n\in\mathbb{N}} P(\tau_{\geq -n}K) \to P(K)$ is an isomorphism of bicomplexes. Since the total complex functor commutes with colimits we conclude that $\varinjlim_{n\in\mathbb{N}} Q(\tau_{\geq -n}K) \to Q(K)$ is an isomorphism.

By the previous lemma, $P(\tau_{\geq -n}K) \to P(\tau_{\geq -(n+1)}K)$ is a bidegreewise split injection hence $Q(\tau_{\geq -n}K) \to Q(\tau_{\geq -(n+1)}K)$ is a degreewise split injection. It follows from Corollary 2.12 that $Q(K)$ is projective cofibrant. Also by the previous lemma, $P(\tau_{\geq -n}K) \to \tau_{\geq -n}K$

is a quasi-isomorphism in each row. It follows from Lemma 3.2 that $Q(\tau_{\geq -n}K) \to \tau_{\geq -n}K$ is a quasi-isomorphism. (3.5) being the sequential colimit of these morphisms, Lemma 3.1 tells us that also (3.5) is a quasi-isomorphism. □

Remark 3.6 Even if this result is not very useful from a practical point of view, it does provide a conceptually satisfying method to compute the derived functor of a left dg Kan extension in the context of §2.4. Indeed, fix a functor $\gamma : \mathcal{C} \to \mathbf{Cpl}(\mathcal{B})$ for a $\mathbf{Mod}(\Lambda)$-cocomplete Λ-linear category $\mathcal{B}$, and assume that γ^* is a left Quillen functor. The image of any $K \in \mathbf{U}\mathcal{C}$ under $\mathrm{L}\gamma^*$ can be computed as follows:

(1) Resolve K by a cofibrant complex QK of semi-representables.
(2) Apply γ to each representable in QK obtaining a bicomplex $\gamma(QK)$ in $\mathcal{B}$.
(3) Take the total complex $\mathrm{Tot}^{\oplus}(\gamma(QK))$.

4. Local model structures

Having dealt with "generators" for universal enriched homotopy theories in §1 and for universal dg homotopy theories in more detail in the subsequent sections, we now turn to "relations". The only sort of relations we will be interested in here are "topological", i. e. induced by a Grothendieck topology on $\mathcal{C}$. Unfortunately we are not able to prove any substantial facts in the general enriched setting which is why we again restrict to the case of dg categories. Here, our main result is completely analogous to the main result of [20] where it is shown that a simplicial presheaf in the Jardine local model structure is fibrant if and only if it is injective fibrant and satisfies descent with respect to hypercovers.

Throughout this section we assume that $\mathcal{C}$ is endowed with a Grothendieck topology τ. Let $\mathbf{Sh}_\tau(\mathcal{C})$ (resp. $\mathbf{Sh}_\tau(\mathcal{C},\Lambda)$) denote the category of τ-sheaves (resp. of sheaves of Λ-modules) on $\mathcal{C}$. The embedding $\mathbf{Sh}_\tau(\mathcal{C}) \to \mathbf{PSh}(\mathcal{C})$ (resp. $\mathbf{Sh}_\tau(\mathcal{C},\Lambda) \to \mathbf{PSh}(\mathcal{C},\Lambda)$) is right adjoint to the sheafification functor a_τ.

4.1. Hypercovers and descent.

Recall ([20, §3]) that a morphism f of presheaves is a *generalized cover* if its sheafification $a_\tau(f)$ is an epimorphism.

Definition 4.1 For any object $c \in \mathcal{C}$ a *τ-hypercover of c* is a simplicial presheaf of sets $c_\bullet$ on $\mathcal{C}$ with an augmentation map $c_\bullet \to c =: c_{-1}$ such that

- c_n is a coproduct of representables for all $n \in \mathbb{N}$, and
- $c_n \to (\mathrm{cosk}_{n-1}\mathrm{sk}_{n-1}c_\bullet)_n$ is a generalized cover for all $n \in \mathbb{N}$.

(To avoid any confusion, the cases $n = 0, 1$ of the second bullet point require $c_0 \to c$ and $d_0 \times d_1 : c_1 \to c_0 \times_c c_0$ to be generalized covers, respectively.) A *refinement* of a hypercover $c_\bullet \to c$ is a hypercover $c'_\bullet \to c$ together with a morphism of simplicial presheaves $c'_\bullet \to c_\bullet$ compatible with the augmentation by c. The class of all τ-hypercovers of c is denoted by $\mathcal{H}_{\tau,c}$. Also set $\mathcal{H}_\tau := \coprod_{c\in\mathcal{C}} \mathcal{H}_{\tau,c}$. A subclass $\mathcal{H}$ of $\mathcal{H}_\tau$ (resp. $\mathcal{H}_{\tau,c}$) is called *dense* if every τ-hypercover (resp. of c) admits a refinement by a hypercover in $\mathcal{H}$.

We refer to [20] for details about hypercovers. In particular, we recall without proof the following important fact.

Fact 4.2 ([20, Pro. 6.7]) *For every $c \in \mathcal{C}$, there exists a dense subset of $\mathcal{H}_{\tau,c}$. Therefore also $\mathcal{H}_\tau$ admits a dense subset.*

In the case of simplicial presheaves the τ-hypercovers provide the "topological" relations in that the hypercover $c_\bullet$ and the representable c are "identified", and we want to translate these relations to the setting of presheaves of complexes. For this notice that given any hypercover $c_\bullet \to c$ we can use the Moore complex (cf. §2.3) to obtain an object $\Lambda(c_\bullet) \in \mathbf{U}\mathcal{C}$ together with a morphism $\Lambda(c_\bullet) \to \Lambda(c)$. Explicitly, $\Lambda(c_\bullet)$ is the complex

$$\cdots \to \Lambda(c_1) \to \Lambda(c_0) \to 0$$

with differentials given by the alternating sum of the face maps, and each $\Lambda(c_i)$ is semi-representable. It follows from Lemma 2.10 that $\Lambda(c_\bullet)$ is projective cofibrant.

Definition 4.3

(1) Let $\mathcal{S}$ be a class of τ-hypercovers. A presheaf $K \in \mathbf{U}\mathcal{C}$ *satisfies $\mathcal{S}$-descent* if for any τ-hypercover $c_\bullet \to c$ in $\mathcal{S}$,

$$K(c) = \underline{\hom}_{\mathrm{dg}}(\Lambda(c), K) \to \underline{\hom}_{\mathrm{dg}}(\Lambda(c_\bullet), K) =: K(c_\bullet)$$

is a quasi-isomorphism of chain complexes.

(2) $K \in \mathbf{U}\mathcal{C}$ *satisfies τ-descent* if it satisfies $\mathcal{H}_\tau$-descent.

Explicitly, $K(c_\bullet)$ is given by the product total complex of the bicomplex

$$K(c_0) \to K(c_1) \to \cdots,$$

where $K(\coprod_{i\in I} d_i)$ for $d_i \in \mathcal{C}$ is defined as $\prod_{i\in I} K(d_i)$.

Remark 4.4 The notion of satisfying descent is homotopy invariant, i. e. given two quasi-isomorphic presheaves of complexes, one satisfies $\mathcal{S}$-descent if and only if the other does. Indeed, as we know from Fact 1.8, $\underline{\hom}_{\mathrm{dg}} : (\mathbf{U}\mathcal{C})^{\mathrm{op}} \times \mathbf{U}\mathcal{C} \to \mathbf{Cpl}(\Lambda)$ is part of a Quillen adjunction of two variables. And since every object in $\mathbf{U}\mathcal{C}$ is fibrant, and since both $\Lambda(c_\bullet)$ and $\Lambda(c)$ are cofibrant (by Fact 2.10), the condition on K to satisfy descent is that

$$\mathrm{R}\,\underline{\hom}_{\mathrm{dg}}(\Lambda(c), K) \to \mathrm{R}\,\underline{\hom}_{\mathrm{dg}}(\Lambda(c_\bullet), K)$$

be an isomorphism in the derived category of Λ. This is different from the situation of simplicial presheaves of sets where $c_\bullet$ is not necessarily projective cofibrant. Thus the interest in *split* hypercovers, cf. [19, Cor. 9.4].

We end this section by the following important result. In terminology to be introduced shortly it tells us that the augmentation morphism $\Lambda(c_\bullet) \to \Lambda(c)$ associated to any τ-hypercover is a τ-local equivalence.

Fact 4.5 ([1, Thm. V, 7.3.2]) *Any τ-hypercover $c_\bullet \to c$ induces identifications*

$$a_\tau \mathrm{H}_n \Lambda(c_\bullet) \cong \begin{cases} a_\tau \Lambda(c) & : n = 0 \\ 0 & : n \neq 0 \end{cases}$$

in $\mathbf{Sh}(\mathcal{C}, \Lambda)$.

4.2. Localization.

Definition 4.6 A morphism f in $\mathbf{U}\mathcal{C}$ is called a *τ-local equivalence* if the induced morphism of homology sheaves $a_\tau \mathrm{H}_n(f)$ is an isomorphism for all $n \in \mathbb{Z}$.

The goal of this section is to prove the following theorem.

Theorem 4.7 *The left Bousfield localization* $\mathbf{U}\mathcal{C}/\tau$ *of* $\mathbf{U}\mathcal{C}$ *with respect to τ-local equivalences exists and satisfies:*

(1) The underlying category of $\mathbf{U}\mathcal{C}/\tau$ *is the one of* $\mathbf{U}\mathcal{C}$*. The cofibrations are also the same. The weak equivalences are the τ-local equivalences.*

(2) $\mathbf{U}\mathcal{C}/\tau$ *is a proper, tractable, cellular, stable model category.*

(3) *The fibrations of* $\mathbf{U}\mathcal{C}/\tau$ *are the fibrations of* $\mathbf{U}\mathcal{C}$ *whose kernel satisfies* τ*-descent. In particular, the fibrant objects of* $\mathbf{U}\mathcal{C}/\tau$ *are the objects satisfying* τ*-descent.*

The model structure on $\mathbf{U}\mathcal{C}/\tau$ *is called the* τ-local model structure.

Remark 4.8 This result was originally one of our main motivations to write the present chapter. The existence of this localization was known before, see [2, Def. 4.4.34], and we use this result in our proof. The main point of the theorem for us was part (3). The analogous description of the fibrant objects for simplicial sets instead of chain complexes is of course the main result of [20], and we deduce our result from theirs.

After having completed this chapter, we learned that also part (3) had appeared in the literature before, see [33]. His proof is different from ours in that he does not reduce to the case of simplicial sets nor uses the theory of Bousfield localizations but proves the axioms of a model structure "by hand".

Let $\mathcal{S} \subset \mathcal{H}_\tau$ be some class of τ-hypercovers. We denote by $\Lambda(\mathcal{S})[\mathbb{Z}]$ the class

$$\{\Lambda(c_\bullet)[n] \to \Lambda(c)[n] \mid c_\bullet \to c \in \mathcal{S}, n \in \mathbb{Z}\}$$

of morphisms in $\mathbf{U}\mathcal{C}$.

Definition 4.9

(1) Recall ([35, Def. 3.1.4]) that an object K in $\mathbf{U}\mathcal{C}$ is called *local* with respect to a class of morphisms $\mathcal{F}$ in $\mathbf{U}\mathcal{C}$ if for each $f \in \mathcal{F}$, the induced morphism of homotopy function complexes $\mathrm{Rmap}(f, K)$ is a weak homotopy equivalence of simplicial sets.

(2) Let $\mathcal{S}$ be a class of τ-hypercovers. We say that $K \in \mathbf{U}\mathcal{C}$ is $\mathcal{S}$*-local* if it is local with respect to $\Lambda(\mathcal{S})[\mathbb{Z}]$.

(3) We say that $K \in \mathbf{U}\mathcal{C}$ is τ*-local* if it is $\mathcal{H}_\tau$-local.

Lemma 4.10 *For a presheaf of complexes* $K \in \mathbf{U}\mathcal{C}$ *and a class* $\mathcal{S}$ *of* τ*-hypercovers the following two conditions are equivalent:*

(1) K *is* $\mathcal{S}$*-local.*

(2) K *satisfies* $\mathcal{S}$*-descent.*

In particular, the following two conditions are equivalent:

(1) K *is* τ*-local.*

(2) K *satisfies* τ*-descent.*

Proof. K is $\mathcal{S}$-local if and only if for any $c_\bullet \to c \in \mathcal{S}$, $n \in \mathbb{Z}$, the morphism of homotopy function complexes

$$\mathrm{Rmap}(\Lambda(c)[n], K) \to \mathrm{Rmap}(\Lambda(c_\bullet)[n], K) \tag{4.11}$$

is a weak equivalence of simplicial sets. But $\mathrm{Rmap}(A, B) \cong \Gamma\tau_{\geq 0}\mathbf{U}\mathcal{C}(A, B)$ by Lemma 2.17. So (4.11) is identified with

$$\Gamma\tau_{\geq -n}K(c) \to \Gamma\tau_{\geq -n}K(c_\bullet),$$

whose m-th homotopy group is thus

$$\mathrm{H}_{m-n}K(c) \to \mathrm{H}_{m-n}K(c_\bullet). \qquad \square$$

We will deduce Theorem 4.7 from the following (cf. [20, Thm. 6.2]).

Theorem 4.12 *Let* $\mathcal{S}$ *be a class of* τ*-hypercovers which contains a dense subset. Then the left Bousfield localization* $\mathbf{U}\mathcal{C}/\mathcal{S}$ *of* $\mathbf{U}\mathcal{C}$ *with respect to* $\Lambda(\mathcal{S})[\mathbb{Z}]$ *exists and coincides with* $\mathbf{U}\mathcal{C}/\tau$.

Proof of Theorem 4.7. Let $\mathcal{S}$ be the class of all τ-hypercovers. By Fact 4.2, $\mathcal{S}$ satisfies the assumption of Theorem 4.12. We know from Corollary 2.1 that $\mathbf{U}\mathcal{C}$ is left-proper, tractable, cellular. These are preserved by left Bousfield localizations by [35, Thm. 4.1.1] and [38, Pro. 4.3]. Since $\mathcal{S}$-local objects are closed under shifts by Lemma 4.10, $\mathbf{U}\mathcal{C}/\mathcal{S}$ and therefore $\mathbf{U}\mathcal{C}/\tau$ are stable model categories (see [8, Pro. 3.6]). Since $\mathbf{U}\mathcal{C}$ is a right proper model category so is $\mathbf{U}\mathcal{C}/\tau$ by [8, Pro. 3.7]. Since all objects are fibrant in $\mathbf{U}\mathcal{C}$, the fibrant objects of $\mathbf{U}\mathcal{C}/\mathcal{S}$ are the τ-local objects. We deduce from Lemma 4.10 and Theorem 4.12 that the fibrant objects of $\mathbf{U}\mathcal{C}/\tau$ are precisely the presheaves satisfying τ-descent. The description of the fibrations in $\mathbf{U}\mathcal{C}/\tau$ then follows from this and [8, Lem. 3.9]. Finally, that the weak equivalences of $\mathbf{U}\mathcal{C}/\tau$ are the τ-local equivalences is proven in [2, Pro. 4.4.32]. □

Assume for the moment that $\mathcal{S}$ in Theorem 4.12 is a set. In this case we know that the left Bousfield localization $\mathbf{U}\mathcal{C}/\mathcal{S}$ (resp. $\Delta^{\mathrm{op}}\,\mathbf{PSh}(\mathcal{C})/\mathcal{S}$) with respect to $\Lambda(\mathcal{S})[\mathbb{Z}]$ (resp. $\mathcal{S}$) exists. Temporarily, we call these model structures the $\mathcal{S}$-local model structures, their fibrations are called $\mathcal{S}$-fibrations, their weak equivalences are called $\mathcal{S}$-equivalences.

Lemma 4.13 *The Dold-Kan correspondence (Proposition 2.16) induces a Quillen adjunction*

$$(N\Lambda, \Gamma\tau_{\geq 0}) : \Delta^{\mathrm{op}}\,\mathbf{PSh}(\mathcal{C})/\mathcal{S} \longrightarrow \mathbf{U}\mathcal{C}/\mathcal{S}.$$

Moreover, $\Gamma\tau_{\geq 0}$ preserves τ-local equivalences.

Proof. Given $f : c_\bullet \to c \in \mathcal{S}$, the morphism $N\Lambda(f)$ factors as

$$N\Lambda(c_\bullet) \to \Lambda(c_\bullet) \xrightarrow{\Lambda(f)} \Lambda(c),$$

where the first arrow is a quasi-isomorphism by Fact 2.14, and the second arrow lies in $\Lambda(\mathcal{S})[\mathbb{Z}]$. Thus the first claim follows from the universal property of localizations. The second claim is also evident since the homotopy groups of $\Gamma\tau_{\geq 0}K$ are the homology groups of K in non-negative degrees, by Fact 2.14. □

Lemma 4.14 *Let $K, K' \in \mathbf{U}\mathcal{C}$ be $\mathcal{S}$-fibrant objects and let $f : K \to K'$ be an $\mathcal{S}$-fibration which is also a τ-local weak equivalence. Then f is a sectionwise trivial fibration, i. e. it is a trivial fibration in the projective model structure on $\mathbf{U}\mathcal{C}$.*

Proof. A morphism $f : K \to K' \in \mathbf{U}\mathcal{C}$ is a trivial fibration if and only if for all $c \in \mathcal{C}$ and all $n \in \mathbb{Z}$, f has the right lifting property with respect to $\partial\Delta^n\Lambda(c) \to \Delta^n\Lambda(c)$ (see Lemma 2.5).

Let $i : (\partial\Delta^1) \otimes c \to \Delta^1 \otimes c$ be the canonical cofibration of simplicial presheaves. Then $N\Lambda(i)$ is also a cofibration and $N\Lambda((\partial\Delta^1) \otimes c) = \partial\Delta^1\Lambda(c)$ and $N\Lambda(\Delta^1 \otimes c) = \Delta^1\Lambda(c)$. Also note that $\partial\Delta^n\Lambda(c) = \partial\Delta^1\Lambda(c)[-n+1]$, and similarly for $\Delta^n\Lambda(c)$. We want to show the existence of a lifting for every diagram of the following form

$$\begin{tikzcd} \partial\Delta^n\Lambda(c) \arrow[r] \arrow[d] & K \arrow[d] \\ \Delta^n\Lambda(c) \arrow[r] & K' \end{tikzcd}$$

Now using shifts this is the same as showing that

$$\begin{tikzcd} \partial\Delta^1\Lambda(c) \arrow[r] \arrow[d] & K[n-1] \arrow[d] \\ \Delta^1\Lambda(c) \arrow[r] & K'[n-1] \end{tikzcd}$$

has a lift. Notice that the right vertical arrow is still an $\mathcal{S}$-fibration.

But using the adjunction of Lemma 4.13 this is the same as showing that

$$\begin{tikzcd} \partial\Delta^1 \otimes c \arrow[r] \arrow[d] & \Gamma\tau_{\geq 0}(K[n-1]) \arrow[d] \\ \Delta^1 \otimes c \arrow[r] & \Gamma\tau_{\geq 0}(K'[n-1]) \end{tikzcd}$$

has a lift, where we know that the right vertical arrow is an $\mathcal{S}$-fibration and τ-local equivalence between $\mathcal{S}$-fibrant objects. Hence by [20, Lem. 6.5] it is a trivial fibration sectionwise. Now $i : (\partial\Delta^1) \otimes c \to \Delta^1 \otimes c$ is a projective cofibration, hence there is a lift in the last diagram above. This finishes the proof. $\square$

Proof of Theorem 4.12. Let $\mathcal{S}$ be as in the theorem, and pick a dense subset $\mathcal{S}'$ of $\mathcal{S}$.

We claim that the $\mathcal{S}'$-local equivalences in $\mathbf{U}\mathcal{C}$ are precisely the τ-local equivalences. Indeed, by Fact 4.5, every $\mathcal{S}'$-local equivalence is a τ-local equivalence. For the converse, we may apply [20, Lem 6.4] together with Lemma 4.14 (we also use the existence of the τ-local model structure, see Remark 4.8). This proves the claim which in turn implies that $\mathbf{U}\mathcal{C}/\mathcal{S}' = \mathbf{U}\mathcal{C}/\tau$.

We deduce that every hypercover in $\mathcal{S}$ is an $\mathcal{S}'$-equivalence hence the localization of $\mathbf{U}\mathcal{C}$ with respect to $\Lambda(\mathcal{S})[\mathbb{Z}]$ exists and coincides with $\mathbf{U}\mathcal{C}/\mathcal{S}'$. $\square$

Let us agree to call a model category $\mathcal{M}$ equipped with a functor $\gamma : \mathcal{C} \to \mathcal{M}$ *τ-local* if for every τ-hypercover $c_\bullet \to c$ in $\mathcal{C}$, $\mathrm{L}\operatorname{colim}_{\Delta^{\mathrm{op}}} \gamma(c_\bullet) \to \gamma(c)$ is an isomorphism in $\mathbf{Ho}(\mathcal{M})$. In line with the viewpoint taken in §1 let us record the following corollary of Theorem 4.7. It asserts that $\mathbf{U}_{\mathrm{dg}}\mathcal{C}/\tau$ is the universal τ-local model dg category associated to $\mathcal{C}$.

Corollary 4.15 *Let $(\mathcal{C}, \tau)$ be a small site. Then there exists a functor $\Lambda : \mathcal{C} \to \mathbf{U}_{\mathrm{dg}}\mathcal{C}/\tau$ into a τ-local model dg category, universal in the sense that for any solid diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Lambda"] \arrow[dr, "\gamma"'] & \mathbf{U}_{\mathrm{dg}}\mathcal{C}/\tau \arrow[d, dotted, "L"] \\ & \mathcal{M} \end{tikzcd}$$

with $\mathcal{M}$ a τ-local model dg category, there exists a left Quillen dg functor L as indicated by the dotted arrow, unique up to a contractible choice, making the diagram commutative up to a weak equivalence $L\Lambda \to \gamma$.

Proof. $\mathbf{U}_{\mathrm{dg}}\mathcal{C}/\tau$ as a dg category is just $\mathbf{U}_{\mathrm{dg}}\mathcal{C}$ and the cofibrations are the same hence to prove that $\mathbf{U}_{\mathrm{dg}}\mathcal{C}/\tau$ is a model dg category, it suffices to see that the pushout-product $i \,\square\, f$ is a τ-weak equivalence for every cofibration i in $\mathbf{Cpl}(\Lambda)$ and every τ-acyclic cofibration $f \in \mathbf{U}\mathcal{C}$. This can be established exactly as in the proof of [9, Thm. 4.46]. (For this step it is not necessary to assume as is done in loc. cit. that the localization is with respect to a set but only that it exists.) The essential point is that $\mathbf{U}\mathcal{C}$ is a tractable model category (by Proposition 2.1).

Next we claim that $\mathrm{L}\operatorname{colim}_{\Delta^{\mathrm{op}}} \Lambda(c_\bullet) \to \Lambda(c)$ is an isomorphism in $\mathbf{Ho}(\mathbf{U}_{\mathrm{dg}}\mathcal{C}/\tau)$. But by Lemma 2.20, this morphism can be identified with $\Lambda(c_\bullet) \to \Lambda(c)$ hence the claim follows from Lemma 4.5.

Given a solid diagram as in the statement of the corollary we know by Corollary 1.11 the existence of a left Quillen dg functor $L : \mathbf{U}_{\mathrm{dg}}\mathcal{C} \to \mathcal{M}$, unique up to contractible choice, making the triangle commutative up to a weak equivalence $Ly \to \gamma$. By the universal property of the localization of model categories together with Theorem 4.12, it now suffices to prove that the left derived functor $\mathrm{L}L$ takes $\Lambda(\mathcal{H}_\tau)[\mathbb{Z}]$ to isomorphisms in $\mathbf{Ho}(\mathcal{M})$. Thus let $c_\bullet \to c \in \mathcal{H}_\tau$ and $n \in \mathbb{Z}$. First notice that L "commutes with shifts" in the sense that

$$L(\bullet[n]) \cong L(S^n \odot \bullet) \cong S^n \odot L(\bullet),$$

and since $\mathcal{M}$ is a model dg category, $S^n \odot \bullet$ preserves weak equivalences. We thus reduce to the case $n = 0$.

Now, again by Lemma 2.20, $\Lambda(c_\bullet)$ can be identified with the homotopy colimit of $\Lambda(c_\bullet)$. Since L is a left Quillen dg functor it will commute with homotopy colimits in the homotopy category. Thus we want the upper row in the following commutative square to be invertible in $\mathbf{Ho}(\mathcal{M})$.

$$\begin{tikzcd} \mathrm{L}\operatorname{colim}_{\Delta^{\mathrm{op}}} L\Lambda(c_\bullet) \arrow[r] \arrow[d] & L\Lambda(c) \arrow[d] \\ \mathrm{L}\operatorname{colim}_{\Delta^{\mathrm{op}}} \gamma(c_\bullet) \arrow[r] & \gamma(c) \end{tikzcd}$$

Our assumptions tell us that the vertical arrows as well as the bottom arrow are isomorphisms so we conclude. □

4.3. Smaller models. Having described explicitly generators and relations for the model dg category $\mathbf{U}\mathcal{C}/\tau$ associated to a small site $(\mathcal{C}, \tau)$, we give in this section two methods to modify the model $\mathbf{U}\mathcal{C}/\tau$ up to Quillen equivalence which are useful in practice. The first consists in replacing presheaves by sheaves, the second allows to reduce the "number" of generators. In both cases therefore we obtain "smaller" models with the same homotopy category. Both modifications are straightforward and have been employed before in the literature.

The category of τ-sheaves of complexes on $\mathcal{C}$, $\mathbf{Sh}_\tau(\mathcal{C}, \mathbf{Cpl}(\Lambda))$, admits the τ-local model structure, obtained by transfer along the right adjoint $\mathbf{Sh}_\tau(\mathcal{C}, \mathbf{Cpl}(\Lambda)) \to \mathbf{PSh}(\mathcal{C}, \mathbf{Cpl}(\Lambda))$ (cf. [2, Cor. 4.4.43]). Since the morphism $K \to a_\tau K$ is a τ-local equivalence for every $K \in \mathbf{U}\mathcal{C}$, the following statement is immediate.

Fact 4.16

$$\mathbf{U}\mathcal{C}/\tau \underset{}{\overset{a_\tau}{\rightleftarrows}} \mathbf{Sh}_\tau(\mathcal{C}, \mathbf{Cpl}(\Lambda))/\tau$$

defines a Quillen equivalence. Their homotopy categories are the derived category of τ-sheaves on $\mathcal{C}$.

It happens frequently that every object $c \in \mathcal{C}$ can be covered by objects belonging to a distinguished strict subcategory $\mathcal{C}'$. Certainly one then expects the model dg categories generated by $\mathcal{C}$ and $\mathcal{C}'$ with the topological relations to be "the same". The following result makes this precise.

Corollary 4.17 *Let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$, and endow it with the topology τ' induced from τ. Assume that every object $c \in \mathcal{C}$ can be covered by objects belonging to $\mathcal{C}'$. Then the (functor underlying the) canonical dg functor*

$$\mathbf{U}\mathcal{C}'/\tau' \longrightarrow \mathbf{U}\mathcal{C}/\tau$$

defines a Quillen equivalence.

Proof. The composition $\mathcal{C}' \xrightarrow{u} \mathcal{C} \to \mathbf{U}\mathcal{C}/\tau$ induces the left Quillen dg functor $u_!$ in the statement by the universal property of $\mathbf{U}\mathcal{C}'/\tau'$ (Corollary 4.15), left-adjoint to the restriction functor u^*. Consider the square of Quillen right functors:

$$\begin{tikzcd}
\mathbf{Sh}_\tau(\mathcal{C}, \mathbf{Cpl}(\Lambda))/\tau \arrow[r, "u^*"] \arrow[d] & \mathbf{Sh}_{\tau'}(\mathcal{C}', \mathbf{Cpl}(\Lambda))/\tau' \arrow[d] \\
\mathbf{U}\mathcal{C}/\tau \arrow[r, "u^*"'] & \mathbf{U}\mathcal{C}'/\tau'
\end{tikzcd}$$

Clearly, it commutes. By the previous fact, the vertical arrows are part of a Quillen equivalence, and the homotopy categories in the top row are the derived categories of τ-sheaves (resp. τ'-sheaves) on $\mathcal{C}$ (resp. $\mathcal{C}'$). By [1, Thm. III.4.1], the top arrow is an equivalence of the underlying categories hence so is the induced functor on their derived categories. □

4.4. Hypercohomology. One might hope that the results obtained so far in this section allow to describe a τ-fibrant replacement directly in terms of hypercovers. In particular, this would lead to an expression for the hypercohomology of complexes of sheaves using hypercovers alone. We have not been able to provide such a fibrant replacement but, as we will now show, the hypercohomology does indeed admit such an expected description. This result should be compared to Verdier's hypercover theorem in [1, Thm. V, 7.4.1]. Our proof once again proceeds by reducing to the case of simplicial (pre)sheaves of sets in [20]. (In the following, we write H^n for H_{-n}.)

Proposition 4.18 *Assume that every τ-hypercover can be refined by a split one. Let $K \in \mathbf{U}\mathcal{C}$ be a presheaf of complexes on $\mathcal{C}$, $c \in \mathcal{C}$, and $n \in \mathbb{Z}$. Then there is a canonical isomorphism of Λ-modules*

$$\mathbb{H}^n_\tau(c, a_\tau K) \cong \operatorname*{colim}_{c_\bullet \to c} \mathrm{H}^n K(c_\bullet),$$

where the left hand side denotes hypercohomology of the complex of τ-sheaves $a_\tau K$ on $\mathcal{C}/c$, and the colimit on the right hand side is over the opposite category of τ-hypercovers of c up to simplicial homotopy (cf. [1, §V.7.3]).

Proof. This follows from the following sequence of isomorphisms:

$$\begin{aligned}
\mathbb{H}^n_\tau(c, a_\tau K) &\cong \hom_{\mathbf{Ho}(\mathbf{U}\mathcal{C}/\tau)}(\Lambda(c), a_\tau K[-n]) && \text{Corollary 2.18} \\
&\cong \hom_{\mathbf{Ho}(\mathbf{U}\mathcal{C}/\tau)}(\Lambda(c), K[-n]) && K \to a_\tau K \ \tau\text{-local equivalence} \\
&\cong \hom_{\mathbf{Ho}(\Delta^{\mathrm{op}}\mathbf{PSh}(\mathcal{C})/\tau)}(c, \Gamma\tau_{\geq -n} K) && \text{Lemma 4.13} \\
&\cong \operatorname*{colim}_{c_\bullet \to c} \pi(c_\bullet, \Gamma\tau_{\geq -n} K) && \text{[20, Thm. 7.6(b)]} \\
&\cong \operatorname*{colim}_{c_\bullet \to c \text{ split}} \pi(c_\bullet, \Gamma\tau_{\geq -n} K) && \text{assumption} \\
&\cong \operatorname*{colim}_{c_\bullet \to c \text{ split}} \hom_{\mathbf{Ho}(\Delta^{\mathrm{op}}\mathbf{PSh}(\mathcal{C}))}(c_\bullet, \Gamma\tau_{\geq -n} K) && \text{split hypercovers cofibrant} \\
&\cong \operatorname*{colim}_{c_\bullet \to c \text{ split}} \hom_{\mathbf{Ho}(\mathbf{U}\mathcal{C})}(N\Lambda(c_\bullet), K[-n]) && \text{Lemma 2.16} \\
&\cong \operatorname*{colim}_{c_\bullet \to c \text{ split}} \hom_{\mathbf{Ho}(\mathbf{U}\mathcal{C})}(\Lambda(c_\bullet), K[-n]) && \text{Lemma 2.14} \\
&\cong \operatorname*{colim}_{c_\bullet \to c \text{ split}} \mathrm{H}^n K(c_\bullet) && \text{Corollary 2.18} \\
&\cong \operatorname*{colim}_{c_\bullet \to c} \mathrm{H}^n K(c_\bullet) && \text{assumption}
\end{aligned}$$

□

Remark 4.19 The hypothesis of the Proposition, i. e. that every hypercover admits a split refinement, is satisfied in many cases, e. g. when $(\mathcal{C}, \tau)$ is a Verdier site, see [20, Thm. 8.6].

Moreover, in these cases the proposition represents another approach to Theorem 4.7. Indeed, the essential point, as we mentioned in Remark 4.8, is the description of the τ-fibrant objects in $\mathbf{U}\mathcal{C}/\tau$. Since $\Lambda(c_\bullet) \to \Lambda(c)$ is a τ-local equivalence for each τ-hypercover $c_\bullet \to c$ (Fact 4.5) it is clear that τ-fibrant objects satisfy τ-descent. Conversely, suppose $K \in \mathbf{U}\mathcal{C}$ satisfies τ-descent and choose a τ-fibrant replacement $f : K \to K'$. Using the previous proposition we will prove that f is a quasi-isomorphism.

Fix $c \in \mathcal{C}$ and $n \in \mathbb{Z}$. Consider the following commutative diagram:

$$\begin{tikzcd}
\operatorname{colim}_{c_\bullet \to c} \mathrm{H}^n K(c_\bullet) \arrow[r, "\sim"] & \hom_{\mathbf{Ho}(\mathbf{U}\mathcal{C}/\tau)}(\Lambda(c), K[-n]) \arrow[d, "f"] \\
\mathrm{H}^n K(c) \arrow[u] \arrow[r, "f"'] & \mathrm{H}^n K'(c)
\end{tikzcd}$$

The left vertical arrow is an isomorphism since K satisfies τ-descent. The right vertical arrow is an isomorphism since K' is τ-fibrant. Thus the claim.

4.5. Complements. In this last paragraph we discuss two further aspects of the local dg homotopy theory: monoidal structures, and closure of fibrant objects under certain operations.

Proposition 4.20 *Assume that either of the following conditions is satisfied:*

(1) $\mathcal{C}$ is cartesian monoidal.

(2) For any objects $c, d \in \mathcal{C}$, $\Lambda(c) \otimes \Lambda(d)$ is projective, and $(\mathcal{C}, \tau)$ has enough points.

Then $\mathbf{U}\mathcal{C}/\tau$ is a symmetric monoidal model category for the objectwise tensor product.

Proof.

(1) If $\mathcal{C}$ is cartesian monoidal, we may adapt the proof of [9, Thm. 4.58]. By [9, Pro. 4.47], it suffices to prove that for each $d \in \mathcal{C}$, and each τ-local $K \in \mathbf{U}\mathcal{C}$, the internal hom object $[\Lambda(d), K]$ is τ-local. Thus let $c_\bullet \to c$ be a τ-hypercover. Using the commutative diagram

$$\begin{tikzcd}
\underline{\hom}_{\mathrm{dg}}(\Lambda(c), [\Lambda(d), K]) \arrow[r] \arrow[d, "\sim"'] & \underline{\hom}_{\mathrm{dg}}(\Lambda(c_\bullet), [\Lambda(d), K)] \arrow[d, "\sim"] \\
\underline{\hom}_{\mathrm{dg}}(\Lambda(c) \otimes \Lambda(d), K) \arrow[r] \arrow[d, "\sim"'] & \underline{\hom}_{\mathrm{dg}}(\Lambda(c_\bullet) \otimes \Lambda(d), K) \arrow[d, "\sim"] \\
\underline{\hom}_{\mathrm{dg}}(\Lambda(c \times d), K) \arrow[r] & \underline{\hom}_{\mathrm{dg}}(\Lambda(c_\bullet \times d), K)
\end{tikzcd}$$

we reduce to showing that $c_\bullet \times d \to c \times d$ is a τ-local equivalence of simplicial presheaves. This follows from the fact that homotopy groups and sheafification commute with finite products.

(2) By Lemma 2.13, $\mathbf{U}\mathcal{C}$ is a symmetric monoidal model category. The result then follows from the proof of [2, Pro. 4.4.63] (whereas the statement in loc. cit. misses the first hypothesis above). □

Our description of τ-fibrant objects in Theorem 4.7 allows one to prove easily that these are closed under various operations. In the following lemmas we discuss two examples.

Lemma 4.21 *Let $K_\bullet$ be a bounded complex of τ-fibrant objects in* $\mathbf{U}\mathcal{C}$. *Then* $\mathrm{Tot}^\oplus K_\bullet \in \mathbf{U}\mathcal{C}$ *is τ-fibrant.*

Proof. Let $c_\bullet \to c$ be a τ-hypercover. We know that for any $l \in \mathbb{Z}$, $K_l(c) \to K_l(c_\bullet)$ is a quasi-isomorphism. Since $K_\bullet$ is bounded below, it follows from Lemma 3.2 that also

$$\mathrm{Tot}^\oplus(K_\bullet(c)) \to \mathrm{Tot}^\oplus(K_\bullet(c_\bullet))$$

is a quasi-isomorphism. Since $K_\bullet$ is bounded (hence $\mathrm{Tot}^\oplus$ and Tot^Π agree), one easily checks that this morphism can be identified with

$$(\mathrm{Tot}^\oplus K_\bullet)(c) \to (\mathrm{Tot}^\oplus K_\bullet)(c_\bullet). \qquad \square$$

Let κ be a regular cardinal. We say that the site $(\mathcal{C}, \tau)$ is *κ-noetherian* if every cover $\{c_i \to c\}_{i\in I}$ has a subcover $\{c_i \to c\}_{i\in J\subset I}$ with $|J| < \kappa$. An $\aleph_0$-noetherian site is called simply *noetherian*, as in [56, §III.3]. Also, recall the notion of Verdier sites from [20, Def. 8.1].

Lemma 4.22 *Let $(\mathcal{C}, \tau)$ be a κ-noetherian Verdier site, $\kappa > \aleph_0$. Then τ-fibrant objects in* $\mathbf{U}\mathcal{C}$ *are closed under κ-filtered colimits.*

Proof. By [20, Rem. 8.7], there is a dense set of τ-hypercovers $\mathcal{S}$ such that for each $c_\bullet \to c \in \mathcal{S}$ and each $n \in \mathbb{N}$, c_n is a coproduct $c_n \cong \coprod_{i\in I_n} c_{n,i}$ with $c_{n,i}$ representable and $|I_n| < \kappa$. By Theorem 4.12 and Lemma 4.10, being τ-fibrant is equivalent to satisfying $\mathcal{S}$-descent. Now let $K : J \to \mathbf{U}\mathcal{C}$ be a κ-filtered diagram of τ-fibrant objects, and $c_\bullet \to c \in \mathcal{S}$. The claim then follows from the isomorphism

$$\begin{aligned}
(\operatorname*{colim}_j K(j))(c_\bullet) &\cong \mathrm{Tot}^\Pi(\operatorname*{colim}_j K(j)_p(c_q))_{p,q}\\
&\cong \mathrm{Tot}^\Pi(\prod_{i\in I_q} \operatorname*{colim}_j K(j)_p(c_{q,i}))_{p,q}\\
&\cong \operatorname*{colim}_j \mathrm{Tot}^\Pi(\prod_{i\in I_q} K(j)_p(c_{q,i}))_{p,q}\\
&\cong \operatorname*{colim}_j(K_j(c_\bullet)),
\end{aligned}$$

as κ-filtered colimits commute with products indexed by cardinals smaller than κ. $\square$

Lemma 4.23 *Let $(\mathcal{C}, \tau)$ be a noetherian Verdier site. Any filtered colimit of bounded above τ-fibrant objects in* $\mathbf{U}\mathcal{C}$ *is τ-fibrant.*

Proof. The proof is essentially the same as in the previous lemma. We must assume bounded above objects so that the product totalization involves only finitely many factors in each degree hence commutes with filtered colimits. $\square$

5. Fibrant replacement

In this section we would like to give an "explicit" fibrant replacement functor in $\mathbf{U}\mathcal{C}/\tau$ using the Godement resolution. It is a direct translation of the analogous construction for simplicial (pre)sheaves in [57, p. 66ff], with, again, the only problem created by the unboundedness of our complexes. We first establish the tools to overcome this difficulty.

5.1. Local model structure and truncation. Let $n \in \mathbb{Z}$ and consider the functor $\Gamma\tau_{\geq n} : \mathbf{U}\mathcal{C} \to \Delta^{\mathrm{op}}\,\mathbf{PSh}(\mathcal{C})$. Applying it objectwise, this generalizes to a functor defined on diagrams with values in $\mathbf{U}\mathcal{C}$ which we still denote by $\Gamma\tau_{\geq n}$.

Lemma 5.1 *The canonical arrow*

$$\mathrm{R}\lim_{\Delta}\Gamma\tau_{\geq n}K \to \Gamma\tau_{\geq n}\mathrm{R}\lim_{\Delta}K$$

is a weak homotopy equivalence for every $K \in (\mathbf{U}\mathcal{C})^{\Delta}$.

Proof. One way to see this is as follows. $\Gamma\tau_{\geq n}$ is a right Quillen functor for the projective model structures on $\mathcal{M} := \mathbf{U}\mathcal{C}$ and $\mathcal{N} := \mathbf{PSh}(\mathcal{C}, \Delta^{\mathrm{op}}\mathrm{Set})$. It follows that the induced morphism of derivators $\mathbb{D}_{\mathcal{M}} \to \mathbb{D}_{\mathcal{N}}$ is continuous (see [14, Pro. 6.12]), in particular it commutes with homotopy limits. The claim now follows from the fact that $\Gamma\tau_{\geq n}$ takes quasi-isomorphisms to weak homotopy equivalences hence doesn't need to be derived. □

Proposition 5.2

(1) For a morphism $f : K \to K'$ in $\mathbf{U}\mathcal{C}$ the following are equivalent:
 (a) f is a τ-local equivalence.
 (b) $\Gamma\tau_{\geq n}f$ is a τ-local equivalence for all $n \in \mathbb{Z}$.
 (c) $\Gamma\tau_{\geq n}f$ is a τ-local equivalence for $n \ll 0$.

(2) For $K \in \mathbf{U}\mathcal{C}$ the following are equivalent:
 (a) K is τ-fibrant.
 (b) $\Gamma\tau_{\geq n}K$ is τ-fibrant for all $n \in \mathbb{Z}$.
 (c) $\Gamma\tau_{\geq n}K$ is τ-fibrant for $n \ll 0$.

Proof.

(1) This is obvious since τ-local equivalences are defined via (the sheafification of) the homology groups which coincide with the homotopy groups after applying Γ.

(2) The implication "(a)⇒(b)" follows from Lemma 4.13. The implication "(b)⇒(c)" is trivial. For the implication "(c)⇒(a)" let $f : K \to K'$ be a τ-fibrant replacement. Again by Lemma 4.13, $\Gamma\tau_{\geq n}(f)$ is a τ-local equivalence between τ-fibrant objects hence it is a sectionwise weak equivalence. It follows that $\tau_{\geq n}(f)$ is a sectionwise weak equivalence. As f is the filtered colimit of $\tau_{\geq n}(f)$, f is a sectionwise weak equivalence. □

5.2. Godement resolution. Now suppose that $(\mathcal{C}, \tau)$ has enough points. This means that there is a set $\mathcal{P}$ of morphisms of sites $p : \mathrm{Set} \to (\mathcal{C}, \tau)$ such that a morphism f of sheaves of sets on $\mathcal{C}$ is an isomorphism if and only if p^*f is an isomorphism for all $p \in \mathcal{P}$. There is an induced morphism of sites $\mathrm{Set}^{\mathcal{P}} \to (\mathcal{C}, \tau)$, and we denote by $(a^*, a_*) : \mathbf{U}\mathcal{C} \to \mathbf{Cpl}(\Lambda)^{\mathcal{P}}$ the induced adjunction. The associated comonad induces functorially for each $K \in \mathbf{U}\mathcal{C}$ a coaugmented cosimplicial object $K \to G^{\bullet}(K)$, where $G^n(K) = (a_*a^*)^{n+1}(K) \in \mathbf{U}\mathcal{C}$. The *Godement resolution* of K is defined to be

$$\mathcal{G}(K) := \mathrm{Tot}^{\Pi}(G^{\bullet}(K))$$

which according to Lemma 2.21 is a model for $\mathrm{R}\lim_{\Delta} G^{\bullet}(K)$.

Recall [57, Def. 1.31] that the site $(\mathcal{C}, \tau)$ is said to be of finite type if "Postnikov towers converge".

Theorem 5.3 *There is a functor $\mathcal{G} : \mathbf{U}\mathcal{C} \to \mathbf{U}\mathcal{C}$ and a natural transformation $\mathrm{id} \to \mathcal{G}$ satisfying:*

(1) $\mathcal{G}$ is an exact functor of abelian categories.

(2) $\mathcal{G}$ takes each presheaf of complexes to a τ-fibrant sheaf of complexes.

(3) $\mathcal{G}$ takes fibrations (i. e. degreewise surjections) to τ-fibrations.

(4) If $(\mathcal{C}, \tau)$ is a finite type site, then $K \to \mathcal{G}(K)$ is a τ-local equivalence for any K.

PROOF.

(1) $\mathcal{G}$ is the composition of exact functors thus exact.

(2) We use Proposition 5.2 to check that $\mathcal{G}(K)$ is τ-fibrant. Thus let $n \in \mathbb{Z}$, and $c_\bullet \to c$ a τ-hypercover. We need to check that

$$\Gamma\tau_{\geq n}\mathcal{G}(K)(c) \to \mathrm{R}\lim_{\Delta} \Gamma\tau_{\geq n}\mathcal{G}(K)(c_\bullet)$$

is a weak homotopy equivalence. This will follow from [57, Pro. 1.59] if we can prove that the canonical arrow

$$\mathcal{G}(\Gamma\tau_{\geq n}K) \to \Gamma\tau_{\geq n}\mathcal{G}(K)$$

is an objectwise weak homotopy equivalence, where the left hand side denotes the Godement resolution for simplicial (pre)sheaves as defined in [57, p. 66], analogous to our construction above. By Lemma 5.1, we see that $\Gamma\tau_{\geq n}$ commutes with $\mathrm{R}\lim_{\Delta}$ up to objectwise weak equivalence, so we reduce to show that it also commutes with $a_* a^*$ up to objectwise weak equivalence.

$a_* a^*$ is applied degreewise and is a composition of left-exact functors hence clearly commutes with $\tau_{\geq n}$. It is also clear that $a_* a^*$ commutes with the Moore complex functor therefore the same holds for the quasi-inverse Γ. Finally, $a_* a^*$ commutes with the forgetful functor $\mathbf{Mod}(\Lambda) \to \mathrm{Set}$.

(3) Let f be an epimorphism with kernel K in $\mathbf{U}\mathcal{C}$. By part (1), $\mathcal{G}(f)$ is an epimorphism with kernel $\mathcal{G}(K)$, which is τ-fibrant by part (2). $\mathcal{G}(f)$ is thus a τ-fibration by Theorem 4.7.

(4) Again, by Proposition 5.2, we need to check that

$$\Gamma\tau_{\geq n}K \to \Gamma\tau_{\geq n}\mathcal{G}(K)$$

is a τ-local equivalence for all $n \in \mathbb{Z}$. But by the same reasoning as in part (2), the target of this morphism is identified (up to sectionwise weak homotopy equivalence) with $\mathcal{G}(\Gamma\tau_{\geq n}K)$ hence the claim follows from [57, Pro. 1.65].

□

IV

AN ISOMORPHISM OF MOTIVIC GALOIS GROUPS

Motives and motivic Galois groups. Let k be a field of characteristic 0. Following Grothendieck, there should be an abelian category $\mathcal{MM}(k)$ of *(mixed) motives* over k together with a functor $M : (\mathrm{Var}/k)^{\mathrm{op}} \to \mathcal{MM}(k)$, associating to each variety X/k its motive $M(X)$, the universal cohomological invariant of X. Every cohomology theory $h : (\mathrm{Var}/k)^{\mathrm{op}} \to \mathcal{A}$ for varieties over k should factor through a realization functor $R_h : \mathcal{MM}(k) \to \mathcal{A}$, i. e. $h(X) = R_h(M(X))$. For some cohomology theories h one would expect this realization functor to present $\mathcal{MM}(k)$ as a neutral Tannakian category with Tannakian dual $\mathcal{G}(k, h)$, a pro-algebraic group called the *motivic Galois group* (of k associated to h). One of the main practical advantages of the Tannakian formalism for motives is that it would allow the translation of arithmetic and geometric questions about k-varieties into questions about (pro-)algebraic groups and their representations. Moreover, the maximal pro-reductive quotient of this group is supposed to coincide with what was classically known as the motivic Galois group, namely the group associated to the Tannakian subcategory of *pure* motives over k (i. e. the universal cohomology theory for *smooth projective* varieties; see [67] for the philosophy underlying this smaller group).

Although this picture is still conjectural, there are candidates for these objects and related constructions. Assume there is an embedding $\sigma : k \hookrightarrow \mathbb{C}$. In this situation there are essentially two existing approaches to motives, one due to Nori and another due to several mathematicians, including Voevodsky. Nori constructed a diagram of pairs of varieties together with the Betti representation into finite dimensional $\mathbb{Q}$-vector spaces, and applied to it his theory of Tannaka duality for diagrams. It yields a universal factorization for the Betti representation through a $\mathbb{Q}$-linear abelian category with a faithful exact $\mathbb{Q}$-linear functor o_{Bti} to finite dimensional $\mathbb{Q}$-vector spaces. After inverting the Tate twist, one obtains the category of Nori motives (over k). This is a Tannakian category with fiber functor o_{Bti}, whose Tannakian dual $\mathcal{G}_{\mathrm{N}}(k)$ is defined to be Nori's motivic Galois group.

On the other hand, there is the better known construction of $\mathbf{DM}(k)$, the triangulated category of Voevodsky motives (this is a candidate not for the category of motives but its derived category). Ayoub constructed a Betti realization functor $\mathrm{Bti}^* : \mathbf{DM}(k) \to \mathbf{D}(\mathbb{Q})$ to the category of graded $\mathbb{Q}$-vector spaces. He also proved that this functor together with its right adjoint Bti_* satisfies the assumptions of his weak Tannakian formalism which in the case at hand endows $\mathrm{Bti}^*\mathrm{Bti}_*\mathbb{Q}$ with the structure of a Hopf algebra. Ayoub's motivic Galois group $\mathcal{G}_{\mathrm{A}}(k)$ is the spectrum of the zeroth homology of this Hopf algebra.

Main result. Our main goal in the present chapter is to prove that the two motivic Galois groups just described are isomorphic, thus answering a question of Ayoub in [5].

Theorem (instance of 8.1) There is an isomorphism of affine pro-algebraic groups over $\mathrm{Spec}(\mathbb{Q})$:

$$\mathcal{G}_{\mathrm{A}}(k) \cong \mathcal{G}_{\mathrm{N}}(k).$$

Let us try to put this result into perspective. In contrast to the approach by Nori which relies essentially on transcendental data, $\mathbf{DM}(k)$ is defined purely in terms of algebro-geometric data and therefore provides a means to understand algebro-geometric invariants of varieties. As explained in [4], the difference between the two approaches is extreme, and one of the ultimate goals in the theory of motives is to create a bridge connecting the two. This goal is considered at the moment to be far out of reach, but the result above can be seen as providing a weak link while sidestepping the more difficult and deep issues. This is possible because Ayoub's construction of the motivic Galois group incorporates transcendental data through the Betti realization. Viewed from a different angle, the result breaks up the ultimate goal into two subgoals which consist in understanding the relation between (compact) Voevodsky motives and comodules over $\mathrm{Bti}^*\mathrm{Bti}_*\mathbb{Q}$ on the one hand, and proving that this Hopf algebra is homologically concentrated in degree 0 on the other hand; see [4] and [5, §2.4] for further discussion.

Even if the link we provide here is a weak one, it can still be seen as an argument for the "correctness" of the two approaches to motives. Moreover, although both constructions of the motivic Galois group are based on some form of Tannaka duality, the precise form is quite different in the two cases (cf. [5, Introduction]); therefore the isomorphism in the theorem can be seen as a surprising phenomenon. Finally, the identification of the two groups allows for transfer of techniques and results, not easily available on both sides without the identification. We plan to use this fact in the future to give a more elementary description of the Kontsevich-Zagier period algebra with fewer generators and relations. More precisely, we intend to show that the algebra considered in [7, §2.2] is canonically isomorphic to the Kontsevich-Zagier period algebra, as was claimed in *loc. cit.*

We would like to remark that a conditional proof of our main result has been given independently by Jon Pridham. In [61, Exa. 3.20] he sketches how the existence of a motivic t-structure (which renders the Betti realization t-exact) would imply the isomorphism of motivic Galois groups. The argument uses the theory of Tannaka duality for dg categories developed in *loc. cit.*

About the proof. As one would expect from the relation between motives and their associated Galois groups, proving our main result involves "comparing" Nori motives with Voevodsky motives. As we remarked above, this is a non-trivial task and we can only hope to relate these two categories indirectly:

- We construct a realization of Nori motives in the category of linear representations of Ayoub's Galois group:

  $$\mathbf{Rep}(\mathcal{G}_{\mathrm{N}}(k)) \to \mathbf{Rep}(\mathcal{G}_{\mathrm{A}}(k)).$$

  The main ingredients used in this construction are the six functors formalism for motives without transfers developed by Ayoub, and its compatibility with the Betti realization, also proved by Ayoub.
- We construct a realization of motives without transfers in the category of graded $\mathcal{O}(\mathcal{G}_{\mathrm{N}}(k))$-comodules:

  $$\mathbf{DA}(k) \to \mathbf{coMod}^{\mathbb{Z}}(\mathcal{O}(\mathcal{G}_{\mathrm{N}}(k))).$$

In this construction the main tool used is the Basic Lemma due to Nori and, independently, to Beilinson.

In fact, we work throughout with arbitrary principal ideal domains as coefficients, not only $\mathbb{Q}$. Since we also use extensively the six functors formalism, we are forced to work with $\mathbf{DA}(k)$, motives without transfers, instead of $\mathbf{DM}(k)$. In any case, the motivic Galois group of Ayoub does not see the difference between these two categories.

The two realizations will induce morphisms between the two Galois groups, and the hard part is to prove that these are inverses to each other. In one direction, we rely heavily on one of the main results of Ayoub's approach to motivic Galois groups, namely a specific model he has given for the object in $\mathbf{DA}(k)$ representing Betti cohomology. Analysing this model closely we can show that the coordinate ring of $\mathcal{G}_{\mathrm{A}}(k)$ as a $\mathcal{G}_{\mathrm{A}}(k)$-representation is generated by $\mathcal{G}_{\mathrm{N}}(k)$-representations $\mathrm{H}^i_{\mathrm{Betti}}(X, Z; \mathbb{Q}(j))$. This will allow us to prove the morphism $\mathcal{G}_{\mathrm{A}}(k) \to \mathcal{G}_{\mathrm{N}}(k)$ a closed immersion. For the other direction we will prove that $\mathcal{G}_{\mathrm{N}}(k) \to \mathcal{G}_{\mathrm{A}}(k)$ is a section to $\mathcal{G}_{\mathrm{A}}(k) \to \mathcal{G}_{\mathrm{N}}(k)$, and here the idea is to reduce all verifications to a class of pairs of varieties whose relative motive in $\mathbf{DA}(k)$ (and its effective version) are easier to handle. We have found that the pairs (X, Z) where X is smooth and Z a simple normal crossings divisor work well for our purposes, and we study their motives without transfers in detail.

Outline. We now give a more detailed account of the chapter. In §1 we recall the construction and basic properties of Nori motives and the associated Galois group. We also state a monoidal version of the universal property of his category of motives the proof of which is given in appendix A. In §2 we recall the construction and basic properties of Morel-Voevodsky motives (or motives without transfer) and the Betti realization. We also explain in detail in which sense the functor Bti^* *is* a Betti realization. We briefly recall the construction of Ayoub's Galois group for Morel-Voevodsky motives in §3.

In §4 we construct motives $\mathcal{R}_{\mathrm{A}}(X, Z, n)$ in $\mathbf{DA}(k)$ for X a variety, $Z \subset X$ a closed subvariety and n a non-negative integer. These motives are defined in terms of the six functors, and have the property that $\mathrm{H}_0(\mathrm{Bti}^* \mathcal{R}_{\mathrm{A}}(X, Z, n)) \cong \mathrm{H}_n(X(\mathbb{C}), Z(\mathbb{C}))$ naturally. Here is where our decision to use the six functors formalism pays off as its compatibility with the Betti realization immediately reduces us to prove the existence of a natural isomorphism between sheaf cohomology and singular cohomology of pairs of (locally compact) topological spaces. We were not able to find the required proofs for this last comparison in the literature, and we therefore decided to provide them in a separate appendix (to wit, appendix B). We end this section by showing how this construction yields a morphism of Hopf algebras $\varphi_{\mathrm{A}} : \mathcal{O}(\mathcal{G}_{\mathrm{N}}(k)) \to \mathcal{O}(\mathcal{G}_{\mathrm{A}}(k))$.

The following two sections 5 and 6 are devoted to defining a morphism in the other direction, at least on the "effective" bialgebras (the Hopf algebras are obtained from these effective bialgebras by inverting a certain element). For this, in §5, we recall Nori's version of the Basic Lemma, and explain how it leads to algebraic cellular decompositions of the singular homology of affine varieties. As an application we obtain a functor from smooth affine schemes to the derived category of effective Nori motives. In §6.1 we show that it can be extended to a functor LC^* defined on the category of effective Morel-Voevodsky motives. This construction relies on our discussion of Kan extensions in the context of dg categories in the previous chapter. This functor is then shown to give rise to the sought after morphism of bialgebras (§6.2).

We also collect additional results on realizing (Morel-)Voevodsky motives in Nori motives which are not strictly necessary for our main theorem but, we believe, of independent interest. In §6.3, we extend our constructions to take into account correspondences thus obtaining a variant of LC^* for effective Voevodsky motives (i. e. effective motives with transfers). Then, in §6.4, we also prove that these realizations pass to the stable categories of motives (with and without transfers). From this we finally deduce mixed Hodge realizations on motives with and without transfers.

The next section is all about explicit computations involving Morel-Voevodsky motives associated to pairs of schemes. The recurrent theme is that these computations are feasible if one restricts to the pairs (X, Z) where X is smooth and Z is a simple normal crossings divisor. We call these almost smooth pairs, and resolution of singularities implies that there are enough of them. This allows us to reduce computations for general pairs to these more manageable ones. In §7.1 we give models for the latter on the effective level, and determine their image under the functor LC^* explicitly. This allows us to compare their comodule structure (with respect to Nori's effective bialgebra mentioned above) to the one of the Betti homology of the pair. As a corollary, we see that the morphism of bialgebras passes to the Hopf algebras $\varphi_{\mathrm{N}} : \mathcal{O}(\mathcal{G}_{\mathrm{A}}(k)) \to \mathcal{O}(\mathcal{G}_{\mathrm{N}}(k))$. In §7.2 we give good models for $\mathcal{R}_{\mathrm{A}}(X, Z, n)$ and their duals on the stable level, when (X, Z) is almost smooth, and we describe their Betti realization.

§8 is the heart of the chapter. Ayoub has given a "singular" model for the object in $\mathbf{DA}(k)$ representing Betti cohomology. Using our description of $\mathcal{R}_{\mathrm{A}}(X, Z, n)$ and performing a close analysis of Ayoub's model we establish that the Hopf algebra $\mathcal{O}(\mathcal{G}_{\mathrm{A}}(k))$ as a comodule over itself is a filtered colimit of Nori motives $\mathrm{H}^i(X(\mathbb{C}), Z(\mathbb{C}); \mathbb{Q}(j))$, where (X, Z) is almost smooth and $i, j \in \mathbb{Z}$. This will be seen to imply surjectivity of φ_{A}, while on the other hand we also prove that $\varphi_{\mathrm{N}}\varphi_{\mathrm{A}}$ is the identity by proving that it is so on motives of almost smooth pairs.

Contents

Notation and conventions. We fix a field k of characteristic 0 together with an embedding $\sigma : k \hookrightarrow \mathbb{C}$. By a scheme we mean a quasi-projective scheme over k. A variety is a reduced scheme. Rings are always assumed commutative and unital. Monoidal categories (resp. functors, transformations) are assumed symmetric and unitary if not stated otherwise. Algebras and coalgebras (also known as monoids and comonoids, respectively) are assumed unitary resp. counitary.

Λ throughout denotes a fixed ring, assumed noetherian if not stated otherwise. The symbol $\mathbf{Cpl}(\Lambda)$ denotes the category of (unbounded) complexes of Λ-modules. As in the previous chapter, our conventions are homological, i. e. the differentials decrease the indices, and the shift operator satisfies $(A[p])_n = A_{p+n}$. For an abelian category $\mathcal{A}$, $\mathbf{D}(\mathcal{A})$ denotes its derived category, and $\mathbf{D}(\Lambda) := \mathbf{D}(\mathbf{Cpl}(\Lambda))$. Also, $\mathcal{A}_{\oplus}$ denotes the category of ind objects in $\mathcal{A}$.

1. Nori's Galois group

We begin by recalling the construction of Nori motives and the associated motivic Galois group (cf [59], [39]). We also describe the universal property of the category of Nori motives in a monoidal setting (Proposition 1.1).

One starts with a diagram (or directed graph) $\mathscr{D}_{\mathrm{N}}$ whose vertices are triples (X, Z, n) where X is a variety, Z is a closed subvariety of X, and n is an integer. There are two types of edges: a single edge from (X, Z, n) to $(Z, W, n-1)$ for any triple $X \supset Z \supset W$, and edges $(X, Z, n) \to (X', Z', n)$ indexed by morphisms $f : X \to X'$ which restrict to morphisms $f : Z \to Z'$.

A representation of such a diagram is simply a morphism of directed graphs $T : \mathscr{D}_{\mathrm{N}} \to \mathcal{C}$ into a category $\mathcal{C}$. Nori's Tannakian theory asserts that associated to any such representation $T : \mathscr{D}_{\mathrm{N}} \to \mathbf{Mod}^{\mathrm{f}}(\Lambda)$ into the category of finitely generated Λ-modules, there is a Λ-linear abelian category $\mathcal{C}(T)$ together with a factorization

$$\mathscr{D}_{\mathrm{N}} \xrightarrow{\tilde{T}} \mathcal{C}(T) \xrightarrow{o} \mathbf{Mod}^{\mathrm{f}}(\Lambda)$$

where o is a faithful exact Λ-linear functor. Moreover, this category is universal for such a factorization. Its construction is easy to describe. A finite (full) subdiagram $\mathscr{F} \subset \mathscr{D}_{\mathrm{N}}$ gives rise to a Λ-algebra $\mathrm{End}(T|_{\mathscr{F}}) \subset \prod_v \mathrm{End}_{\Lambda}(T(v))$ of families of compatible (with respect to the edges) endomorphisms indexed over the vertices of $\mathscr{F}$. We then set

$$\mathcal{C}(T) = \varinjlim_{\mathscr{F} \subset \mathscr{D}_{\mathrm{N}}} \mathbf{Mod}^{\mathrm{f}}(\mathrm{End}(T|_{\mathscr{F}})),$$

this (2-)colimit being over the finite subdiagrams. This is applied to the homology representation $\mathrm{H}_{\bullet} : \mathscr{D}_{\mathrm{N}} \to \mathbf{Mod}^{\mathrm{f}}(\Lambda)$ which takes a vertex (X, Z, n) to the relative singular homology Λ-module $\mathrm{H}_n(X^{\mathrm{an}}, Z^{\mathrm{an}}; \Lambda)$ of the associated topological spaces on the $\mathbb{C}$-points (this uses $\sigma : k \hookrightarrow \mathbb{C}$). To the single edge $(X, Z, n) \to (Z, W, n-1)$ it associates the boundary map of the long exact sequence of a triple $\mathrm{H}_n(X^{\mathrm{an}}, Z^{\mathrm{an}}) \to \mathrm{H}_{n-1}(Z^{\mathrm{an}}, W^{\mathrm{an}})$, and to an edge $(X, Z, n) \to (X', Z', n)$ corresponding to $f : X \to X'$ it associates the morphism in homology induced by $f^{\mathrm{an}} : X^{\mathrm{an}} \to X'^{\mathrm{an}}$. (Here, and in the following we refrain from writing the coefficients in the homology when these can be guessed from the context. Also, we sometimes write $\mathrm{H}_{\bullet}(X, Z)$ instead of $\mathrm{H}_{\bullet}(X^{\mathrm{an}}, Z^{\mathrm{an}})$.) The resulting category $\mathcal{C}(\mathrm{H}_{\bullet})$ is denoted by $\mathbf{HM}^{\mathrm{eff}}$, the category of *effective homological Nori motives*.

In case Λ is a principal ideal domain and T takes values in finitely generated free Λ-modules, the dual $\mathcal{A}(T|_{\mathscr{F}}) = \mathrm{End}(T|_{\mathscr{F}})^{\vee}$ carries a canonical coalgebra structure for any finite subdiagram $\mathscr{F} \subset \mathscr{D}_{\mathrm{N}}$. Moreover, $\mathcal{C}(T)$ can then be described as the category $\mathbf{coMod}^{\mathrm{f}}(\mathcal{A}(T))$ of $\mathcal{A}(T)$-comodules in $\mathbf{Mod}^{\mathrm{f}}(\Lambda)$,[1] where

$$\mathcal{A}(T) = \varinjlim_{\mathscr{F} \subset \mathscr{D}_{\mathrm{N}}} \mathcal{A}(T|_{\mathscr{F}}).$$

We will need monoidal versions of these constructions. For this let us restrict to the full subdiagram $\mathscr{D}_{\mathrm{N}}^{\mathrm{g}}$ of $\mathscr{D}_{\mathrm{N}}$ consisting of *good pairs*, i. e. vertices (X, Z, n) with $X \backslash Z$ smooth and $\mathrm{H}_\bullet(X, Z; \mathbb{Z})$ a free abelian group concentrated in degree n. It follows essentially from the Basic Lemma (recalled in §5) that $\mathcal{C}(\mathrm{H}_\bullet|_{\mathscr{D}_{\mathrm{N}}^{\mathrm{g}}})$ is canonically equivalent to $\mathbf{HM}^{\mathrm{eff}}$ (see [59, Pro. 3.2] or [39, Cor. 1.7] for a proof). Moreover, on $\mathscr{D}_{\mathrm{N}}^{\mathrm{g}}$ there is a “commutative product structure with unit” in the sense of [39] induced by the cartesian product of varieties, and $\mathrm{H}_\bullet|_{\mathscr{D}_{\mathrm{N}}^{\mathrm{g}}}$ is canonically a u. g. m. representation (see appendix A for a recollection on these notions). This endows $\mathbf{HM}^{\mathrm{eff}}$ with a monoidal structure such that the functor o mapping to $\mathbf{Mod}^{\mathrm{f}}(\Lambda)$ is monoidal ([59, Thm. 4.1], [39, Pro. B.16]). As in the non-monoidal case it has a universal property which we state in the following instance of Nori's Tannaka duality theorem in the monoidal setting A.1 (cf. also [12]).

Corollary 1.1 *Let Λ be a principal ideal domain. Suppose we are given a right exact monoidal abelian Λ-linear category $\mathcal{A}$[2] together with a monoidal faithful exact Λ-linear functor $o : \mathcal{A} \to \mathbf{Mod}^{\mathrm{f}}(\Lambda)$ and a u. g. m. representation $T : \mathscr{D}_{\mathrm{N}}^{\mathrm{g}} \to \mathcal{A}$ such that the following diagram of solid arrows commutes.*

$$\begin{tikzcd}
\mathscr{D}_{\mathrm{N}}^{\mathrm{g}} \arrow[r, "T"] \arrow[d, "\tilde{\mathrm{H}}_\bullet"'] & \mathcal{A} \arrow[d, "o"] \\
\mathbf{HM}^{\mathrm{eff}} \arrow[r, "o"'] \arrow[ur, dotted] & \mathbf{Mod}^{\mathrm{f}}(\Lambda)
\end{tikzcd}$$

Then there exists a monoidal functor $\mathbf{HM}^{\mathrm{eff}} \to \mathcal{A}$ (unique up to unique monoidal isomorphism), represented by the dotted arrow in the diagram rendering the two triangles commutative (up to monoidal isomorphism).

Moreover, this functor is faithful exact Λ-linear.

This monoidal structure endows the coalgebra $\mathbf{H}_{\mathrm{N}}^{\mathrm{eff}} := \mathbf{H}_{\mathrm{N},\Lambda}^{\mathrm{eff}} := \mathcal{A}(\mathrm{H}_\bullet|_{\mathscr{D}_{\mathrm{N}}^{\mathrm{g}}})$ with the structure of a commutative algebra turning it into a (commutative) bialgebra ([59, §4.2], [39, Pro. B.16]). The coordinate ring $\mathbf{H}_{\mathrm{N}}$ (or $\mathbf{H}_{\mathrm{N},\Lambda}$) of *Nori's motivic Galois group* (denoted by $\mathcal{G}_{\mathrm{N}}$, or $\mathcal{G}_{\mathrm{N},\Lambda}$) is the commutative Hopf algebra obtained from $\mathbf{H}_{\mathrm{N}}^{\mathrm{eff}}$ by localizing (as an algebra) with respect to an element $s_{\mathrm{N}} \in \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}}$ which can be described as follows (cf. [59, p. 13]). Choose an isomorphism

$$\mathrm{H}_1(\mathbb{G}_m, \{1\}) \xrightarrow{\sim} \Lambda. \tag{1.2}$$

Then $s_{\mathrm{N}} \in \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}}$ is the image of $1 \in \Lambda$ under the composition

$$\Lambda \underset{\sim}{\xleftarrow{(1.2)}} \mathrm{H}_1(\mathbb{G}_m, \{1\}) \xrightarrow{\mathrm{ca}} \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}} \otimes \mathrm{H}_1(\mathbb{G}_m, \{1\}) \underset{\sim}{\xrightarrow{(1.2)}} \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}} \otimes \Lambda \cong \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}},$$

where ca denotes the coaction of $\mathbf{H}_{\mathrm{N}}^{\mathrm{eff}}$ on $\mathrm{H}_1(\mathbb{G}_m, \{1\})$. Clearly, s_{N} does not depend on the choice of (1.2).

The category $\mathbf{HM} := \mathbf{HM}_\Lambda := \mathbf{coMod}^{\mathrm{f}}(\mathbf{H}_{\mathrm{N},\Lambda})$ is the category of *(homological) Nori motives*.

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[1] For our conventions regarding comodules see appendix C.

[2] Hence $(A, \otimes)$ is a monoidal abelian Λ-linear category such that $\otimes$ is right exact Λ-linear in each variable.

2. Betti realization for Morel-Voevodsky motives

As explained in the introduction, we will work with motives without transfers in order to use the six functors formalism. Based on Voevodsky's original construction of the triangulated category of motives with transfers $\mathbf{DM}(k)$, and following an insight by Morel, this formalism has been worked out by Ayoub in [2]. In this section we briefly recall the construction of this category of motives, and the associated Betti realization from [3] and [5]. We also prove a few results not stated there explicitly. In particular we give a dg model for the Betti realization.

2.1. Effective motives. The category of effective motives without transfers may profitably be viewed from the perspective of *universal dg homotopy theories* as introduced in the previous chapter. Here, we start with a small category $\mathcal{C}$ with finite products, endowed with a Grothendieck topology τ and $I \in \mathcal{C}$ an "object parametrizing homotopies"; also fix any ring Λ. The category $\mathbf{U}\mathcal{C} = \mathbf{PSh}(\mathcal{C}, \mathbf{Cpl}(\Lambda))$ of presheaves on $\mathcal{C}$ with values in complexes of Λ-modules can be endowed with three model structures (among others):

- the *projective model structure* whose fibrations (resp. weak equivalences) are objectwise epimorphisms (resp. quasi-isomorphisms); its homotopy category is just the derived category $\mathbf{D}(\mathbf{U}\mathcal{C})$;
- the *projective τ-local model structure* arises from the projective model structure by Bousfield localization with respect to τ-hypercovers; its homotopy category is equivalent to the derived category of τ-sheaves on $\mathcal{C}$.
- the *projective (I,τ)-local model structure* arises as a further Bousfield localization with respect to arrows $\Lambda(I \times Y)[i] \to \Lambda(Y)[i]$, where $\Lambda : \mathcal{C} \to \mathbf{U}\mathcal{C}$ denotes the "Yoneda embedding", and $Y \in \mathcal{C}$ and $i \in \mathbb{Z}$ are arbitrary; its homotopy category is a Λ-linear unstable (or "effective") I-homotopy theory of $(\mathcal{C},\tau)$. (The reader will have no difficulties formulating the universal property of this model structure analogous to III.4.15.)

In each case, the model category is stable and monoidal (for the objectwise tensor product) hence the homotopy categories are triangulated monoidal. The following examples will be of interest to us (notation is explained subsequently):

$\mathcal{C}$	τ	I	Λ-linear unstable I-homotopy theory
Sm_X	Nis or ét	$\mathbb{A}^1_X$	$\mathbf{DA}^{\mathrm{eff}}(X) = \mathbf{DA}^{\mathrm{eff}}_{\Lambda}(X)$
SmAff_X	Nis or ét	$\mathbb{A}^1_X$	$\mathbf{DA}^{\mathrm{eff}}_{\mathrm{aff}}(X) = \mathbf{DA}^{\mathrm{eff}}_{\mathrm{aff},\Lambda}(X)$
AnSm_X	usu	$\mathbb{D}^1_X$	$\mathbf{AnDA}^{\mathrm{eff}}(X) = \mathbf{AnDA}^{\mathrm{eff}}_{\Lambda}(X)$
Open_X	usu	X	$\mathbf{D}(X,\Lambda)$

Here, in the first two examples, X is a scheme, and Sm_X (resp. SmAff_X) denotes the category of smooth schemes over X (resp. which are affine) endowed with the Nisnevich or the étale topology. In case $X = \mathrm{Spec}(k)$, we denote this category by Sm (resp. SmAff). $\mathbf{DA}^{\mathrm{eff}}(X)$ is called the category of *effective Morel-Voevodsky X-motives*. In the few cases where the topology chosen plays any role, we will make this explicit. Also, if $X = \mathrm{Spec}(k)$ then we simply write $\mathbf{DA}^{\mathrm{eff}}$. The canonical inclusion $\mathrm{SmAff}_X \to \mathrm{Sm}_X$ induces a triangulated monoidal equivalence $\mathbf{DA}^{\mathrm{eff}}_{\mathrm{aff}}(X) \xrightarrow{\sim} \mathbf{DA}^{\mathrm{eff}}(X)$, by III.4.17.

In the third example above, X is a complex analytic space, i. e. a "complex space" in the sense of [24] which is supposed to be denumerable at infinity, and AnSm_X denotes the category of complex analytic spaces smooth over X with the topology usu given by open covers. If X is the terminal object $\star$, then AnSm_X is denoted simply by $\mathrm{Man}_{\mathbb{C}}$. $\mathbb{D}^1$ denotes the open unit disk considered as a complex analytic space. As above, the $\mathbb{D}^1$-homotopy theory is denoted by $\mathbf{AnDA}^{\mathrm{eff}}$ in case $X = \star$.

Finally, in the fourth example X denotes a topological space, Open_X the category associated to the preorder of open subsets of X. It is endowed with the topology usu given by open covers. The (X, usu)-local and the usu-local model structures evidently agree, and their homotopy category $\mathbf{D}(X) = \mathbf{D}(X, \Lambda)$ is (canonically identified with) the derived category of sheaves on X.

The Betti realization will now link these examples, as follows. For a complex analytic space X there is an obvious inclusion $\iota_X : \mathrm{Open}_X \to \mathrm{AnSm}_X$ which defines a morphism of sites and induces a Quillen equivalence ([3, Thm. 1.8])

$$(\iota_X^*, \iota_{X*}) : \mathbf{U}(\mathrm{Open}_X)/\mathrm{usu} \to \mathbf{U}(\mathrm{AnSm}_X)/(\mathbb{D}^1_X, \mathrm{usu}).$$

If $X = \mathrm{Spec}(k)$, the left adjoint takes a complex to the associated constant presheaf and is denoted by $(\bullet)_{\mathrm{cst}}$, while the right adjoint is the global sections functor and accordingly denoted by Γ (it shouldn't be confused with the functor appearing in the Dold-Kan correspondence of the last chapter which won't make any appearance in the present chapter anymore).

Any scheme Y gives rise to a complex analytic space Y^{an}, namely the topological space $(Y \times_{k,\sigma} \mathbb{C})(\mathbb{C})$ with the natural complex analytic structure. We obtain an analytification functor $\mathrm{An}_X : \mathrm{Sm}_X \to \mathrm{AnSm}_{X^{\mathrm{an}}}$ which induces Quillen adjunctions

$$(\mathrm{An}_X^*, \mathrm{An}_{X*}) : \mathbf{U}(\mathrm{Sm}_X) \to \mathbf{U}(\mathrm{AnSm}_{X^{\mathrm{an}}})$$

for the corresponding model structures considered above. The left adjoint An_X^* in fact preserves (I, τ)-local weak equivalences (see [5, Rem. 2.57]).

Finally, Ayoub defines the *effective Betti realization* as the composition

$$\mathrm{Bti}^{\mathrm{eff},*} : \mathbf{DA}^{\mathrm{eff}}(X) \xrightarrow{\mathrm{An}_X^*} \mathbf{AnDA}^{\mathrm{eff}}(X) \xrightarrow[\sim]{\mathrm{R}\iota_{X*}} \mathbf{D}(X).$$

By construction, this is a triangulated monoidal functor.

2.2. Motives. Motives will be obtained from effective motives by a stabilization process which we again describe in the abstract setting first. Let $\mathcal{M}$ be a cellular left-proper monoidal model category and $T \in \mathcal{M}$ a cofibrant object. The category $\mathbf{Spt}_T^{\Sigma}\mathcal{M}$ of symmetric T-spectra in $\mathcal{M}$ admits the following two model structures (among others):

- the *projective unstable model structure* whose fibrations (resp. weak equivalences) are levelwise fibrations (resp. weak equivalences);
- the *projective stable model structure* arises from the unstable one by Bousfield localization with respect to morphisms $\mathrm{Sus}_T^{n+1}(T \otimes K) \to \mathrm{Sus}_T^n(K)$ for cofibrant objects $K \in \mathcal{M}$.

Here, $(\mathrm{Sus}_T^i, \mathrm{Ev}_i) : \mathbf{U}\mathcal{C} \to \mathbf{Spt}_T^{\Sigma}\mathbf{U}\mathcal{C}$ denotes the canonical adjunction, Ev_i being evaluation at level i. For the details (also concerning the existence of the model structures) we refer to [37]. Again, the model categories are both monoidal, and if $\mathcal{M}$ was stable then so is $\mathbf{Spt}_T^{\Sigma}\mathcal{M}$. If not mentioned explicitly otherwise, when we refer to *the* model structure on $\mathbf{Spt}_T^{\Sigma}\mathcal{M}$ we mean the stable one. In $\mathbf{Spt}_T^{\Sigma}\mathcal{M}$, tensoring with T becomes a Quillen equivalence, and it should be thought of as the universal such model category although this is not quite true in the obvious sense (cf. [37, §9]). Still, we call it the T-stabilization of $\mathcal{M}$.

In the algebraic geometric examples above we choose T_X to be a cofibrant replacement of $\Lambda(\mathbb{A}^1_X)/\Lambda(\mathbb{G}_{m,X})$. The resulting T_X-stable $\mathbb{A}^1$-homotopy theory of (Sm_X, τ), $\mathbf{DA}(X) = \mathbf{DA}_\Lambda(X)$, is the category of *Morel-Voevodsky X-motives*. As before the affine version $\mathbf{DA}_{\mathrm{aff}}(X)$ is canonically equivalent. Again, we leave the topology implicit most of the time, and in case

$X = k$ we also write $\mathbf{DA}$. There is the notion of a compact motive, namely a motive in $\mathbf{DA}$ which is compact in the sense of additive categories, and the full subcategory of compact motives forms a thick triangulated subcategory.

In the analytic setting, stabilization is performed with respect to a cofibrant replacement T_X of the quotient presheaf $\Lambda(\mathbb{A}^{1,\mathrm{an}}_X)/\Lambda(\mathbb{G}^{\mathrm{an}}_{m,X})$. The resulting homotopy category is denoted by $\mathbf{AnDA}(X) = \mathbf{AnDA}_\Lambda(X)$ (and again simply $\mathbf{AnDA}$ in case $X = \star$). [3, Lem. 1.10] together with [37, Thm. 9.1] show that the adjunction

$$(\mathrm{Sus}^0_{T_X}, \mathrm{Ev}_0) : \mathbf{U}(\mathrm{AnSm}_X)/(\mathbb{D}^1_X, \mathrm{usu}) \to \mathbf{Spt}^\Sigma_{T_X}\mathbf{U}(\mathrm{AnSm}_X)/(\mathbb{D}^1_X, \mathrm{usu})$$

defines a Quillen equivalence. Moreover, the analytification functor passes to the level of symmetric spectra and preserves stable (I,τ)-local equivalences. Ayoub then defines the *Betti realization* to be the composition

$$\mathrm{Bti}^* : \mathbf{DA}(X) \xrightarrow{\mathrm{An}^*_X} \mathbf{AnDA}(X) \xrightarrow[\sim]{\mathrm{R}\iota_{X*}\mathrm{REv}_0} \mathbf{D}(X).$$

Again, it is a triangulated monoidal functor.

We recall that the six functors constitute a formalism on the categories $\mathbf{DA}(X)$ for schemes X which associates to any morphism of schemes $f : X \to Y$ adjunctions

$$(\mathrm{L}f^*, \mathrm{R}f_*) : \mathbf{DA}(Y) \to \mathbf{DA}(X), \qquad (\mathrm{L}f_!, f^!) : \mathbf{DA}(X) \to \mathbf{DA}(Y), \tag{2.1}$$

and which endows $\mathbf{DA}(X)$ with a closed monoidal structure

$$(\otimes^{\mathrm{L}}, \mathrm{R}\,\underline{\mathrm{Hom}}).^{3}$$

All these functors are triangulated. The formalism governs the relation between them, e. g. under what conditions two of these functors can be identified or when they commute. Some of these relations are given explicitly in [2, Sch. 1.4.2]. We will also heavily use the part concerning duality. Recall that on compact motives there is a contravariant autoequivalence $(\bullet)^\vee$ which exchanges the two adjunctions in (2.1) so that for example $(\mathrm{R}f_*M)^\vee \cong \mathrm{L}f_!M^\vee$ for any compact motive M (see [2, Thm. 2.3.75]).

The same formalism is available in the analytic (see [3]) and in the topological setting (at least if the topological space is locally compact, see e. g. [43]). The main result of Ayoub in [3] is that the Betti realization is compatible with these, at least if one restricts to compact motives.

2.3. dg enhancement. In the remainder of the section we will exhibit the (effective) Betti realization as the derived functor of a left Quillen dg functor (Proposition 2.2). This will be used in section 6 to construct a motivic realization $\mathbf{DA}^{\mathrm{eff}} \to \mathbf{D}(\mathbf{HM}^{\mathrm{eff}}_\oplus)$.

Let X be a complex analytic space. Denote by $\mathrm{Sg}(X)$ the complex of singular chains in X (with Λ-coefficients). This extends to a lax monoidal functor Sg on topological spaces in virtue of the Eilenberg-Zilber map (cf. [18, VI, 12]). Its "left dg Kan extension" (rather, the functor underlying the left dg Kan extension of Fact III.1.1) is denoted by

$$\mathrm{Sg}^* : \mathbf{U}(\mathrm{Man}_\mathbb{C}) \to \mathbf{Cpl}(\Lambda).$$

It possesses an induced lax-monoidal structure, by Lemma III.1.2. Moreover, for each complex manifold X, $\mathrm{Sg}(X)$ is projective cofibrant hence $\bullet \otimes \mathrm{Sg}(X)$ is a left Quillen functor. It follows from Lemma III.1.5 that Sg^* is also a left Quillen functor with respect to the projective model structures.

[3] In the literature the symbols L and R indicating that some left or right derivation takes place are often dropped from the notation. For us however, the distinction between the derived and underived functors will be important which is why we stick to the clumsier notation.

Proposition 2.2 LSg^* *takes* $(\mathbb{D}^1, \mathrm{usu})$*-local equivalences to quasi-isomorphisms and the induced functor*

$$\mathbf{DA}^{\mathrm{eff}} \xrightarrow{\mathrm{An}^*} \mathbf{AnDA}^{\mathrm{eff}} \xrightarrow{\mathrm{LSg}^*} \mathbf{D}(\Lambda)$$

is isomorphic to $\mathrm{Bti}^{\mathrm{eff},*}$ *as monoidal triangulated functor. In particular, the following triangle commutes up to a monoidal isomorphism:*

$$\begin{tikzcd} \mathrm{Sm} \arrow[r, "\Lambda(\bullet)"] \arrow[dr, "\mathrm{Sg}\circ\mathrm{An}"'] & \mathbf{DA}^{\mathrm{eff}} \arrow[d, "\mathrm{Bti}^{\mathrm{eff},*}"] \\ & \mathbf{D}(\Lambda) \end{tikzcd}$$

Proof. In fact, we will deduce the first statement from the second.

For this let us recall the "singular analytic complexes" constructed in [5, §2.2.1]. We denote by $\mathbb{D}^1(r)$ the open disk of radius r centered at the origin (thus $\mathbb{D}^1 = \mathbb{D}^1(1)$) and by $\mathbb{D}^n(r)$ the n-fold cartesian product ($n \geq 0$). Letting $r > 1$ vary we obtain pro-complex manifolds $\overline{\mathbb{D}}^n = (\mathbb{D}^n(r))_{r>1}$. There is an obvious way to endow the family $(\overline{\mathbb{D}}^n)_{n\geq 0}$ with the structure of a cocubical object in the category of pro-complex manifolds (see [5, Déf. 2.19]). For any complex manifold X one then deduces a cubical Λ-module $\underline{\mathrm{hom}}(\overline{\mathbb{D}}^\bullet, X)$, where the latter in degree n is given by $\varinjlim_{r>1} \Lambda\mathrm{Man}_{\mathbb{C}}(\mathbb{D}^n(r), X)$. The associated simple complex (see [5, Déf. A.4]) is called the singular analytic complex associated to X, and is denoted by $\mathrm{Sg}^{\mathbb{D}}(X)$. It clearly extends to a functor

$$\mathrm{Sg}^{\mathbb{D}} : \mathrm{Man}_{\mathbb{C}} \to \mathbf{Cpl}(\Lambda),$$

and admits a natural lax monoidal structure induced by the association

$$(a : \mathbb{D}^m(r) \to X, b : \mathbb{D}^n(r) \to Y) \longmapsto (a \times b : \mathbb{D}^{m+n}(r) \to X \times Y).$$

We would now like to prove that $\mathrm{Sg}^{\mathbb{D}}$ and Sg are monoidally quasi-isomorphic, and for this we need a third, intermediate singular complex.

For any real number $r > 1$, denote by $\mathbb{I}^1(r)$ the open interval $(-r, r)$. Set $\mathbb{I}^1 = \mathbb{I}^1(1)$. There is an obvious analytic embedding $\mathbb{I}^1(r) \to \mathbb{D}^1(r)$ of real analytic manifolds. Denote by $\mathbb{I}^n(r)$ the n-fold cartesian product of $\mathbb{I}^1(r)$. Letting $r > 1$ vary we obtain a pro-real analytic manifold $\overline{\mathbb{I}}^n$. There is an obvious embedding of pro-real analytic manifolds $\overline{\mathbb{I}}^n \to \overline{\mathbb{D}}^n$ for each n, and by restriction this induces the structure of a cocubical object in pro-real analytic manifolds on $\overline{\mathbb{I}}^\bullet$.

Denoting by i the inclusion $\mathrm{Man}_{\mathbb{C}} \hookrightarrow \mathrm{Man}_{\mathbb{R}}^{\omega}$ we obtain a monoidal natural transformation

$$\mathrm{Sg}^{\mathbb{D}} \to \mathrm{Sg}^{\mathbb{I}} \circ i, \tag{2.3}$$

and it suffices to prove that this is sectionwise a quasi-isomorphism. Indeed, a similar argument as in [54, App. A, §2, Thm. 2.1] shows that the right hand side is monoidally quasi-isomorphic to the analogous functor of cubical complexes of *continuous* functions. And the latter is in turn monoidally quasi-isomorphic to Sg, by [30, Thm. 5.1].

Following [5], we denote the "left dg Kan extension" of $\mathrm{Sg}^{\mathbb{D}}$ again by the same symbol $\mathrm{Sg}^{\mathbb{D}} : \mathbf{U}\mathrm{Man}_{\mathbb{C}} \to \mathbf{Cpl}(\Lambda)$. (That this indeed coincides with the functor in [5] follows from Lemma III.2.23. Cocontinuity is a consequence of [5, Lem. A.3].) [5, Thm. 2.23] together with [5, Cor. 2.26, 2.27] show that $\mathrm{Sg}^{\mathbb{D}}$ takes $(\mathbb{D}^1, \mathrm{usu})$-local equivalences to quasi-isomorphisms. The same argument also shows that $\mathrm{Sg}^{\mathbb{I}}$ takes $(\mathbb{I}, \mathrm{usu})$-local equivalences to quasi-isomorphisms. Now, let's start with a complex manifold X. We want to prove that (2.3) applied to X is a quasi-isomorphism. For this, choose a usu-hypercover $X_\bullet \to X$ of complex manifolds such that each representable in each degree is contractible. This can also be considered as a usu-hypercover of real analytic manifolds, and by what we just discussed, the two horizontal arrows in the following commutative square are quasi-isomorphisms:

$$\begin{tikzcd} \mathrm{Sg}^{\mathbb{D}}(X_\bullet) \arrow[r] \arrow[d] & \mathrm{Sg}^{\mathbb{D}}(X) \arrow[d] \\ \mathrm{Sg}^{\mathbb{I}}(i(X_\bullet)) \arrow[r] & \mathrm{Sg}^{\mathbb{I}}(i(X)) \end{tikzcd}$$

By Lemma III.3.2, we reduce to show that for any contractible complex manifold Y, $\mathrm{Sg}^{\mathbb{D}}(Y) \to \mathrm{Sg}^{\mathbb{I}}(i(Y))$ is a quasi-isomorphism, which is easy.

By [5, Cor. 2.26, Pro. 2.83], $\mathrm{Bti}^{\mathrm{eff},*}$ is isomorphic to $\mathrm{Sg}^{\mathbb{D}} \circ \mathrm{An}^*$ as triangulated monoidal functor hence the discussion above implies the second statement of the proposition. The first statement can now be deduced as follows. From the monoidal quasi-isomorphism $\mathrm{Sg}^{\mathbb{D}} \sim \mathrm{Sg}$ we obtain triangulated monoidal isomorphisms (cf. Lemma III.1.2)

$$\mathrm{Sg}^{\mathbb{D}} \cong \mathrm{LSg}^{\mathbb{D}} \cong \mathrm{L}(\mathrm{Sg})^* : \mathbf{D}(\mathbf{U}\mathrm{Man}_{\mathbb{C}}) \to \mathbf{D}(\Lambda).$$

Indeed, the second isomorphism can be checked on representables (these are compact generators of $\mathbf{D}(\mathbf{U}\mathrm{Man}_{\mathbb{C}})$ by Lemma III.2.22) and these objects are cofibrant. □

Remark 2.4 Using the topological singular complex we can construct an explicit fibrant model for the unit spectrum in **AnDA** as follows. Denote by $\mathrm{Sg}^\vee$ the presheaf of complexes on $\mathrm{Man}_{\mathbb{C}}$ which takes a complex manifold X to $\mathrm{Sg}(X)^\vee$. Let $U = (\mathbb{P}^1 \times \mathbb{P}^1) \backslash \Delta(\mathbb{P}^1)$, and let u be a rational point of U over $\mathbb{P}^1 \times \{\infty\}$. As in [5, §2.3.1], we can use $T^{\mathrm{an}} = \Lambda(U^{\mathrm{an}})/\Lambda(u^{\mathrm{an}})$ to form symmetric spectra (hence T^{an} is a cofibrant replacement of $\Lambda(\mathbb{A}^{1,\mathrm{an}})/\Lambda(\mathbb{G}_m^{\mathrm{an}})$). Fix $\hat{\beta} \in \mathrm{Sg}^\vee_{-2}(U^{\mathrm{an}}, u^{\mathrm{an}}; \Lambda)$ whose class in $\mathrm{H}^2(U^{\mathrm{an}}, u^{\mathrm{an}}; \Lambda) \cong \Lambda$ is a generator. Define a symmetric T^{an}-spectrum $\mathbf{Sg}^\vee$ which in level n is $\mathrm{Sg}^\vee[-2n]$ with the trivial Σ_n-action, and whose bonding maps are given by the adjoints of the quasi-isomorphism

$$\hat{\beta} \times \bullet : \mathrm{Sg}^\vee(X)[-2n] \to \mathrm{Sg}^\vee((U,u) \times X)[-2(n+1)]$$

for any complex manifold X.

The canonical morphism $\Lambda_{\mathrm{cst}} \to \mathrm{Sg}^\vee$ induces by adjunction a morphism of symmetric spectra $\mathrm{Sus}^{\mathrm{o}}_{T^{\mathrm{an}}} \Lambda_{\mathrm{cst}} \to \mathbf{Sg}^\vee$ which in level n is given by the composition

$$\begin{aligned} (T^{\mathrm{an}})^{\otimes n} \otimes \Lambda_{\mathrm{cst}} &\to (T^{\mathrm{an}})^{\otimes n} \otimes \mathrm{Sg}^\vee \\ &\xrightarrow{\mathrm{id} \otimes (\hat{\beta} \times \bullet)^n} (T^{\mathrm{an}})^{\otimes n} \otimes \underline{\mathrm{hom}}((T^{\mathrm{an}})^{\otimes n}, \mathrm{Sg}^\vee[-2n]) \\ &\xrightarrow{\mathrm{ev}} \mathrm{Sg}^\vee[-2n]. \end{aligned}$$

The first arrow is a usu-local equivalence, the second arrow is a quasi-isomorphism, and the third is a $(\mathbb{D}^1, \mathrm{usu})$-local equivalence since T^{an} is invertible in $\mathbf{AnDA}^{\mathrm{eff}}$, by [3, Lem. 1.10]. It follows that $\mathrm{Sus}^{\mathrm{o}}_{T^{\mathrm{an}}} \Lambda_{\mathrm{cst}} \to \mathbf{Sg}^\vee$ is a levelwise $(\mathbb{D}^1, \mathrm{usu})$-local equivalence. Since the source is an Ω-spectrum so is $\mathbf{Sg}^\vee$. Also, since Λ_{cst} is $\mathbb{D}^1$-local so is $\mathbf{Sg}^\vee$ levelwise. Finally, for any usu-hypercover $X_\bullet \to X$ of a complex manifold X, $\mathrm{Sg}^\vee(X) \to \mathrm{Sg}^\vee(X_\bullet)$ is a quasi-isomorphism which proves that $\mathbf{Sg}^\vee$ is levelwise usu-fibrant.

Summing up, we have proved that $\mathbf{Sg}^\vee$ is a projective stable $(\mathbb{D}^1, \mathrm{usu})$-fibrant replacement of $\mathrm{Sus}^{\mathrm{o}}_{T^{\mathrm{an}}} \Lambda_{\mathrm{cst}}$.

3. Ayoub's Galois group

We recall here the construction of Ayoub's bialgebra in [5]. In section 1 of that paper he develops a weak Tannaka duality theory which allows to factor certain monoidal functors $f : \mathcal{M} \to \mathcal{E}$ between monoidal categories universally as

$$\mathcal{M} \xrightarrow{\tilde{f}} \mathbf{coMod}(H) \xrightarrow{o} \mathcal{E} \tag{3.1}$$

for a commutative bialgebra $H \in \mathcal{E}$, where o is the forgetful functor, and where both functors in the factorization are monoidal. This was applied in [5] to the monoidal (effective) Betti realization functor

$$\mathrm{Bti}^* : \mathbf{DA} \to \mathbf{D}(\Lambda) \qquad (\text{resp. } \mathrm{Bti}^{\mathrm{eff},*} : \mathbf{DA}^{\mathrm{eff}} \to \mathbf{D}(\Lambda))$$

yielding the *stable (resp. effective) motivic bialgebra*

$$\mathcal{H}_{\mathrm{A}} \in \mathbf{D}(\Lambda) \qquad (\text{resp. } \mathcal{H}_{\mathrm{A}}^{\mathrm{eff}} \in \mathbf{D}(\Lambda)).$$

It is shown in [5], that $\mathcal{H}_{\mathrm{A}}$ is a Hopf algebra. Also, the bialgebras do not depend (up to canonical isomorphism) on the topology chosen. Explicitly, the bialgebras as objects in $\mathbf{D}(\Lambda)$ are given by $\mathcal{H}_{\mathrm{A}} = \mathrm{Bti}^*\mathrm{Bti}_*\Lambda$ and $\mathcal{H}_{\mathrm{A}}^{\mathrm{eff}} = \mathrm{Bti}^{\mathrm{eff},*}\mathrm{Bti}_*^{\mathrm{eff}}\Lambda$.

We said above that these bialgebras enjoy a universal property; let us recall the precise statement for the effective case (an analogous statement holds in the stable situation but we will not use this).

Fact 3.2 ([5, Pro. 1.55]) *Suppose we are given a commutative bialgebra K in $\mathbf{D}(\Lambda)$ and a commutative diagram in the category of monoidal categories*

$$\begin{tikzcd} \mathbf{DA}^{\mathrm{eff}} \arrow[r, "f"] \arrow[dr, "\mathrm{Bti}^{\mathrm{eff},*}"'] & \mathbf{coMod}(K) \arrow[d, "o"] \\ & \mathbf{D}(\Lambda) \end{tikzcd}$$

where o is the forgetful functor, such that $f(A_{\mathrm{cst}})$ is the trivial K-comodule associated to A, for any $A \in \mathbf{D}(\Lambda)$. Then there exists a unique morphism of bialgebras $\mathcal{H}_{\mathrm{A}}^{\mathrm{eff}} \to K$ making the following diagram commutative:

$$\begin{tikzcd} \mathbf{DA}^{\mathrm{eff}} \arrow[r, "f"] \arrow[d, "\widetilde{\mathrm{Bti}}^{\mathrm{eff},*}"'] & \mathbf{coMod}(K) \arrow[d, "o"] \\ \mathbf{coMod}(\mathcal{H}_{\mathrm{A}}^{\mathrm{eff}}) \arrow[ur] \arrow[r, "o"'] & \mathbf{D}(\Lambda) \end{tikzcd}$$

Now, consider the functor $\mathrm{H}_0 : \mathbf{D}(\Lambda) \to \mathbf{Mod}(\Lambda)$ which associates to a complex its 0th homology. By abuse of notation, we set $\mathbf{H}_{\mathrm{A}}^{\mathrm{eff}} \in \mathbf{Mod}(\Lambda)$ to be $\mathrm{H}_0\mathcal{H}_{\mathrm{A}}^{\mathrm{eff}}$, and $\mathbf{H}_{\mathrm{A}} \in \mathbf{Mod}(\Lambda)$ to be $\mathrm{H}_0\mathcal{H}_{\mathrm{A}}$. By [5, Cor. 2.105], the homology of $\mathcal{H}_{\mathrm{A}}^{\mathrm{eff}}$ and $\mathcal{H}_{\mathrm{A}}$ is concentrated in non-negative degrees and it follows that the bialgebra structures descend to $\mathbf{H}_{\mathrm{A}}^{\mathrm{eff}}$ and $\mathbf{H}_{\mathrm{A}}$, and $\mathbf{H}_{\mathrm{A}}$ is still a Hopf algebra. It is the coordinate ring of the *motivic Galois group of Ayoub*, denoted by $\mathcal{G}_{\mathrm{A}}$ (or $\mathcal{G}_{\mathrm{A},\Lambda}$ if the coefficient ring is not clear from the context).

By [5, Thm. 2.14], $\mathcal{H}_{\mathrm{A}}$ (resp. $\mathbf{H}_{\mathrm{A}}$) is obtained by localization from $\mathcal{H}_{\mathrm{A}}^{\mathrm{eff}}$ (resp. $\mathbf{H}_{\mathrm{A}}^{\mathrm{eff}}$), as follows. Choose an isomorphism

$$\mathrm{Bti}^{\mathrm{eff},*}(T[2]) \xrightarrow{\sim} \Lambda. \tag{3.3}$$

We then let $s_{\mathrm{A}} \in \mathcal{H}^{\mathrm{eff}}_{\mathrm{A}}$ be the image of $1 \in \Lambda$ under the composition

$$\Lambda \xleftarrow[\sim]{(3.3)} \mathrm{Bti}^{\mathrm{eff},*}(T[2]) \xrightarrow{\mathrm{ca}} \mathcal{H}^{\mathrm{eff}}_{\mathrm{A}} \otimes \mathrm{Bti}^{\mathrm{eff},*}(T[2]) \xrightarrow[\sim]{(3.3)} \mathcal{H}^{\mathrm{eff}}_{\mathrm{A}} \otimes \Lambda \cong \mathcal{H}^{\mathrm{eff}}_{\mathrm{A}},$$

where ca denotes the coaction of $\mathcal{H}^{\mathrm{eff}}_{\mathrm{A}}$ on $\mathrm{Bti}^{\mathrm{eff},*}(T[2])$. Clearly, s_{A} does not depend on the choice of the isomorphism (3.3). By [5, Thm. 2.14], $\mathcal{H}_{\mathrm{A}}$ is the sequential homotopy colimit of the diagram

$$\mathcal{H}^{\mathrm{eff}}_{\mathrm{A}} \xrightarrow{s_{\mathrm{A}}\times\bullet} \mathcal{H}^{\mathrm{eff}}_{\mathrm{A}} \xrightarrow{s_{\mathrm{A}}\times\bullet} \cdots.$$

Applying H_0 we see that $\mathbf{H}_{\mathrm{A}} = \mathbf{H}^{\mathrm{eff}}_{\mathrm{A}}[s_{\mathrm{A}}^{-1}]$ as an algebra.

In order to apply the results on the category of $\mathbf{H}^{(\mathrm{eff})}_{\mathrm{A}}$-comodules in appendix C we need the following result.

Lemma 3.4 *Let Λ be a principal ideal domain. Then $\mathbf{H}^{\mathrm{eff}}_{A}$ and $\mathbf{H}_{A}$ are flat Λ-modules.*

Proof. The proof is the same in both cases; we do it for $\mathbf{H}^{\mathrm{eff}}_{\mathrm{A}}$. By [6, Cor. 1.27], $\mathcal{H}^{\mathrm{eff}}_{\mathrm{A}}$ sits in a distinguished triangle

$$C' \to \mathcal{H}^{\mathrm{eff}}_{\mathrm{A}} \to C \to C'[-1],$$

where C is a complex in $\mathbf{D}(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q})$. (Explicitly, $C' = C^0(\mathrm{Gal}(\bar{k}, k), \Lambda)$, the Λ-module of locally constant functions on the absolute Galois group of k with values in Λ, which maps canonically to $\mathcal{H}^{\mathrm{eff}}_{\mathrm{A}}$. Essentially due to the Rigidity Theorem of Suslin-Voevodsky, this map is a quasi-isomorphism for torsion coefficients.) Looking at the associated long exact sequence in homology one sees that all homologies of $\mathcal{H}_{\mathrm{A}}$ must be torsion-free thus flat. $\square$

Remark 3.5 The Betti realization can be constructed in a similar way also for Voevodsky motives (see [6, §1.1.2]), and the same weak Tannakian formalism applies to give two bialgebras in $\mathbf{D}(\Lambda)$. It is proved in [6, Thm. 1.13] that they are canonically isomorphic to $\mathcal{H}^{\mathrm{eff}}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{A}}$, respectively. In the case of the Hopf algebras (and this is the case we are chiefly interested in), this follows from the fact that the canonical functor

$$\mathbf{DA}^{\text{ét}}(k) \to \mathbf{DM}^{\text{ét}}(k)$$

is an equivalence.

4. Motivic representation

The goal of this section is to factor the homology representation $\mathrm{H}_\bullet : \mathscr{D}_{\mathrm{N}} \to \mathbf{Mod}^{\mathrm{f}}(\Lambda)$ through the Betti realization $\mathrm{H}_0 \circ \mathrm{Bti}^* : \mathbf{DA} \to \mathbf{Mod}(\Lambda)$ (Propositions 4.3, 4.7) in order to obtain a morphism of bialgebras $\varphi_{\mathrm{A}} : \mathbf{H}_{\mathrm{N}} \to \mathbf{H}_{\mathrm{A}}$. Let us see how to derive a solution $\mathcal{R}_{\mathrm{A}} : \mathscr{D}_{\mathrm{N}} \to \mathbf{DA}$ to this task.

We saw in the previous section that for *smooth* schemes X, $\mathrm{Bti}^{\mathrm{eff},*}\Lambda(X)$ computes the Betti homology of X. A first guess might be that for *any* scheme X, $\mathrm{Bti}^{\mathrm{eff},*}\Lambda\,\mathrm{hom}(\bullet, X)$ also does. This is not true however even in simple non-smooth cases. Instead we notice that, for X with smooth structure morphism $\pi : X \to k$, there are canonical isomorphisms

$$\mathrm{LSus}^0_T\Lambda(X) \cong \mathrm{L}\pi_\#\pi^*\Lambda \cong \mathrm{L}\pi_!\pi^!\Lambda$$

in $\mathbf{DA}$. The last expression makes sense for *any* scheme X, and we will prove below that the Betti realization of this object indeed computes the Betti homology of X. We should remark that there is nothing original about this idea. The object $\mathrm{L}\pi_!\pi^!\Lambda$ (and not the presheaf $\Lambda\,\mathrm{hom}(\bullet, X)$) is commonly considered to be the "correct" representation of X in $\mathbf{DA}$, and is therefore also called the *(homological) motive of* X. The six functors formalism also allows to naturally define a relative motive associated to a pair of schemes, and this will yield the representation $\mathcal{R}_{\mathrm{A}}$ we were looking for.

4.1. Construction. Let (X, Z, n) be a vertex in Nori's diagram of pairs. Fix the following notation:

$$Z \xrightarrow{i} X \xleftarrow{j} U, \qquad \pi : X \to k,$$

where $U = X \backslash Z$ is the open complement. Set

$$\mathcal{R}_{\mathrm{A}}(X, Z, n) = \mathrm{L}\pi_! \mathrm{R}j_* j^* \pi^! \Lambda[n] \in \mathbf{DA}.$$

This extends to a representation $\mathcal{R}_{\mathrm{A}} : \mathscr{D}_{\mathrm{N}} \to \mathbf{DA}$ as follows:

- The first type of edge in $\mathscr{D}_{\mathrm{N}}$ is $(X, Z, n) \to (Z, W, n-1)$. We have the following distinguished ("localization") triangle (of endofunctors) in $\mathbf{DA}(X)$ (and similarly for the pair (Z, W)):

$$i_! i^! \xrightarrow{\mathrm{adj}} \mathrm{id} \xrightarrow{\mathrm{adj}} \mathrm{R}j_* j^* \xrightarrow{\partial} i_! i^! [-1]. \tag{4.1}$$

  (Here, as in the sequel, adj denotes the unit or counit of an adjunction.) Applying $\mathrm{L}\pi_!$ and evaluating at $\pi^! \Lambda[n]$, we therefore obtain a morphism

$$\begin{aligned} \mathcal{R}_{\mathrm{A}}(\partial) : \mathcal{R}_{\mathrm{A}}(X, Z, n) &\xrightarrow{\partial} \mathcal{R}_{\mathrm{A}}(Z, \varnothing, n-1) \\ &\xrightarrow{\mathrm{adj}} \mathcal{R}_{\mathrm{A}}(Z, W, n-1). \end{aligned}$$

- The second type of edge $(X, Z, n) \to (X', Z', n)$ is induced from a morphism of varieties $f : X \to X'$ with $f(Z) \subset Z'$. We have the following commutative diagram of solid arrows in (endofunctors of) $\mathbf{DA}(X')$:

$$\begin{tikzcd} \mathrm{L}f_! i_! i^! f^! \arrow[r] \arrow[d, "\mathrm{adj}"] & \mathrm{L}f_! f^! \arrow[r] \arrow[d, "\mathrm{adj}"] & \mathrm{L}f_! \mathrm{R}j_* j^* f^! \arrow[r] \arrow[d, dotted] & \mathrm{L}f_! i_! i^! f^! [-1] \arrow[d, "\mathrm{adj}"] \\ i'_! i'^! \arrow[r] & \mathrm{id} \arrow[r] & \mathrm{R}j'_* j'^* \arrow[r] & i'_! i'^! [-1] \end{tikzcd}$$

  where the rows are distinguished ("localization") triangles, and where the dotted arrow is the unique morphism making the vertical arrows into a morphism of triangles. (Uniqueness follows from the isomorphism $\mathrm{L}f_! i_! \cong i'_! \mathrm{L}(f|_Z)_!$ and the fact that there are no non-zero morphisms from $i'_!$ to $\mathrm{R}j'_*$.) After applying $\mathrm{L}\pi'_!$, shifting by n, and evaluating at $\pi'^! \Lambda$ this dotted arrow gives the morphism $\mathcal{R}_{\mathrm{A}}(f) : \mathcal{R}_{\mathrm{A}}(X, Z, n) \to \mathcal{R}_{\mathrm{A}}(X', Z', n)$ associated to f.

We will prove in a moment that this representation has the expected properties. Before doing so we would like to recall the following classical result.

Fact 4.2 ([34]) *Let (X, Z) be a pair of varieties. Then its analytification $(X^{\mathrm{an}}, Z^{\mathrm{an}})$ is a locally finite CW-pair. In particular, X^{an} and Z^{an} are paracompact, locally contractible, and locally compact.*

Proposition 4.3 *Suppose that Λ is a principal ideal domain. Then there is an isomorphism of representations*

$$\begin{tikzcd} \mathscr{D}_{N} \arrow[r, "\mathrm{H}_0 \widetilde{\mathrm{Bti}}^* \mathcal{R}_A"] \arrow[dr, "\mathrm{H}_\bullet"'] & \mathbf{coMod}^{\mathrm{f}}(\mathbf{H}_A) \arrow[d, "o"] \\ & \mathbf{Mod}^{\mathrm{f}}(\Lambda) \end{tikzcd}$$

Proof. By Fact 4.2 and B.3 the complex of relative singular cochains $\mathrm{Sg}(X^{\mathrm{an}}, Z^{\mathrm{an}}; \Lambda)^\vee$ provides a model for $\mathrm{R}\pi^{\mathrm{an}}_* j^{\mathrm{an}}_! \Lambda$ in $\mathbf{D}(\Lambda)$. We claim that the complex $\mathrm{Sg}(X^{\mathrm{an}}, Z^{\mathrm{an}})$ defines a strongly dualizable object in $\mathbf{D}(\Lambda)$. Indeed, using the distinguished triangle

$$\mathrm{Sg}(Z^{\mathrm{an}}) \to \mathrm{Sg}(X^{\mathrm{an}}) \to \mathrm{Sg}(X^{\mathrm{an}}, Z^{\mathrm{an}}) \to \mathrm{Sg}(Z^{\mathrm{an}})[-1],$$

we reduce to prove it for $\mathrm{Sg}(X^{\mathrm{an}})$. As we will see in section 5, this complex is quasi-isomorphic to a bounded complex of finitely generated free Λ-modules, thus it is a strongly dualizable object.[4]

Therefore the canonical map from $\mathrm{Sg}(X^{\mathrm{an}}, Z^{\mathrm{an}})$ to its double dual is a quasi-isomorphism, and we obtain the following sequence of isomorphisms in $\mathbf{Mod}(\Lambda)$, for every $n \in \mathbb{Z}$:

$$\begin{aligned}
\mathrm{H}_n(X^{\mathrm{an}}, Z^{\mathrm{an}}) &= \mathrm{H}_n \mathrm{Sg}(X^{\mathrm{an}}, Z^{\mathrm{an}}) & \\
&\cong \mathrm{H}_n(\mathrm{Sg}(X^{\mathrm{an}}, Z^{\mathrm{an}})^{\vee\vee}) & \\
&\cong \mathrm{H}_n((\mathrm{R}\pi^{\mathrm{an}}_* \mathrm{R}j^{\mathrm{an}}_! \Lambda)^\vee) & &\text{see appendix B} \\
&\cong \mathrm{H}_n \mathrm{Bti}^*((\mathrm{R}\pi_* j_! \Lambda)^\vee) & &\text{by the main results of [3]} \\
&\cong \mathrm{H}_n \mathrm{Bti}^* \mathrm{L}\pi_! \mathrm{R}j_* j^* \pi^! \Lambda & &\text{by duality} \\
&\cong \mathrm{H}_0 \mathrm{Bti}^* \mathcal{R}_{\mathrm{A}}(X, Z, n) & &\text{since } \mathrm{Bti}^* \text{ is triangulated.}
\end{aligned}$$

This defines the isomorphism in the proposition. We have to check that it is compatible with the two types of edges in $\mathscr{D}_{\mathrm{N}}$.

Let $f : (X, Z, n) \to (X', Z', n)$ be an edge in $\mathscr{D}_{\mathrm{N}}$. Compatibility with respect to f will follow from the commutativity of the outer rectangle in the following diagram (namely, after applying H_n and noticing that Bti^* commutes with shifts):

$$\begin{tikzcd}
\mathrm{Sg}(X^{\mathrm{an}}, Z^{\mathrm{an}}) \arrow[r, "\mathrm{Sg}(f)"] \arrow[d, dash, "\sim"'] & \mathrm{Sg}(X'^{\mathrm{an}}, Z'^{\mathrm{an}}) \arrow[d, dash, "\sim"] \\
\mathrm{Sg}(X^{\mathrm{an}}, Z^{\mathrm{an}})^{\vee\vee} \arrow[r, "\mathrm{Sg}(f)^{\vee\vee}"] \arrow[d, dash, "\sim"'] & \mathrm{Sg}(X'^{\mathrm{an}}, Z'^{\mathrm{an}})^{\vee\vee} \arrow[d, dash, "\sim"] \\
(\mathrm{R}\pi^{\mathrm{an}}_* j^{\mathrm{an}}_! \Lambda)^\vee \arrow[r, "\mathcal{R}^{\mathrm{an},\vee}_{\mathrm{A}}(f)^\vee"] \arrow[d, dash, "\sim"'] & (\mathrm{R}\pi'^{\mathrm{an}}_* j'^{\mathrm{an}}_! \Lambda)^\vee \arrow[d, dash, "\sim"] \\
\mathrm{Bti}^*(\mathrm{R}\pi_* j_! \Lambda)^\vee \arrow[r, "\mathcal{R}^{\vee}_{\mathrm{A}}(f)^\vee"] \arrow[d, dash, "\sim"'] & \mathrm{Bti}^*(\mathrm{R}\pi'_* j'_! \Lambda)^\vee \arrow[d, dash, "\sim"] \\
\mathrm{Bti}^* \mathrm{L}\pi_! \mathrm{R}j_* j^* \pi^! \Lambda \arrow[r, "\mathcal{R}_{\mathrm{A}}(f)"] & \mathrm{Bti}^* \mathrm{L}\pi'_! \mathrm{R}j'_* j'^* \pi'^! \Lambda
\end{tikzcd}$$

We will describe the arrows as we go along proving each square commutative. Starting at the bottom, $\mathcal{R}^\vee_{\mathrm{A}}(f) : j'_! \Lambda \to \mathrm{R}f_* j_! \Lambda$ is defined dually to $\mathcal{R}_{\mathrm{A}}(f)$. It is then clear that the bottom square commutes. The same definition (using the six functors formalism, that is) in the analytic setting gives rise to the arrow $\mathcal{R}^{\mathrm{an},\vee}_{\mathrm{A}}(f)$ in the third row. Again, by the main results of [3], the third square commutes as well. Commutativity of the first square is clear, while commutativity of the second square follows from Lemma B.4.

[4] In the notation of section 7.1, this quasi-isomorphic complex is $C^{\mathcal{Y}}(X, \varnothing)$ for a finite affine open cover $\mathcal{Y}$ of X.

Let (X, Z, n) be a vertex in $\mathscr{D}_{\mathrm{N}}$ and $W \subset Z$ a closed subvariety, giving rise to the edge $(X, Z, n) \to (Z, W, n-1)$. Compatibility with respect to this edge will follow from commutativity of the diagram

$$\begin{tikzcd}
\mathrm{H}_n(X,Z) \arrow[r, "\partial"] \arrow[d, "\sim"'] & \mathrm{H}_{n-1}(Z) \arrow[r] \arrow[d, "\sim"'] & \mathrm{H}_{n-1}(Z,W) \arrow[d, "\sim"] \\
\mathrm{H}_0\mathrm{Bti}^*\mathcal{R}_{\mathrm{A}}(X,Z,n) \arrow[r, "\partial"'] & \mathrm{H}_0\mathrm{Bti}^*\mathcal{R}_{\mathrm{A}}(Z,\varnothing,n-1) \arrow[r, "\mathrm{adj}"'] & \mathrm{H}_0\mathrm{Bti}^*\mathcal{R}_{\mathrm{A}}(Z,W,n-1)
\end{tikzcd}$$

where the vertical arrows are the isomorphisms constructed above, and where the horizontal arrows on the right are induced by $(Z, \varnothing, n-1) \to (Z, W, n-1)$. In particular, the right square commutes by what we have shown above, and we reduce to prove commutativity of the left square. It can be decomposed as follows (before applying H_n, using again that Bti^* commutes with shifts; the horizontal arrows will be made explicit below):

$$\begin{tikzcd}
\mathrm{Sg}(X^{\mathrm{an}}, Z^{\mathrm{an}}) \arrow[r, "\partial"] \arrow[d, "\sim"'] & \mathrm{Sg}(Z^{\mathrm{an}})[-1] \arrow[d, "\sim"] \\
\mathrm{Sg}(X^{\mathrm{an}}, Z^{\mathrm{an}})^{\vee\vee} \arrow[r, "\delta^\vee"] \arrow[d, "\sim"'] & (\mathrm{Sg}(Z^{\mathrm{an}})[-1])^{\vee\vee} \arrow[d, "\sim"] \\
(\mathrm{R}\pi^{\mathrm{an}}_* j^{\mathrm{an}}_! \Lambda)^\vee \arrow[r, "\delta^\vee"] \arrow[d, "\sim"'] & (\mathrm{R}\pi^{\mathrm{an}}_* i^{\mathrm{an}}_* \Lambda[1])^\vee \arrow[d, "\sim"] \\
\mathrm{Bti}^*(\mathrm{R}\pi_* j_! \Lambda)^\vee \arrow[r, "\delta^\vee"] \arrow[d, "\sim"'] & \mathrm{Bti}^*(\mathrm{R}\pi_* i_* \Lambda[1])^\vee \arrow[d, "\sim"] \\
\mathrm{Bti}^*\mathrm{L}\pi_!\mathrm{R}j_* j^* \pi^! \Lambda \arrow[r, "\mathcal{R}_{\mathrm{A}}(\partial)"] & \mathrm{Bti}^*\mathrm{L}\pi_! i_! i^! \pi^! \Lambda[-1]
\end{tikzcd} \tag{4.4}$$

Starting at the bottom, the morphism δ arises from the distinguished triangle of motives over X:

$$i_*\Lambda[1] \xrightarrow{\delta} j_!\Lambda \to \Lambda \to i_*\Lambda. \tag{4.5}$$

Taking the dual we obtain the other localization triangle (4.1). Thus commutativity of the bottom square follows.

We can consider the exact same distinguished triangle as (4.5) in the analytic setting. This gives rise to the arrow $\delta^\vee$ in the third row of (4.4). Thus commutativity of the third square in (4.4) follows from the fact that the compatibility of the Betti realization with the six functors formalism is also compatible with the triangulations.

By Lemma B.6 in the appendix, the second square in (4.4) commutes if we take $\delta^\vee$ in the second row to be induced by the short exact sequence of singular cochain complexes. We leave it as an exercise to prove that this renders the top square in (4.4) commutative after applying H_n. ☐

4.2. Monoidality. Our next goal is to prove that the isomorphism of the proposition preserves the u. g. m. structures of the two representations (restricted to $\mathscr{D}^{\mathrm{g}}_{\mathrm{N}}$; cf. appendix A). But first we must define this structure on the representation $\mathrm{H}_0\widetilde{\mathrm{Bti}}^*\mathcal{R}_{\mathrm{A}}$.

Let (X_1, Z_1) and (X_2, Z_2) be two pairs of varieties, and set $\overline{X} = X_1 \times X_2$, $\overline{Z}_1 = Z_1 \times X_2$, $\overline{Z}_2 = X_1 \times Z_2$, $\overline{Z} = \overline{Z}_1 \cup \overline{Z}_2$. There is a canonical morphism (a motivic "cup product")

$$\mathrm{R}\overline{\pi}_* \overline{j}_{1!}\Lambda \otimes^{\mathrm{L}} \mathrm{R}\overline{\pi}_* \overline{j}_{2!}\Lambda \to \mathrm{R}\overline{\pi}_*(\overline{j}_{1!}\Lambda \otimes^{\mathrm{L}} \overline{j}_{2!}\Lambda) \cong \mathrm{R}\overline{\pi}_* \overline{j}_! \Lambda \tag{4.6}$$

and we obtain (for $\overline{n} = n_1 + n_2$)

$$\begin{aligned} \tilde{\tau} : \mathcal{R}_{\mathrm{A}}(\overline{X}, \overline{Z}, \overline{n}) &\xrightarrow{(4.6)^\vee} (\mathcal{R}_{\mathrm{A}}(\overline{X}, \overline{Z}_1, 0) \otimes^{\mathrm{L}} \mathcal{R}_{\mathrm{A}}(\overline{X}, \overline{Z}_2, 0))[\overline{n}] \underset{\sim}{\xrightarrow{\gamma}} \\ \mathcal{R}_{\mathrm{A}}(\overline{X}, \overline{Z}_1, n_1) \otimes^{\mathrm{L}} \mathcal{R}_{\mathrm{A}}(\overline{X}, \overline{Z}_2, n_2) &\xrightarrow{\mathcal{R}_{\mathrm{A}}(p_1) \otimes^{\mathrm{L}} \mathcal{R}_{\mathrm{A}}(p_2)} \mathcal{R}_{\mathrm{A}}(X_1, Z_1, n_1) \otimes^{\mathrm{L}} \mathcal{R}_{\mathrm{A}}(X_2, Z_2, n_2) \end{aligned}$$

where $p_i : \overline{X} \to X_i$ denotes the projection onto the ith factor. One word about the isomorphism γ: In the category of complexes there are two natural choices for γ, by following one of the two paths in the following square:

$$\begin{tikzcd} (\bullet_1 \otimes \bullet_2)[\overline{n}] \arrow[r] \arrow[d] & (\bullet_1 \otimes \bullet_2[n_2])[n_1] \arrow[d] \\ (\bullet_1[n_1] \otimes \bullet_2)[n_2] \arrow[r] & \bullet_1[n_1] \otimes \bullet_2[n_2] \end{tikzcd}$$

This square commutes up to the sign $(-1)^{n_1 \cdot n_2}$. We choose the γ which is the identity in degree 0. (Which of the two paths we choose thus depends on the sign conventions for the tensor product and shift in the category of chain complexes.)

We can now define the u. g. m. structure on $\mathrm{H}_0 \widetilde{\mathrm{Bti}}^* \mathcal{R}_{\mathrm{A}}$ as the following composition (for any $v_1, v_2 \in \mathscr{D}_{\mathrm{N}}$):

$$\begin{aligned} \tau_{(v_1, v_2)} : \mathrm{H}_0 \widetilde{\mathrm{Bti}}^* \mathcal{R}_{\mathrm{A}}(v_1 \times v_2) &\xrightarrow{\tilde{\tau}} \mathrm{H}_0 \widetilde{\mathrm{Bti}}^* (\mathcal{R}_{\mathrm{A}}(v_1) \otimes^{\mathrm{L}} \mathcal{R}_{\mathrm{A}}(v_2)) \\ &\xrightarrow{\sim} \mathrm{H}_0(\widetilde{\mathrm{Bti}}^* \mathcal{R}_{\mathrm{A}}(v_1) \otimes^{\mathrm{L}} \widetilde{\mathrm{Bti}}^* \mathcal{R}_{\mathrm{A}}(v_2)) \to \mathrm{H}_0 \widetilde{\mathrm{Bti}}^* \mathcal{R}_{\mathrm{A}}(v_1) \otimes \mathrm{H}_0 \widetilde{\mathrm{Bti}}^* \mathcal{R}_{\mathrm{A}}(v_2). \end{aligned}$$

Proposition 4.7 *Assume that Λ is a principal ideal domain. Then:*

(1) *The morphisms $\tau_{(v_1, v_2)}$ define a u. g. m. structure on the representation $\mathrm{H}_0 \widetilde{\mathrm{Bti}}^* \mathcal{R}_A : \mathscr{D}_N^{\mathrm{g}} \to$ $\mathbf{coMod}^{\mathrm{f}}(\mathbf{H}_A)$.*
(2) *The isomorphism of the previous proposition is compatible with the u. g. m. structures, i. e. it induces an isomorphism of u. g. m. representations*

$$\begin{tikzcd} \mathscr{D}_N^{\mathrm{g}} \arrow[r, "\mathrm{H}_0 \widetilde{\mathrm{Bti}}^* \mathcal{R}_A"] \arrow[dr, "\mathrm{H}_\bullet"'] & \mathbf{coMod}^{\mathrm{f}}(\mathbf{H}_A) \arrow[d, "o"] \\ & \mathbf{Mod}^{\mathrm{f}}(\Lambda) \end{tikzcd}$$

Proof. For the first part we need to check that in $\mathbf{coMod}^{\mathrm{f}}(\mathbf{H}_{\mathrm{A}})$, some morphisms are invertible and some diagrams commute. Both these properties can be checked after applying $o : \mathbf{coMod}^{\mathrm{f}}(\mathbf{H}_{\mathrm{A}}) \to \mathbf{Mod}^{\mathrm{f}}(\Lambda)$. Since the corresponding properties are true for the representation $\mathrm{H}_\bullet$, we see that to prove the proposition, it suffices to show that the isomorphism of the previous proposition takes $o \circ \tau$ to the Künneth isomorphism.

Write $\nu_1, \nu_2, \overline{\nu}$ for the motives $\mathcal{R}_\Lambda(X_1, Z_1, 0), \mathcal{R}_\Lambda(X_2, Z_2, 0)$ and $\mathcal{R}_\Lambda(\overline{X}, \overline{Z}, 0)$, respectively. Consider then the following diagram:[5]

$$\begin{tikzcd}
\mathrm{H}_{\overline{n}}\mathrm{Bti}^*\overline{\nu} \arrow[r, "\gamma^{-1}\circ\tilde{\tau}"'] \arrow[d, "\sim"'] & \mathrm{H}_{\overline{n}}(\mathrm{Bti}^*\nu_1 \otimes^{\mathrm{L}} \mathrm{Bti}^*\nu_2) \arrow[r, "\sim", "\gamma"'] \arrow[d, "\sim"'] & \mathrm{H}_{n_1}\mathrm{Bti}^*\nu_1 \otimes \mathrm{H}_{n_2}\mathrm{Bti}^*\nu_2 \arrow[d, "\sim"] \\
\mathrm{H}_{\overline{n}}(\overline{X}, \overline{Z}) \arrow[r, "\sim", "\text{"AW"}"'] & \mathrm{H}_{\overline{n}}(\mathrm{Sg}(X_1, Z_1) \otimes \mathrm{Sg}(X_2, Z_2)) \arrow[r, "\sim", "\gamma"'] & \mathrm{H}_{n_1}(X_1, Z_1) \otimes \mathrm{H}_{n_2}(X_2, Z_2)
\end{tikzcd}$$

where we have written $\overline{n}$ for the sum $n_1 + n_2$. The right square clearly commutes. The bottom horizontal arrow on the left is induced by the Alexander-Whitney map (it is really a zig-zag on the level of complexes) and γ in the bottom right induces the canonical isomorphism of the (algebraic) Künneth formula, hence it follows that the composition of the arrows in the bottom row is nothing but the (topological) Künneth isomorphism. On the other hand, the composition of the arrows in the top row is τ. Hence we are reduced to prove commutativity of the left square in the diagram above, and it suffices to do so before applying $\mathrm{H}_{\overline{n}}$.

We now write $\overline{\nu}_i$ for the motive $\mathcal{R}_\Lambda(\overline{X}, \overline{Z}_i, 0)$. Decompose $\tilde{\tau}$ according to its definition, and use the fact that the Alexander-Whitney map admits a similar decomposition in $\mathbf{D}(\Lambda)$:

$$\begin{tikzcd}
\mathrm{Bti}^*\overline{\nu} \arrow[r, "(4.6)^\vee"] \arrow[dd, "\sim"] & \mathrm{Bti}^*(\overline{\nu}_1 \otimes^{\mathrm{L}} \overline{\nu}_2) \arrow[r, "\mathcal{R}_\Lambda(p_1)\otimes^{\mathrm{L}}\mathcal{R}_\Lambda(p_2)"] \arrow[d, "\sim"'] & \mathrm{Bti}^*(\nu_1 \otimes^{\mathrm{L}} \nu_2) \arrow[d, "\sim"'] \\
 & \mathrm{Bti}^*\overline{\nu}_1 \otimes^{\mathrm{L}} \mathrm{Bti}^*\overline{\nu}_2 \arrow[r, "\mathcal{R}_\Lambda(p_1)\otimes^{\mathrm{L}}\mathcal{R}_\Lambda(p_2)"] \arrow[d, "\sim"'] & \mathrm{Bti}^*\nu_1 \otimes^{\mathrm{L}} \mathrm{Bti}^*\nu_2 \arrow[d, "\sim"'] \\
\mathrm{Sg}(\overline{X}, \overline{Z}) \arrow[r, "\text{"AW-diag"}"] & \mathrm{Sg}(\overline{X}, \overline{Z}_1) \otimes \mathrm{Sg}(\overline{X}, \overline{Z}_2) \arrow[r, "\mathrm{Sg}(p_1)\otimes\mathrm{Sg}(p_2)"] & \mathrm{Sg}(X_1, Z_1) \otimes \mathrm{Sg}(X_2, Z_2)
\end{tikzcd}$$

We wrote "AW-diag" for the Alexander-Whitney diagonal approximation which is a zig-zag of morphisms of complexes (see below). It is clear that the upper right square commutes, as does the lower right square by the proof of the previous proposition. For the square on the left, notice that (4.6) equally defines a morphism in the category **AnDA**. Thus we now denote by $\overline{\nu}, \overline{\nu}_i$ the same expressions in terms of the four functors in **AnDA** instead of **DA**. Then the proof of the proposition will be complete if we can prove commutative the following diagram (in which all vertical arrows are the canonical invertible ones):

$$\begin{tikzcd}
\overline{\nu} \arrow[rr, "(4.6)^\vee"] \arrow[d, dash] & & \overline{\nu}_1 \otimes^{\mathrm{L}} \overline{\nu}_2 \arrow[d, dash] \\
(\overline{\pi}_*\overline{j}_!\Lambda)^\vee \arrow[rr, "(4.6)^\vee"] \arrow[d, dash] & & (\overline{\pi}_*\overline{j}_{1!}\Lambda \otimes^{\mathrm{L}} \overline{\pi}_*\overline{j}_{2!}\Lambda)^\vee \arrow[d, dash] \\
\mathrm{Sg}(\overline{X}, \overline{Z})^{\vee\vee} \arrow[d, dash] & \mathrm{Sg}(\overline{X}, \overline{Z}_1 + \overline{Z}_2)^{\vee\vee} \arrow[l, "\sim"'] \arrow[r, "\smile^\vee"] \arrow[d, dash] & (\mathrm{Sg}(\overline{X}, \overline{Z}_1)^\vee \otimes \mathrm{Sg}(\overline{X}, \overline{Z}_2)^\vee)^\vee \arrow[d, dash] \\
\mathrm{Sg}(\overline{X}, \overline{Z}) & \mathrm{Sg}(\overline{X}, \overline{Z}_1 + \overline{Z}_2) \arrow[l, "\sim"'] \arrow[r, "\text{AW-diag}"] & \mathrm{Sg}(\overline{X}, \overline{Z}_1) \otimes \mathrm{Sg}(\overline{X}, \overline{Z}_2)
\end{tikzcd}$$

[5]Here, as in the sequel, we often write $\mathrm{Sg}(X, Z)$ for $\mathrm{Sg}(X^{\mathrm{an}}, Z^{\mathrm{an}})$.

Here, $\mathrm{Sg}(\overline{X}, \overline{Z}_1 + \overline{Z}_2)$ denotes the free Λ-module on simplices in $\overline{X}$ which are neither contained in $\overline{Z}_1$ nor in $\overline{Z}_2$. The first rectangle clearly commutes, the second does so by Lemma B.8 (which may be applied because of Fact 4.2), the bottom right square is well-known to commute (see e. g. [18, VII, 8]), and the bottom left one obviously commutes as well. ☐

Using the proposition we obtain the following commutative (up to a u. g. m. isomorphism) rectangle of u. g. m. representations:

$$\begin{tikzcd} \mathscr{D}_{\mathrm{N}}^{\mathrm{g}} \arrow[r, "\mathrm{H}_0\widetilde{\mathrm{Bti}}^*\mathcal{R}_{\mathrm{A}}"] \arrow[d, "\tilde{\mathrm{H}}_\bullet"'] & \mathbf{coMod}^{\mathrm{f}}(\mathbf{H}_{\mathrm{A}}) \arrow[d, "o"] \\ \mathbf{coMod}^{\mathrm{f}}(\mathbf{H}_{\mathrm{N}}^{\mathrm{eff}}) \arrow[r, "o"'] \arrow[ur, dotted] & \mathbf{Mod}^{\mathrm{f}}(\Lambda) \end{tikzcd} \tag{4.8}$$

Still assuming that Λ is a principal ideal domain we know, by Lemma 3.4 together with Fact C.1, that $\mathbf{coMod}^{\mathrm{f}}(\mathbf{H}_{\mathrm{A}})$ is an abelian category. Hence Corollary 1.1 yields a monoidal functor $\overline{\varphi_{\mathrm{A}}}$ represented by the dotted arrow in the diagram (4.8), rendering it commutative (up to monoidal isomorphism). It then follows from [64, II, 3.3.1] that $\overline{\varphi_{\mathrm{A}}}$ necessarily arises from a map of bialgebras $\varphi_{\mathrm{A}} : \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}} \to \mathbf{H}_{\mathrm{A}}$.

Using the commutativity of (4.8) one easily checks that $\varphi_{\mathrm{A}}(s_{\mathrm{N}}) = s_{\mathrm{A}} \in \mathbf{H}_{\mathrm{A}}$ hence φ_{A} factors through $\mathbf{H}_{\mathrm{N}}$:

$$\begin{tikzcd} \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}} \arrow[r, "\varphi_{\mathrm{A}}"] \arrow[d, "\iota"'] & \mathbf{H}_{\mathrm{A}} \\ \mathbf{H}_{\mathrm{N}} \arrow[ur, "\varphi_{\mathrm{A}}"'] & \end{tikzcd} \tag{4.9}$$

5. Basic Lemma, and applications

Now we would like to construct a morphism φ_{N} in the other direction. The construction relies on Nori's functor which associates to an affine variety a complex in $\mathbf{HM}^{\mathrm{eff}}$ computing its homology, and which in turn relies on the "Basic Lemma". We recall them both in this section (basically following [23]), while we prove the existence of φ_{N} in the next section.

We first recall the Basic Lemma in the form Nori formulated it [59, Thm. 2.1]; it was independently proven by Beilinson in a more general context [10, Lem. 3.3].

Fact 5.1 (Basic Lemma) *Let X be an affine variety of dimension n, and $W \subset X$ a closed subvariety of dimension $\leq n-1$. Then there exists a closed subvariety $W \subset Z \subset X$ of dimension $\leq n-1$ such that $\mathrm{H}_\bullet(X^{\mathrm{an}}, Z^{\mathrm{an}}; \mathbb{Z})$ is a free abelian group concentrated in degree n.*

We call a pair (X, Z, n) *very good* (cf. [39, Def. D.1]) if either X is affine of dimension n, Z is of dimension $\leq n-1$, $X \backslash Z$ is smooth, and $\mathrm{H}_\bullet(X^{\mathrm{an}}, Z^{\mathrm{an}}; \mathbb{Z})$ is a free abelian group concentrated in degree n, or if $X = Z$ is affine of dimension less than n. Thus the Basic Lemma implies that any pair (X, W, n) with $\dim(X) = n > \dim(W)$ can be embedded into a very good pair (X, Z, n).

Nori applied this result to construct "cellular decompositions" of affine varieties as follows. Let X be an affine variety. A *filtration of* X is an increasing sequence $F_\bullet = (F_i X)_{i \in \mathbb{Z}}$ of closed subvarieties of X such that

- $\dim(F_i X) \leq i$ for all i (in particular, $F_{-1}X = \varnothing$),
- $F_n X = X$ for some $n \in \mathbb{Z}$.

The minimal $n \in \mathbb{Z}$ such that $F_n X = X$ is called the *length* of $F_\bullet$ (by convention, for $X = \varnothing$ this length is defined to be $-\infty$). Clearly the filtrations of X form a directed set. A filtration $F_\bullet$ is called *very good* if $(F_i X, F_{i-1} X, i)$ is a very good pair for each i. The following result says that very good filtrations form a cofinal set.

Corollary 5.2 *Let X be an affine variety, and $F_\bullet$ a filtration of X. Then there exists a very good filtration $G_\bullet \supset F_\bullet$ of X. In particular, every affine variety of dimension n admits a very good filtration of length n.*

Proof. We do induction on the length n of $F_\bullet$. Every filtration of length $n = -\infty$ or $n = 0$ is very good. Assume now $n > 0$. Set $G_i X = X$ for all $i \geq n$. If $\dim(X) < n$, let $G_{n-1} X = X$. If $\dim(X) = n$ then, applying the Basic Lemma to the pair $(X, F_{n-1} X)$, we obtain a closed subvariety $F_{n-1} X \subset Z \subset X$ such that (X, Z, n) is very good, and we set $G_{n-1} X = Z$ in this case. Now apply the induction hypothesis to the filtration $\varnothing \subset F_0 X \subset \cdots \subset F_{n-2} X \subset G_{n-1} X$. ☐

To any filtration $F_\bullet$ of X we associate the complex $\mathrm{H}_\bullet(X, F_\bullet) = \mathrm{H}_\bullet(X, F_\bullet; \Lambda)$,

$$\mathrm{H}_n(X^{\mathrm{an}}, F_{n-1} X^{\mathrm{an}}) \to \mathrm{H}_{n-1}(F_{n-1} X^{\mathrm{an}}, F_{n-2} X^{\mathrm{an}}) \to \cdots \to \mathrm{H}_0(F_0 X^{\mathrm{an}}, \varnothing),$$

concentrated in the range of degrees $[0, n]$, where the differentials are the boundary maps from the homology sequence of a triple. It follows that $\mathrm{H}_\bullet(X, F_\bullet)$ can be considered as an object of $\mathbf{Cpl}(\mathbf{HM}^{\mathrm{eff}})$. For a very good filtration, this complex computes the singular homology of X^{an} for the same reason that cellular homology and singular homology agree. (For a more precise statement see Fact 5.3 below.) It then follows from the corollary that also

$$C(X) := \varinjlim_{F_\bullet} \mathrm{H}_\bullet(X, F_\bullet) \in \mathbf{Cpl}(\mathbf{HM}^{\mathrm{eff}}_\oplus)$$

computes singular homology of X^{an}.

Given a morphism of affine varieties $f : X \to Y$, and a filtration $F_\bullet$ on X, we obtain a filtration

$$\overline{f(X)} \supset \overline{f(F_{n-1} X)} \supset \cdots \supset \overline{f(F_0 X)} \supset \varnothing$$

of $\overline{f(X)}$. Let $m = \dim(Y)$, and define a filtration $G_\bullet$ on Y by

$$G_i Y = \begin{cases} \overline{f(F_i X)} & : i < m \\ Y & : i \geq m. \end{cases}$$

This induces a morphism $\mathrm{H}_\bullet(X, F_\bullet) \to \mathrm{H}_\bullet(Y, G_\bullet)$ in $\mathbf{Cpl}(\mathbf{HM}^{\mathrm{eff}})$. It follows that C defines a functor $\mathrm{AffVar} \to \mathbf{Cpl}(\mathbf{HM}^{\mathrm{eff}}_\oplus)$ on affine varieties.

Now given filtrations $F_\bullet$ and $G_\bullet$ on affine varieties X and Y, respectively, we form the filtration $(F \times G)_\bullet$ on $X \times Y$, setting $(F \times G)_i(X \times Y)$ to be $\cup_{p+q=i} F_p X \times G_q Y$. There is a canonical morphism $\mathrm{H}_\bullet(X, F_\bullet) \otimes \mathrm{H}_\bullet(Y, G_\bullet) \to \mathrm{H}_\bullet(X \times Y, (F \times G)_\bullet)$ which induces a morphism $C(X) \otimes C(Y) \to C(X \times Y)$. One can check that this endows C with a lax monoidal structure.

To go further we have to make precise the relation between the functors C and $\mathrm{Sg} \circ \mathrm{An}$. For this, following [23], we consider the subcomplex $P(X)$ of $\mathrm{Sg}(X^{\mathrm{an}})$ which in degree p consists of singular p-chains in X^{an} whose image is contained in Z^{an} for some closed subvariety $Z \subset X$ of dimension $\leq p$, and whose boundary lies in W^{an} for some closed subvariety $W \subset X$ of dimension $\leq p-1$. Such a singular chain defines a homology class in $\mathrm{H}_p(Z^{\mathrm{an}}, W^{\mathrm{an}}; \Lambda)$ hence there is a canonical map $P(X) \twoheadrightarrow oC(X)$ (here, as usual, o forgets the comodule structure). The following result follows from the Basic Lemma and some linear algebra.

Fact 5.3 ([23, Lem. 4.14]) *Let X be an affine variety. Both maps of chain complexes of Λ-modules*

$$oC(X) \leftarrow P(X) \rightarrow \mathrm{Sg}(X^{\mathrm{an}})$$

are quasi-isomorphisms.

It is clear that P defines a functor AffVar $\to \mathbf{Cpl}(\Lambda)$ and that the two maps above are natural in X. Moreover P comes with a canonical lax monoidal structure induced from the one on Sg (the Eilenberg-Zilber transformation), and which is compatible with the one on C defined before.

Corollary 5.4 *The maps of the previous lemma define monoidal transformations between lax monoidal functors*

$$oC \leftarrow P \rightarrow \mathrm{Sg} \circ \mathrm{An}$$

from AffVar *to* $\mathbf{Cpl}(\Lambda)$. *If Λ is a principal ideal domain then after composing with the canonical (lax monoidal) functor $\mathbf{Cpl}(\Lambda) \to \mathbf{D}(\Lambda)$ these become monoidal transformations of monoidal functors.*

Proof. Given affine varieties X and Y, we have a commutative diagram:

$$\begin{tikzcd}
oC(X) \otimes oC(Y) \arrow[d] & P(X) \otimes P(Y) \arrow[l] \arrow[r] \arrow[d] & \mathrm{Sg}(X^{\mathrm{an}}) \otimes \mathrm{Sg}(Y^{\mathrm{an}}) \arrow[d] \\
oC(X \times Y) & P(X \times Y) \arrow[l] \arrow[r] & \mathrm{Sg}(X^{\mathrm{an}} \times Y^{\mathrm{an}})
\end{tikzcd}$$

Sg takes values in (complexes of) free Λ-modules, and $oC(X)$ is a direct limit of (complexes of) finitely generated free Λ-modules (by Corollary 5.2) hence is itself a complex of flat Λ-modules. If Λ is a principal ideal domain then P necessarily takes values in (complexes of) free Λ-modules as well. In conclusion, under the assumptions of the corollary, all tensor factors in the diagram above are flat.

It follows that all horizontal arrows in the diagram above are quasi-isomorphisms, and that in the upper row, all tensor products are equal to their derived versions. The corollary now follows from the fact that the right-most vertical arrow is a quasi-isomorphism. ☐

Remark 5.5 There are several ways to extend the functor C to all varieties, as explained in [59, p. 9]. However, for our purposes this will not be necessary as in the end we are interested only in the induced functor $\mathbf{DA}^{\mathrm{eff}} \to \mathbf{D}(\mathbf{HM}^{\mathrm{eff}}_{\oplus})$, and here we can use the equivalence between $\mathbf{DA}^{\mathrm{eff}}$ and $\mathbf{DA}^{\mathrm{eff}}_{\mathrm{aff}}$.

6. Motivic realization

We would now like to explain how the lax monoidal functor C constructed in the previous section induces a functor on categories of motives. The case of effective motives is treated in §6.1 and as an application we deduce a morphism of bialgebras $\varphi_{\mathrm{N}} : \mathbf{H}^{\mathrm{eff}}_{\mathrm{A}} \to \mathbf{H}^{\mathrm{eff}}_{\mathrm{N}}$ in §6.2. In §6.3 and §6.4 we treat the case of effective motives with transfers and non-effective motives, respectively.

6.1. Construction.

First we define the functor

$$C^{*} : \mathbf{U}\mathrm{SmAff} \to \mathbf{Cpl}(\mathbf{HM}^{\mathrm{eff}}_{\oplus})$$

by the coend formula for a "left dg Kan extension" (cf. §III.1.1)

$$K \longmapsto \int^{X \in \mathrm{SmAff}} K(X) \otimes C(X),$$

where the comodule structure is induced from the one on the right tensor factor. Recall that the coend appearing in the definition is nothing but the coequalizer of the diagram

$$\bigoplus_{X\to Y} K(Y)\otimes C(X) \rightrightarrows \bigoplus_{X} K(X)\otimes C(X),$$

where the two arrows are induced by the functoriality of K and C, respectively. We will prove below that C^* is left Quillen for the projective model structure on the domain (even induced from the injective model structure on $\mathbf{Cpl}(\Lambda)$) and the injective model structure on the codomain (cf. Fact C.2).

Proposition 6.1 *Let Λ be a principal ideal domain. $\mathrm{L}C^*$ inherits a monoidal structure, and takes $(\mathbb{A}^1,\tau)$-local equivalences to quasi-isomorphisms. Moreover, it makes the following square commutative up to monoidal triangulated isomorphism.*

$$\begin{tikzcd}
\mathbf{DA}^{\mathrm{eff}}_{\mathrm{aff}} \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\mathrm{L}C^*"'] & \mathbf{DA}^{\mathrm{eff}} \arrow[d, "\mathrm{Bti}^{\mathrm{eff},*}"] \\
\mathbf{D}(\mathbf{HM}^{\mathrm{eff}}_{\oplus}) \arrow[r, "\mathbf{D}(o)"'] & \mathbf{D}(\Lambda)
\end{tikzcd}$$

Remark 6.2 In [59, p. 9], Nori remarks that for an arbitrary variety X and an affine open cover $\mathcal{U}=(U_1,\dots,U_q)$ of X, the complex

$$\mathrm{Tot}(\cdots \to \oplus_{1\le i_1<\cdots<i_p\le q} C(U_{i_1}\cap\cdots\cap U_{i_p}) \to \cdots) \tag{6.3}$$

"computes the homology of X". This can also be explained using the proposition, at least if X is smooth. Namely, in that case, the complex

$$\cdots \to \oplus_{1\le i_1<\cdots<i_p\le q}\Lambda(U_{i_1}\cap\cdots\cap U_{i_p}) \to \cdots$$

defines a cofibrant replacement of $\Lambda(X)$ (as we will see in Lemma 7.1), and C^* applied to it is just (6.3), as follows from Lemma III.2.23. Hence the proposition tells us that the underlying complex of Λ-modules in (6.3) is nothing but $\mathrm{Bti}^{\mathrm{eff},*}\Lambda(X)\cong \mathrm{Sg}(X^{\mathrm{an}})$ (by Proposition 2.2).

We will come back to this explicit description of $\mathrm{L}C^*$ in section 7.1 where (6.3) is denoted by $C^{\mathcal{U}}(X)$.

Proof of Proposition 6.1. Just as C admits a left Kan extension, so do P, Sg and An:

$$\begin{tikzcd}
\mathrm{SmAff} \arrow[r] \arrow[d, "C"'] & \mathbf{U}\mathrm{SmAff} \arrow[dl, "C^*"] \arrow[ddl, "P^*"] \arrow[dd, "\mathrm{An}^*"] \\
\mathbf{Cpl}(\mathbf{HM}^{\mathrm{eff}}_{\oplus}) \arrow[d, "o"'] & \\
\mathbf{Cpl}(\Lambda) & \mathbf{U}\mathrm{Man}_{\mathbb{C}} \arrow[l, "\mathrm{Sg}^*"]
\end{tikzcd}$$

Endow $\mathbf{Cpl}(\Lambda)$ and $\mathbf{Cpl}(\mathbf{HM}^{\mathrm{eff}}_{\oplus})$ with the injective model structures (cf. Fact C.2), and the presheaf categories with the projective model structures deduced from the injective model structure on $\mathbf{Cpl}(\Lambda)$. We then use Lemma III.1.5 to prove that all these Kan extensions are left Quillen functors. For Sg, P, and C this follows from the fact that they take values in complexes of flat objects (see the proof of Corollary 5.4) hence the tensor product with these complexes is a left Quillen functor for the injective model structure. For An, this is because evaluation at a smooth affine scheme X clearly preserves (trivial) fibrations.

Also, C^* as well as P^* and $\mathrm{Sg}^*\mathrm{An}^* \cong (\mathrm{Sg}\circ\mathrm{An})^*$ inherit canonically lax monoidal structures (Proposition III.1.2). From Corollary 5.4 we deduce monoidal transformations

$$o\circ C^* \leftarrow P^* \rightarrow \mathrm{Sg}^*\circ\mathrm{An}^* \tag{6.4}$$

of lax monoidal functors defined on **U**SmAff taking values in $\mathbf{Cpl}(\Lambda)$. They give rise to monoidal triangulated transformations between the corresponding left derived functors (this uses Lemma C.4), and to prove that these transformations are invertible, it suffices to check it on objects of the form $\Lambda(X)$, where $X \in \mathrm{SmAff}$ (by Lemma III.2.22 these compactly generate the derived category). These objects are cofibrant, and we conclude since the maps

$$o\circ C^*\Lambda(X) \leftarrow P^*\Lambda(X) \rightarrow \mathrm{Sg}^*\circ\mathrm{An}^*\Lambda(X)$$

are identified with the quasi-isomorphisms

$$o\circ C(X) \leftarrow P(X) \rightarrow \mathrm{Sg}\circ\mathrm{An}(X).$$

We have now constructed a diagram of lax monoidal triangulated functors

$$\begin{tikzcd}
\mathbf{D}(\mathbf{HM}^{\mathrm{eff}}_{\oplus}) \arrow[d, "\mathbf{D}(o)"'] & \mathbf{D}(\mathbf{U}\mathrm{SmAff}) \arrow[l, "\mathrm{L}C^*"'] \arrow[d, "\mathrm{An}^*"] \arrow[dl, "\mathrm{L}P^*"] \\
\mathbf{D}(\Lambda) & \mathbf{D}(\mathbf{U}\mathrm{Man}_{\mathbb{C}}) \arrow[l, "\mathrm{LSg}^*"]
\end{tikzcd}$$

which commutes up to monoidal triangulated isomorphism. Using the identification $\mathbf{DA}^{\mathrm{eff}}_{\mathrm{aff}} \xrightarrow{\sim} \mathbf{DA}^{\mathrm{eff}}$ the result therefore follows from Proposition 2.2. ☐

6.2. Bialgebra morphism. Since $\mathbf{H}^{\mathrm{eff}}_{\mathrm{N}}$ is a flat Λ-module, $\mathbf{Cpl}(\mathbf{HM}^{\mathrm{eff}}_{\oplus})$ is canonically equivalent to the category of $\mathbf{H}^{\mathrm{eff}}_{\mathrm{N}}$-comodules in $\mathbf{Cpl}(\Lambda)$ (Fact C.1), and we can consider the following composition of monoidal functors:

$$\mathcal{R}_{\mathrm{N}} : \mathbf{DA}^{\mathrm{eff}} \simeq \mathbf{DA}^{\mathrm{eff}}_{\mathrm{aff}} \xrightarrow{\mathrm{L}C^*} \mathbf{D}(\mathbf{HM}^{\mathrm{eff}}_{\oplus}) \rightarrow \mathbf{coMod}(\mathbf{H}^{\mathrm{eff}}_{\mathrm{N}})^{\mathbf{D}(\Lambda)},$$

where the last term denotes the category of $\mathbf{H}^{\mathrm{eff}}_{\mathrm{N}}$-comodules in $\mathbf{D}(\Lambda)$. The upshot of the discussion so far is that we obtain a diagram

$$\begin{tikzcd}
\mathbf{DA}^{\mathrm{eff}} \arrow[r, "\mathcal{R}_{\mathrm{N}}"] \arrow[dr, "\mathrm{Bti}^{\mathrm{eff},*}"'] & \mathbf{coMod}(\mathbf{H}^{\mathrm{eff}}_{\mathrm{N}})^{\mathbf{D}(\Lambda)} \arrow[d, "o"] \\
& \mathbf{D}(\Lambda)
\end{tikzcd}$$

of monoidal functors which commutes up to monoidal isomorphism. To apply Fact 3.2 we still need to verify the following Lemma.

Lemma 6.5 *Let $K \in \mathbf{D}(\Lambda)$, Λ a principal ideal domain. Then the coaction of $\mathbf{H}^{\mathrm{eff}}_{\mathrm{N}}$ on $\mathcal{R}_{\mathrm{N}}(K_{\mathrm{cst}})$ is trivial.*

Proof. We may assume that K is projective cofibrant consisting of free Λ-modules in each degree (for example by Proposition III.3.4). K_{cst} is projective cofibrant, and in each degree consists of a direct sum of $\Lambda(\mathrm{Spec}(k))$, the presheaf represented by $\mathrm{Spec}(k)$. Hence $\mathrm{L}C^*(K_{\mathrm{cst}}) = C^*(K_{\mathrm{cst}})$ is a complex which in each degree consists of a direct sum of $C(\mathrm{Spec}(k))$ on which $\mathbf{H}^{\mathrm{eff}}_{\mathrm{N}}$ coacts trivially. Hence $\mathbf{H}^{\mathrm{eff}}_{\mathrm{N}}$ also coacts trivially on $\mathcal{R}_{\mathrm{N}}(K_{\mathrm{cst}})$. ☐

Corollary 6.6 *Assume that Λ is a principal ideal domain. There is a morphism of bialgebras $\mathcal{H}_A^{\mathrm{eff}} \to \mathbf{H}_N^{\mathrm{eff}}$ inducing $\varphi_N : \mathbf{H}_A^{\mathrm{eff}} \to \mathbf{H}_N^{\mathrm{eff}}$ and rendering the following diagrams commutative up to monoidal isomorphism:*

$$\begin{tikzcd}
\mathbf{DA}^{\mathrm{eff}} \arrow[r, "\mathcal{R}_N"] \arrow[d, "\widetilde{\mathrm{Bti}}^{\mathrm{eff},*}"'] & \mathbf{coMod}(\mathbf{H}_N^{\mathrm{eff}})^{\mathbf{D}(\Lambda)} \arrow[d, "o"] \\
\mathbf{coMod}(\mathcal{H}_A^{\mathrm{eff}}) \arrow[r, "o"'] \arrow[ur] & \mathbf{D}(\Lambda)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{DA}^{\mathrm{eff}} \arrow[r, "\mathrm{H}_0\mathcal{R}_N"] \arrow[d, "\mathrm{H}_0\widetilde{\mathrm{Bti}}^{\mathrm{eff},*}"'] & \mathbf{coMod}(\mathbf{H}_N^{\mathrm{eff}}) \arrow[d, "o"] \\
\mathbf{coMod}(\mathbf{H}_A^{\mathrm{eff}}) \arrow[r, "o"'] \arrow[ur, "\overline{\varphi_N}"'] & \mathbf{Mod}(\Lambda)
\end{tikzcd}$$

There are two ways to obtain similar statements in the stable setting. The easier one is to check that $\mathcal{H}_{\mathrm{A}}^{\mathrm{eff}} \to \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}}$ passes to the localizations $\mathcal{H}_{\mathrm{A}} \to \mathbf{H}_{\mathrm{N}}$ and consider the composition

$$\mathbf{DA} \xrightarrow{\widetilde{\mathrm{Bti}}^*} \mathbf{coMod}(\mathcal{H}_{\mathrm{A}}) \to \mathbf{coMod}(\mathbf{H}_{\mathrm{N}})^{\mathbf{D}(\Lambda)}.$$

This will be sufficient for our main theorem, and we will pursue it in section 7.1. However, it might seem more natural and lead to stronger results to extend the construction of C^* to the level of spectra and derive the resulting functor. This will be done in §6.4.

6.3. Transfers.

The remainder of §6 will not be strictly necessary for our main theorem but the results obtained here are of independent interest. In §6.3, our goal is Proposition 6.7 where we prove that LC^* extends to effective motives with transfers.

Recall ([59, §3.1]) that to a finite correspondence $X \to S^d Y$ of degree d between affine schemes, Nori associates a morphism $C(X) \to C(Y)$, defined as the composition

$$C(X) \to C(S^d Y) \xleftarrow{\sim} C(Y^d)_{\Sigma_d} \xrightarrow{\sum_{i=1}^d C(p_i)} C(Y),$$

where $(\bullet)_{\Sigma_d}$ denotes the Σ_d-coinvariants, and where the $p_i : Y^d \to Y$ are the canonical projections. As proved in [31], this induces a functor $C_{\mathrm{tr}} : \mathrm{SmAffCor} \to \mathbf{Cpl}(\mathbf{HM}_{\oplus}^{\mathrm{eff}})$ on smooth affine correspondences, and the same procedure as above yields a left Quillen functor $C_{\mathrm{tr}}^* : \mathbf{U}\mathrm{SmAffCor} \to \mathbf{Cpl}(\mathbf{HM}_{\oplus}^{\mathrm{eff}})$ for the projective model structure on the domain and the injective model structure on the codomain.

Proposition 6.7 *Let Λ be a principal ideal domain. $\mathrm{LC}_{\mathrm{tr}}^*$ inherits a monoidal structure, and takes $(\mathbb{A}^1, \tau)$-local equivalences to quasi-isomorphisms. Moreover, it fits into the following diagram, commutative up to monoidal triangulated isomorphism.*

$$\begin{tikzcd}
\mathbf{DA}_{\mathrm{aff}}^{\mathrm{eff}} \arrow[r, "\mathrm{LC}^*"] \arrow[d] & \mathbf{D}(\mathbf{HM}_{\oplus}^{\mathrm{eff}}) \\
\mathbf{DM}_{\mathrm{aff}}^{\mathrm{eff}} \arrow[ur, "\mathrm{LC}_{\mathrm{tr}}^*"'] &
\end{tikzcd}$$

The vertical arrow is given by "adding transfers". By Lemma III.4.17 there is a canonical triangulated monoidal equivalence $\mathbf{DM}_{\mathrm{aff}}^{\mathrm{eff}} \simeq \mathbf{DM}^{\mathrm{eff}}$.

Proof. It is proved in [31] that the lax monoidal structure on C is natural with respect to finite correspondences. It follows that C_{tr}^* and $\mathrm{LC}_{\mathrm{tr}}^*$ inherit lax monoidal structures. To check that the latter is *strong* monoidal it suffices to prove that $C_{\mathrm{tr}}^*\Lambda_{\mathrm{tr}}(X) \otimes C_{\mathrm{tr}}^*\Lambda_{\mathrm{tr}}(Y) \to C_{\mathrm{tr}}^*(\Lambda_{\mathrm{tr}}(X) \otimes \Lambda_{\mathrm{tr}}(Y))$ is a quasi-isomorphism for any smooth affine schemes X and Y (because the image of the Yoneda embedding $\Lambda_{\mathrm{tr}} : \mathrm{SmAffCor} \to \mathbf{D}(\mathbf{U}(\mathrm{SmAffCor}))$ compactly generates the whole triangulated category; see Lemma III.2.22). But this morphism is canonically identified with $C(X) \otimes C(Y) \to C(X \times Y)$ which we know to be a quasi-isomorphism.

For the first claim we need to prove that the right adjoint $C_{\mathrm{tr},*}$ takes a fibrant object K to an $(\mathbb{A}^1, \tau)$-fibrant presheaf of complexes. In other words, we need to check that

- $C_{\mathrm{tr},*}K$ satisfies descent with respect to τ-hypercovers; and
- $C_{\mathrm{tr},*}K(X) \to C_{\mathrm{tr},*}K(\mathbb{A}^1_X)$ is a quasi-isomorphism for every smooth affine scheme X.

But we can restrict $C_{\mathrm{tr},*}K$ to the site SmAff without affecting any of the two conditions hence they are both satisfied since this is true for C_*K (by Proposition 6.1).

We just used that the composition $\mathrm{SmAff} \to \mathrm{SmAffCor} \xrightarrow{C_{\mathrm{tr}}} \mathbf{Cpl}(\mathbf{HM}^{\mathrm{eff}})$ coincides with C which yields an isomorphism $C^*_{\mathrm{tr}} a_{\mathrm{tr}} \cong C^*$ and therefore also a triangulated isomorphism

$$\mathrm{L}C^*_{\mathrm{tr}}\mathrm{L}a_{\mathrm{tr}} \cong \mathrm{L}C^* : \mathbf{DA}^{\mathrm{eff}}_{\mathrm{aff}} \to \mathbf{D}(\mathbf{HM}^{\mathrm{eff}}_{\oplus}), \tag{6.8}$$

where a_{tr} denotes the functor which "adds transfers", left adjoint to o_{tr}, "forgetting transfers". This concludes the proof of the proposition. ☐

6.4. Stabilization. In this subsection we will develop the stable motivic realizations for motives with and without transfers in parallel. Statements containing the symbol (tr) thus have two obvious interpretations.

For any flat complex of comodules K, there is an injective stable model structure on the category of symmetric K-spectra, by Proposition C.3. Denote by T, as in section 2, a cofibrant replacement of $\Lambda(\mathbb{A}^1)/\Lambda(\mathbb{G}_m)$ and set $T_{\mathrm{tr}} = a_{\mathrm{tr}}T$. Notice that, canonically, $C^*_{\mathrm{tr}}T_{\mathrm{tr}} \cong C^*T$.

Lemma 6.9

(1) *The canonical morphism of bialgebras $\iota : \mathbf{H}^{\mathrm{eff}}_N \to \mathbf{H}_N$ induces a functor*

$$\mathbf{Spt}^{\Sigma}_{C^*T}\,\mathbf{Cpl}(\mathbf{HM}^{\mathrm{eff}}_{\oplus}) \xrightarrow{\bar\iota} \mathbf{Spt}^{\Sigma}_{\iota C^*T}\,\mathbf{Cpl}(\mathbf{HM}_{\oplus})$$

which preserves stable weak equivalences.

(2) *There is a canonical Quillen equivalence*

$$(\mathrm{Sus}^{\mathrm{o}}_{\iota C^*T}, \mathrm{Ev}_{\mathrm{o}}) : \mathbf{Cpl}(\mathbf{HM}_{\oplus}) \to \mathbf{Spt}^{\Sigma}_{\iota C^*T}\,\mathbf{Cpl}(\mathbf{HM}_{\oplus}).$$

Proof. The functor is obtained by applying $\bar\iota$ levelwise (cf. the following proof for the details, or [2, Déf. 4.3.16]). The first part is then obvious, and the second part follows from [37, Thm. 9.1] since, as proved in the following section, tensoring with $C^*T[2]$ (and hence with C^*T) is a Quillen equivalence. ☐

We will prove in the following proposition that $C^*_{(\mathrm{tr})}$ induces a left Quillen functor $C^*_{(\mathrm{tr}),s}$ on the level of spectra. Thus we may define the compositions

$$\mathcal{R}_{\mathrm{N},s} : \mathbf{DA} \simeq \mathbf{DA}_{\mathrm{aff}} \xrightarrow{\mathrm{L}C^*_s} \mathbf{Hot}(\mathbf{Spt}^{\Sigma}_{C^*T}\,\mathbf{Cpl}(\mathbf{HM}^{\mathrm{eff}}_{\oplus})) \xrightarrow{\mathrm{REv}_{\mathrm{o}}\circ\bar\iota} \mathbf{D}(\mathbf{HM}_{\oplus}),$$

$$\mathcal{R}_{\mathrm{N},\mathrm{tr},s} : \mathbf{DM} \simeq \mathbf{DM}_{\mathrm{aff}} \xrightarrow{\mathrm{L}C^*_{\mathrm{tr},s}} \mathbf{Hot}(\mathbf{Spt}^{\Sigma}_{C^*T}\,\mathbf{Cpl}(\mathbf{HM}^{\mathrm{eff}}_{\oplus})) \xrightarrow{\mathrm{REv}_{\mathrm{o}}\circ\bar\iota} \mathbf{D}(\mathbf{HM}_{\oplus}).$$

These are triangulated functors, and we will prove that they are in addition monoidal, at least if Λ is a field.

Proposition 6.10

(1) *The functors C^* and C^*_{tr} induce canonically lax monoidal left Quillen functors*

$$C^*_s : \mathbf{Spt}^{\Sigma}_{T}\mathbf{U}(\mathrm{SmAff})/(\mathbb{A}^1, \tau) \to \mathbf{Spt}^{\Sigma}_{C^*T}\,\mathbf{Cpl}(\mathbf{HM}^{\mathrm{eff}}_{\oplus}),$$

$$C^*_{\mathrm{tr},s} : \mathbf{Spt}^{\Sigma}_{T_{\mathrm{tr}}}\mathbf{U}(\mathrm{SmAffCor})/(\mathbb{A}^1, \tau) \to \mathbf{Spt}^{\Sigma}_{C^*T}\,\mathbf{Cpl}(\mathbf{HM}^{\mathrm{eff}}_{\oplus}).$$

(2) *The following triangles commute up to triangulated isomorphisms*

$$\begin{tikzcd} \mathbf{DA} \arrow[r, "\mathcal{R}_{N,s}"] \arrow[dr, "\mathrm{Bti}^*"'] & \mathbf{D}(\mathbf{HM}_\oplus) \arrow[d, "\mathbf{D}(o)"] \\ & \mathbf{D}(\Lambda) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{DA} \arrow[r, "\mathcal{R}_{N,s}"] \arrow[d, "\mathrm{L}a_{\mathrm{tr}}"'] & \mathbf{D}(\mathbf{HM}_\oplus) \\ \mathbf{DM} \arrow[ur, "\mathcal{R}_{N,\mathrm{tr},s}"'] & \end{tikzcd} \tag{6.11}$$

(3) *If Λ is a field then $\mathcal{R}_{N,(\mathrm{tr}),s}$ is monoidal, and the triangles in (6.11) commute up to monoidal isomorphisms.*

(4) *The Nori realization functors restrict to functors*

$$\mathcal{R}_{N,s} : \mathbf{DA}_{\mathrm{ct}} \to \mathbf{D}^{\mathrm{b}}(\mathbf{HM}), \qquad \mathcal{R}_{N,\mathrm{tr},s} : \mathbf{DM}_{\mathrm{ct}} \to \mathbf{D}^{\mathrm{b}}(\mathbf{HM})$$

on the categories of constructible motives.

Recall that the category of constructible motives is the thick subcategory generated by smooth schemes.

Proof. We will prove the first part for C^* but the case with transfers is literally the same. C^* together with the natural transformation $\theta : C^*T \otimes C^*(\bullet) \to C^*(T \otimes \bullet)$ induces a functor

$$C^*_s : \mathbf{Spt}^\Sigma_T \mathbf{U}\mathrm{SmAff} \to \mathbf{Spt}^\Sigma_{C^*T}\, \mathbf{Cpl}(\mathbf{HM}^{\mathrm{eff}}_\oplus)$$

(cf. [2, Déf. 4.3.16]). Explicitly, it takes a symmetric T-spectrum $\mathbf{E}$ to the symmetric C^*T-spectrum which in level n is given by $C^*(\mathbf{E}_n)$ and whose bonding maps are given by

$$C^*T \otimes C^*(\mathbf{E}_n) \xrightarrow{\theta} C^*(T \otimes \mathbf{E}_n) \to C^*(\mathbf{E}_{n+1}),$$

the second arrow being induced by the bonding map of $\mathbf{E}$. The lax monoidal structure on C^* induces canonically a lax monoidal structure on C^*_s.

It is clear that C^*_s is cocontinuous hence admits a right adjoint, by the adjoint functor theorem for locally presentable categories. Let f be a projective cofibration in $\mathbf{Spt}^\Sigma_T\mathbf{U}\mathrm{SmAff}$. Then f is in particular levelwise a cofibration ([2, Cor. 4.3.23]) and by the discussion in the previous section, C^* takes these to monomorphisms. Thus $C^*_s(f)$ is a monomorphism. The same argument shows that C^*_s takes projective cofibrations which are levelwise $(\mathbb{A}^1, \tau)$-local equivalences to monomorphisms which are levelwise quasi-isomorphisms. In other words, C^*_s is a left Quillen functor for the *unstable* model structures. To prove the first part of the proposition, it remains to prove that C^*_s takes the morphism

$$\zeta^D_n : \mathrm{Sus}^{n+1}_T(T \otimes D) \to \mathrm{Sus}^n_T D$$

to a stable equivalence for every cofibrant object D and every $n \geq 0$ (cf. [37, Def. 8.7]). But in the unstable homotopy category we can factor the image of ζ^D_n as follows:

$$\begin{aligned} C^*_s\mathrm{Sus}^{n+1}_T(T \otimes D) &\leftarrow \mathrm{Sus}^{n+1}_{C^*T} C^*(T \otimes D) \\ &\leftarrow \mathrm{Sus}^{n+1}_{C^*T}(C^*T \otimes C^*D) \\ &\to \mathrm{Sus}^n_{C^*T} C^*D \\ &\to C^*_s \mathrm{Sus}^n_T D. \end{aligned}$$

The first, second and fourth arrows are all levelwise quasi-isomorphisms because $\mathrm{L}C^*$ is monoidal on the level of derived categories. Moreover, the third arrow is a stable equivalence by definition.

We now come to the second part of the proposition. Commutativity of the triangle on the right follows from (the proof of) Proposition 6.7. For the triangle on the left, recall that $\mathrm{Sg}^* : \mathbf{U}(\mathrm{Man}_{\mathbb{C}})/(\mathbb{D}^1, \mathrm{usu}) \to \mathbf{Cpl}(\Lambda)$ is a lax monoidal left Quillen functor. As for C^* above this implies that there is an induced lax monoidal left Quillen functor Sg^*_s on the level of spectra (for the projective, respectively injective stable model structures). The Betti realization can then also be described as the following composition:

$$\mathbf{DA} \xrightarrow{\mathrm{An}^*} \mathbf{AnDA} \xrightarrow[\sim]{\mathrm{LSg}^*_s} \mathbf{Hot}(\mathbf{Spt}^{\Sigma}_{\mathrm{Sg}^*\mathrm{An}^*T}\,\mathbf{Cpl}(\Lambda)) \xrightarrow[\sim]{\mathrm{REv}_0} \mathbf{D}(\Lambda).$$

Analogously, $\mathbf{D}(o)\mathcal{R}_{\mathrm{N},s}$ can be described as the composition

$$\mathbf{DA}_{\mathrm{aff}} \xrightarrow{\mathrm{LC}^*_s} \mathbf{Hot}(\mathbf{Spt}^{\Sigma}_{C^*T}\,\mathbf{Cpl}(\mathbf{coMod}(\mathbf{H}^{\mathrm{eff}}_{\mathrm{N}}))) \xrightarrow{\mathbf{D}(o)} \mathbf{Hot}(\mathbf{Spt}^{\Sigma}_{oC^*T}\,\mathbf{Cpl}(\Lambda)) \xrightarrow[\sim]{\mathrm{REv}_0} \mathbf{D}(\Lambda).$$

One is then essentially reduced to compare $\mathbf{D}(o)\mathrm{LC}^*_s$ and $\mathrm{LSg}^*_s\mathrm{An}^*$ which is done, as in the effective case, by means of the intermediate functor P.

We come to the third part, and assume now that Λ is a field. Using Lemma C.5 together with [37, Thm. 8.11] we see that the categories occurring in the definition of $\mathcal{R}_{\mathrm{N},(\mathrm{tr}),s}$ all carry induced monoidal structures. By the previous lemma, $\mathrm{REv}_0 \circ \bar{\iota}$ is lax monoidal, as is $\mathrm{LC}^*_{(\mathrm{tr}),s}$ by the first part of the proposition. It follows that $\mathcal{R}_{\mathrm{N},(\mathrm{tr}),s}$ is a lax monoidal functor, and the comparisons in part (2) are compatible with these lax monoidal structures.

Monoidality of $\mathcal{R}_{\mathrm{N},s}$ now follows from monoidality of Bti^* and the fact that the derived forgetful functor is conservative. Monoidality of $\mathcal{R}_{\mathrm{N},\mathrm{tr},s}$ in the étale case follows from this since $\mathrm{L}a_{\mathrm{tr}}$ is an equivalence of categories (cf. [5, Cor. B.14]). Finally, the Nisnevich realization factors through the étale realization via a monoidal functor.

The last part of the proposition holds because $\mathcal{R}_{\mathrm{N},(\mathrm{tr}),s}$ takes a smooth affine scheme into $\mathbf{D}^{\mathrm{b}}(\mathbf{HM})$. (For this we use that $\mathbf{D}^{\mathrm{b}}(\mathbf{HM})$ is a full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathbf{HM}_{\oplus})$; see [44, Pro. 8.6.11 and Thm. 15.3.1.(i)].) ☐

Remark 6.12 During the preparation of the present chapter, Ivorra in [41] independently defined such a motivic realization for étale motives without transfers. While his construction is more general in that it applies also to a relative case (involving his generalization of Nori motives to "perverse Nori motives" over a base), he does not consider monoidality of the functor nor its behaviour with respect to transfers.

There is a monoidal exact mixed Hodge realization for Nori motives whose composition with the forgetful functor yields the forgetful functor on Nori motives. Composing its derived counterpart with $\mathcal{R}_{\mathrm{N},(\mathrm{tr}),s}$ from the previous proposition yields the following immediate corollary.

Corollary 6.13 *There are* mixed Hodge realization *functors*

$$\mathcal{R}_H : \mathbf{DA}_{\mathbb{Q}} \longrightarrow \mathbf{D}(\mathbf{MHS}^{\mathrm{pol}}_{\mathbb{Q},\oplus}), \qquad \mathcal{R}_{H,\mathrm{tr}} : \mathbf{DM}_{\mathbb{Q}} \longrightarrow \mathbf{D}(\mathbf{MHS}^{\mathrm{pol}}_{\mathbb{Q},\oplus})$$

to the derived category of (not necessarily finite dimensional) polarizable mixed $\mathbb{Q}$-Hodge structures satisfying the following properties:

(1) They are triangulated monoidal.

(2) They make the following triangles commute up to monoidal triangulated isomorphisms.

$$\begin{tikzcd} \mathbf{DA}_{\mathbb{Q}} \arrow[r, "\mathcal{R}_H"] \arrow[dr, "\mathrm{Bti}^*"'] & \mathbf{D}(\mathbf{MHS}^{\mathrm{pol}}_{\mathbb{Q},\oplus}) \arrow[d, "\mathbf{D}(o)"] \\ & \mathbf{D}(\mathbb{Q}) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{DA}_{\mathbb{Q}} \arrow[r, "\mathcal{R}_H"] \arrow[d, "\mathrm{L}a_{\mathrm{tr}}"'] & \mathbf{D}(\mathbf{MHS}^{\mathrm{pol}}_{\mathbb{Q},\oplus}) \\ \mathbf{DM}_{\mathbb{Q}} \arrow[ur, "\mathcal{R}_{H,\mathrm{tr}}"'] & \end{tikzcd}$$

(3) They restrict to triangulated monoidal functors

$$\mathcal{R}_H : \mathbf{DA}_{\mathbb{Q},\mathrm{ct}} \longrightarrow \mathbf{D}^{\mathrm{b}}(\mathbf{MHS}^{\mathrm{pol}}_{\mathbb{Q}}), \qquad \mathcal{R}_{H,\mathrm{tr}} : \mathbf{DM}_{\mathbb{Q},\mathrm{ct}} \longrightarrow \mathbf{D}^{\mathrm{b}}(\mathbf{MHS}^{\mathrm{pol}}_{\mathbb{Q}})$$

on the categories of constructible motives.

7. Almost smooth pairs

In the sequel we will want to manipulate the Morel-Voevodsky motives of pairs of varieties (X, Z), and describe their images under certain functors explicitly. This is easy if both X and Z are smooth but turns out to be rather difficult in general. What we need is a class of pairs which on the one hand are close enough to smooth ones so that explicit computations are feasible, and on the other hand flexible enough so that we are able to reduce our arguments from general pairs to this smaller class. This is provided by the class of *almost smooth* pairs, i. e. pairs of varieties (X, Z) where X is smooth and Z a simple normal crossings divisor. By resolution of singularities and excision, every good pair receives a morphism from an almost smooth one which induces isomorphisms in Betti homology. In this section, we will give rather explicit motivic models for almost smooth pairs, both on the effective and the stable level, and compute their images under various functors. One immediate consequence of our discussion here is that the morphism of bialgebras φ_{N} passes to the stable level.

7.1. Effective level. (X, Z) will now be our running notation for an almost smooth pair. We always denote the irreducible components of Z by $Z_1, \dots, Z_p$ and endow them with the reduced structure. The (smooth) intersection of Z_i and Z_j is denoted by Z_{ij}, and similarly for intersections of more than two components. The presheaf $\Lambda(X, Z)$ is defined to be the cokernel of the morphism $\oplus_{i=1}^{p} \Lambda(Z_i) \to \Lambda(X)$.

In addition, let $\mathcal{Y} = (Y_1, \dots, Y_q)$ be an open affine cover of X. For any functor F : SmAff $\to \mathbf{Cpl}(\mathcal{C})$ into the category of complexes on an abelian category $\mathcal{C}$, we define $F^{\mathcal{Y}}(X, Z_\bullet) \in \mathbf{Cpl}(\mathcal{C})$ to be the (sum) total complex of the tricomplex whose (i, j, k)-th term is

$$\bigoplus_{a_0 < \cdots < a_i,\ b_1 < \cdots < b_j} F_k\left(Y_{a_0 \cdots a_i} \cap Z_{b_1 \cdots b_j}\right),$$

where by convention the empty intersection of the Z_i's is X. This can also be understood as the mapping cone of the morphism

$$F^{\mathcal{Y} \cap Z_\bullet}(Z_\bullet) \to F^{\mathcal{Y}}(X)$$

with an obvious interpretation of the first term. If F is defined on all smooth schemes, we set $F(X, Z_\bullet)$ to be $F^{(X)}(X, Z_\bullet)$, and if F is defined on all affine varieties, we can similarly define $F^{\mathcal{Y}}(X, Z)$.

For example, we can consider the presheaf of complexes $\Lambda^{\mathcal{Y}}(X, Z_\bullet)$ with the canonical map to $\Lambda(X, Z)$. This defines a cofibrant replacement as we now prove.

Lemma 7.1 *The canonical morphism* $\Lambda^{\mathcal{Y}}(X, Z_\bullet) \to \Lambda(X, Z)$ *is a cofibrant replacement for the τ-local model structure, and the complexes*

$$C^* \Lambda^{\mathcal{Y}}(X, Z_\bullet) \xrightarrow{\sim} C^{\mathcal{Y}}(X, Z_\bullet) \xrightarrow{\sim} C^{\mathcal{Y}}(X, Z)$$

all provide models for $\mathrm{LC}^* \Lambda(X, Z)$.

Proof. To prove the first statement consider the following morphism of distinguished triangles in the derived category of presheaves on smooth schemes:

$$\begin{tikzcd}
\Lambda^{\mathcal{Y}}(Z_\bullet) \arrow[r] \arrow[d] & \Lambda^{\mathcal{Y}}(X) \arrow[r] \arrow[d] & \Lambda^{\mathcal{Y}}(X, Z_\bullet) \arrow[r] \arrow[d] & \Lambda^{\mathcal{Y}}(Z_\bullet)[-1] \arrow[d] \\
\Lambda(Z_\bullet) \arrow[r] & \Lambda(X) \arrow[r] & \Lambda(X, Z) \arrow[r] & \Lambda(Z_\bullet)[-1]
\end{tikzcd}$$

(It should be clear what the first term denotes although we haven't formally defined it above. That the second row arises from a short exact sequence of complexes of presheaves (and hence is indeed a distinguished triangle) is [68, 2.1.4]; see [65, Lem. 1.4] for a proof.) The second vertical arrow is a τ-local equivalence as is the left vertical arrow by induction on the number of irreducible components of Z. It follows that the third vertical arrow is a τ-local equivalence as well. Since $\Lambda^{\mathcal{Y}}(X, Z_\bullet)$ is a bounded below complex of representables, it is projective cofibrant (Fact III.2.10).

We now come to the second statement of the lemma. It is clear that the first arrow is invertible. For the second arrow consider the following diagram:

$$\begin{tikzcd}
oC^{\mathcal{Y}}(X, Z_\bullet) \arrow[d] & P^{\mathcal{Y}}(X, Z_\bullet) \arrow[l] \arrow[r] \arrow[d] & (\mathrm{Sg})^{\mathcal{Y}}(X, Z_\bullet) \arrow[d] \\
oC^{\mathcal{Y}}(X, Z) & P^{\mathcal{Y}}(X, Z) \arrow[l] \arrow[r] & (\mathrm{Sg})^{\mathcal{Y}}(X, Z)
\end{tikzcd}$$

By the discussion in section 5, we know that the horizontal arrows are all quasi-isomorphisms. Since the right-most vertical arrow is a quasi-isomorphism so is the left-most. ☐

Define the following zig-zag of morphisms of complexes of Λ-modules:

$$oC^{\mathcal{Y}}(X, Z) \leftarrow P^{\mathcal{Y}}(X, Z) \to (\mathrm{Sg})^{\mathcal{Y}}(X, Z) \to \mathrm{Sg}(X, Z). \tag{7.2}$$

Lemma 7.3 *Assume that (X, Z) is an almost smooth pair, and that Λ is a principal ideal domain. Then (7.2) induces an isomorphism of $\mathbf{H}_N^{\mathrm{eff}}$-comodules*

$$\mathrm{H}_n \mathcal{R}_N \Lambda(X, Z) \xrightarrow{\sim} \mathrm{H}_n(X, Z)$$

for all $n \in \mathbb{Z}$.

Proof. By Jouanolou's trick there exists a smooth affine variety X' and a Zariski locally trivial morphism $p : X' \to X$ whose fibers are isomorphic to affine space. Setting $Z_i' = Z_i \times_X X'$ we obtain an almost smooth pair (X', Z') with X' affine, and a morphism $p : (X', Z') \to (X, Z)$ which induces an isomorphism in singular homology.

Let $\mathcal{Y}'$ be the pullback of the affine cover to X' and consider the following commutative diagram, where all the arrows are the canonical ones:

$$\begin{tikzcd}
oC^{\mathcal{Y}}(X,Z) & P^{\mathcal{Y}}(X,Z) \arrow[l] \arrow[r] & (\mathrm{Sg})^{\mathcal{Y}}(X,Z) \\
oC^{\mathcal{Y}'}(X',Z') \arrow[u] \arrow[d] & P^{\mathcal{Y}'}(X',Z') \arrow[l] \arrow[r] \arrow[u] \arrow[d] & (\mathrm{Sg})^{\mathcal{Y}'}(X',Z') \arrow[u] \arrow[d] \\
oC(X',Z') & P(X',Z') \arrow[l] \arrow[r] \arrow[d] & \mathrm{Sg}(X',Z') \arrow[d] \\
 & P(X')/P(Z') \arrow[r] & \mathrm{Sg}(X')/\mathrm{Sg}(Z')
\end{tikzcd}$$

By the discussion in section 5, we know that the top horizontal arrows are both quasi-isomorphisms. All vertical arrows are quasi-isomorphisms. We thus reduce to prove that the zig-zag of morphisms $oC(X',Z') \leftarrow P(X',Z') \rightarrow \mathrm{Sg}(X')/\mathrm{Sg}(Z')$ induces an $\mathbf{H}_{\mathrm{N}}^{\mathrm{eff}}$-comodule (iso)morphism in the n-th homology. Writing (X,Z) for (X',Z'), this is expressed by commutativity of the following diagram, where the vertical arrows are the coaction of $\mathbf{H}_{\mathrm{N}}^{\mathrm{eff}}$ on the objects in question:

$$\begin{tikzcd}
\mathrm{H}_n oC(X,Z) \arrow[d, "\mathrm{ca}"'] & \mathrm{H}_n P(X,Z) \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{H}_n(X,Z) \arrow[d, "\mathrm{ca}"] \\
\mathrm{H}_n oC(X,Z) \otimes \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}} & \mathrm{H}_n P(X,Z) \otimes \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}} \arrow[l, "\sim"] \arrow[r, "\sim"'] & \mathrm{H}_n(X,Z) \otimes \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}}
\end{tikzcd}$$

Start with any $[(f,g)] \in \mathrm{H}_n P(X,Z)$. Thus there exist $X_n \subset X$, $Z_{n-1} \subset Z$ closed subvarieties of dimension at most n and $n-1$, respectively, such that $f \in \mathrm{Sg}_n(X_n)$, $g = \pm\partial f \in \mathrm{Sg}_{n-1}(Z_{n-1})$ (depending on the sign conventions for the mapping cone). It is then clear from the definition of the natural transformations $oC \leftarrow P \rightarrow \mathrm{Sg}$ that we reduce to prove commutativity of the following diagram

$$\begin{tikzcd}
\mathrm{H}_n oC(X,Z) \arrow[d, "\mathrm{ca}"'] & \mathrm{H}_n(X_n, Z_{n-1}) \arrow[l] \arrow[r] \arrow[d, "\mathrm{ca}"'] & \mathrm{H}_n(X,Z) \arrow[d, "\mathrm{ca}"] \\
\mathrm{H}_n oC(X,Z) \otimes \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}} & \mathrm{H}_n(X_n, Z_{n-1}) \otimes \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}} \arrow[l] \arrow[r] & \mathrm{H}_n(X,Z) \otimes \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}}
\end{tikzcd}$$

which is obvious. ☐

This lemma will be important later on as well but one immediate application is that it allows us to extend the morphism of bialgebras $\varphi_{\mathrm{N}} : \mathbf{H}_{\mathrm{A}}^{\mathrm{eff}} \rightarrow \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}}$ constructed in section 6 to a morphism $\mathbf{H}_{\mathrm{A}} \rightarrow \mathbf{H}_{\mathrm{N}}$. Indeed, we see that there is the following isomorphism of $\mathbf{H}_{\mathrm{N}}^{\mathrm{eff}}$-comodules:

$$\begin{aligned}
\mathrm{H}_0 \overline{\varphi_{\mathrm{N}} \widetilde{\mathrm{Bti}}}^{\mathrm{eff},*}(T[2]) &\xrightarrow{\sim} \mathrm{H}_0 \mathcal{R}_{\mathrm{N}}(T[2]) && \text{by Cor. 6.6} \\
&\xrightarrow{\sim} \mathrm{H}_0 \mathcal{R}_{\mathrm{N}} \Lambda(\mathbb{G}_m, \{1\})[1] \\
&\xrightarrow{\sim} \mathrm{H}_1(\mathbb{G}_m, \{1\}),
\end{aligned}$$

the last isomorphism by the previous lemma. One deduces easily that $\varphi_{\mathrm{N}}(s_{\mathrm{A}}) = s_{\mathrm{N}} \in \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}}$ and hence the morphism $\mathcal{H}_{\mathrm{A}}^{\mathrm{eff}} \rightarrow \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}}$ from Corollary 6.6 passes to the localization and induces the following commutative squares:

$$\begin{tikzcd} \mathcal{H}_{\mathrm{A}}^{\mathrm{eff}} \arrow[r] \arrow[d, "\iota"'] & \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}} \arrow[d, "\iota"] \\ \mathcal{H}_{\mathrm{A}} \arrow[r] & \mathbf{H}_{\mathrm{N}} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{H}_{\mathrm{A}}^{\mathrm{eff}} \arrow[r, "\varphi_{\mathrm{N}}"] \arrow[d, "\iota"'] & \mathbf{H}_{\mathrm{N}}^{\mathrm{eff}} \arrow[d, "\iota"] \\ \mathbf{H}_{\mathrm{A}} \arrow[r, "\varphi_{\mathrm{N}}"'] & \mathbf{H}_{\mathrm{N}} \end{tikzcd}$$

Remark 7.4 In particular, we can now define a stable version of the functor $\mathcal{R}_{\mathrm{N}}$ constructed in the effective case in section 6 (still assuming that Λ is a principal ideal domain). Indeed, we set it to be the composition

$$\mathcal{R}_{\mathrm{N}} : \mathbf{DA} \xrightarrow{\widetilde{\mathrm{Bti}}^*} \mathbf{coMod}(\mathcal{H}_{\mathrm{A}}) \to \mathbf{coMod}(\mathbf{H}_{\mathrm{N}})^{\mathbf{D}(\Lambda)}.$$

As the composition of two monoidal functors, $\mathcal{R}_{\mathrm{N}}$ is again monoidal. It follows also that the diagrams analogous to the ones in Corollary 6.6 commute

$$\begin{tikzcd} \mathbf{DA} \arrow[r, "\mathcal{R}_{\mathrm{N}}"] \arrow[d, "\widetilde{\mathrm{Bti}}^*"'] & \mathbf{coMod}(\mathbf{H}_{\mathrm{N}})^{\mathbf{D}(\Lambda)} \arrow[d, "o"] \\ \mathbf{coMod}(\mathcal{H}_{\mathrm{A}}) \arrow[ur] \arrow[r, "o"'] & \mathbf{D}(\Lambda) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{DA} \arrow[r, "\mathrm{H}_0\mathcal{R}_{\mathrm{N}}"] \arrow[d, "\mathrm{H}_0\widetilde{\mathrm{Bti}}^*"'] & \mathbf{coMod}(\mathbf{H}_{\mathrm{N}}) \arrow[d, "o"] \\ \mathbf{coMod}(\mathbf{H}_{\mathrm{A}}) \arrow[ur, "\overline{\varphi_{\mathrm{N}}}"'] \arrow[r, "o"'] & \mathbf{Mod}(\Lambda) \end{tikzcd}$$

as does the following square:

$$\begin{tikzcd} \mathbf{DA}_{\mathrm{aff}}^{\mathrm{eff}} \arrow[r, "\mathcal{R}_{\mathrm{N}}"] \arrow[d, "\mathrm{LSus}_T^0"'] & \mathbf{coMod}(\mathbf{H}_{\mathrm{N}}^{\mathrm{eff}})^{\mathbf{D}(\Lambda)} \arrow[d, "\bar{\iota}"] \\ \mathbf{DA} \arrow[r, "\mathcal{R}_{\mathrm{N}}"'] & \mathbf{coMod}(\mathbf{H}_{\mathrm{N}})^{\mathbf{D}(\Lambda)} \end{tikzcd}$$

7.2. Stable level. We continue our study of almost smooth pairs but now we work in the context of non-effective motives. For such a pair, we will provide a rather explicit model for both its homological as well as cohomological Morel-Voevodsky motive in Theorem 7.5, and then similarly for its analytification in Theorem 7.9. Subsequently we prove that the Betti realization is in some sense compatible with these models (Lemmas 7.10 and 7.12).

Let (X, Z) be an almost smooth pair. The inclusion of the complement $U = X \backslash Z \to X$ is denoted by j. Recall (from [5, §2.2.4]) the following constructions. Given a presheaf K of complexes on smooth schemes, one defines $K(X, Z)$ to be the kernel of the map $K(X) \to \prod_{i=1}^{p} K(Z_i)$. The endofunctor $\underline{\mathrm{hom}}((X, Z), \bullet)$ is defined as the right adjoint to tensoring with $\Lambda(X, Z)$. Explicitly,

$$\underline{\mathrm{hom}}((X, Z), K)(Y) = K(Y \times X, Y \times Z)$$

for any presheaf of complexes K and for any smooth scheme Y. $\underline{\mathrm{hom}}((X, Z), \bullet)$ canonically extends to an endofunctor on symmetric T-spectra of presheaves of complexes.

In general, we denote the internal hom in symmetric T-spectra by $\underline{\mathrm{Hom}}$. We note that for a complex of presheaves K and a symmetric T-spectrum $\mathbf{E}$, the object $\underline{\mathrm{Hom}}(\mathrm{Sus}_T^0 K, \mathbf{E})$ admits the following simple description. In level n, it is given by $\underline{\mathrm{Hom}}(K, \mathbf{E}_n)$, the action of Σ_n is on $\mathbf{E}_n$, and the bonding maps are given by the composition

$$T \otimes \underline{\mathrm{Hom}}(K, \mathbf{E}_n) \to \underline{\mathrm{Hom}}(K, T \otimes \mathbf{E}_n) \to \underline{\mathrm{Hom}}(K, \mathbf{E}_{n+1}),$$

where the first arises from the adjunction $(\otimes, \underline{\mathrm{Hom}})$, and the second uses the bonding maps from $\mathbf{E}$. To emphasize this description we write simply $\underline{\mathrm{Hom}}(K, \mathbf{E})$ for this symmetric T-spectrum.

Using the notation from §7.1, $\Lambda(X, Z_\bullet)$ denotes the augmented complex

$$\cdots \to \bigoplus_{i_1 < \cdots < i_l} \Lambda(Z_{i_1 \cdots i_l}) \to \cdots \to \bigoplus_i \Lambda(Z_i) \to \Lambda(X),$$

the last term being in homological degree 0.

For the next result, recall that on presheaves of complexes on smooth schemes there is also an *injective* $(\mathbb{A}^1, \tau)$-local model structure, obtained by $(\mathbb{A}^1, \tau)$-localization from the "injective model structure"([2, Déf. 4.5.12]). The cofibrations and weak equivalences of the latter are defined objectwise. One deduces then the existence of an "injective stable $(\mathbb{A}^1, \tau)$-local model structure" on symmetric T-spectra as described in section 2 (cf. [2, Déf. 4.5.21]).

Theorem 7.5 *Let (X, Z) be almost smooth.*

(1) Let $\mathbf{E}$ be a projective stable $(\mathbb{A}^1, \tau)$-fibrant symmetric T-spectrum of presheaves of complexes on Sm. *Then $\underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E})$ provides a model for $\mathrm{R}\pi_* j_! \mathbf{E}|_U$ in* **DA**. *Moreover, this identification is functorial in* $\mathbf{E}$.

(2) If $\mathbf{E}$ is injective stable $(\mathbb{A}^1, \tau)$-fibrant instead, then one can replace $\underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E})$ by $\underline{\mathrm{hom}}((X, Z), \mathbf{E})$ in the statement above.

(3) For $\mathbf{E}$ an injective stable $(\mathbb{A}^1, \tau)$-fibrant replacement of the unit spectrum, $\underline{\mathrm{hom}}((X, Z), \mathbf{E})$ provides a model for $\mathcal{R}_A(X, Z, 0)^\vee$ in **DA**.

(4) In **DA**, $\mathrm{LSus}^0_T \Lambda(X, Z) \cong \mathcal{R}_A(X, Z, 0)$ *canonically.*

Proof. We first prove the second part. Let $K_\bullet(\mathbf{E})$ be the object

$$\mathbf{E}|_X \to \oplus_i i_{i*} \mathbf{E}|_{Z_i} \to \oplus_{i<j} i_{ij*} \mathbf{E}|_{Z_{ij}} \to \cdots.$$

There is a canonical morphism

$$j_! \mathbf{E}|_U \to \mathrm{Tot}(K_\bullet(\mathbf{E})) =: K(\mathbf{E}) \tag{7.6}$$

(the totalization functor is applied levelwise; up to canonical isomorphism it doesn't matter whether $\mathrm{Tot}^\oplus$ or Tot^Π is used). We claim that it is a stable $(\mathbb{A}^1, \tau)$-local equivalence, and the target is projective stable $(\mathbb{A}^1, \tau)$-fibrant.

For the first claim, one can use conservativity of the couple $(j^*, \mathrm{L}i^*)$, $i : Z \hookrightarrow X$ being the closed immersion. It is obvious that j^* applied to (7.6) is an equivalence while in the case of $\mathrm{L}i^*$ it is an easy induction argument on the number of irreducible components of Z. For the second claim, we need to prove two things, namely that the target is levelwise projective $(\mathbb{A}^1, \tau)$-fibrant, and an Ω-spectrum. For the first of these, fix a level n and set $E = \mathbf{E}_n$. We know that for each l, $K_l(E)$ is τ-fibrant hence so is $K(E)$ by Lemma III.4.21. The same argument shows that $K(E)$ is $\mathbb{A}^1$-local.

We now prove that $K(\mathbf{E})$ is an Ω-spectrum. Since $K_l(\mathbf{E}_n) \to \underline{\mathrm{hom}}(T, K_l(\mathbf{E}_{n+1}))$ is an $(\mathbb{A}^1, \tau)$-local equivalence for each l so is the totalization

$$\mathrm{Tot}(K_\bullet(\mathbf{E}_n)) \to \mathrm{Tot}(\underline{\mathrm{hom}}(T, K_\bullet(\mathbf{E}_{n+1}))) = \underline{\mathrm{hom}}(T, \mathrm{Tot}(K_\bullet(\mathbf{E}_{n+1}))).$$

Hence this proves that $K(\mathbf{E})$ is a projective stable $(\mathbb{A}^1, \tau)$-fibrant object.

We also find an $(\mathbb{A}^1, \tau)$-local equivalence

$$\Lambda(X, Z_\bullet) \to \Lambda(X, Z) \tag{7.7}$$

between injective cofibrant objects hence $\underline{\mathrm{Hom}}(\bullet, \mathbf{E})$ will transform (7.7) into an (un)stable $(\mathbb{A}^1, \tau)$-local equivalence. It follows that in **DA**,

$$
\begin{aligned}
\underline{\mathrm{hom}}((X,Z),\mathbf{E}) &= \underline{\mathrm{Hom}}(\Lambda(X,Z),\mathbf{E}) \\
&\xrightarrow{\sim} \underline{\mathrm{Hom}}(\Lambda(X,Z_\bullet),\mathbf{E}) \\
&= \mathrm{Tot}(\underline{\mathrm{Hom}}(\Lambda(X),\mathbf{E}) \to \oplus_i \underline{\mathrm{Hom}}(\Lambda(Z_i),\mathbf{E}) \to \cdots) \\
&= \mathrm{Tot}(\pi_*\pi^*\mathbf{E} \to \oplus_i \pi_{Z_i *}\pi^*_{Z_i}\mathbf{E} \to \cdots) \\
&= \pi_*\mathrm{Tot}(K_\bullet(\mathbf{E})) \\
&\xleftarrow{\sim} \mathrm{R}\pi_* j_! \mathbf{E}|_U.
\end{aligned}
\tag{7.8}
$$

Finally, functoriality in $\mathbf{E}$ is clear.

Part three of the theorem is an immediate consequence of this and the identification $\mathcal{R}_{\mathrm{A}}(X, Z, 0)^\vee = \mathrm{R}\pi_* j_! \mathbb{1}$. Part four then follows by duality. The first part of the theorem can be deduced as follows: Given $\mathbf{E}$ as in the statement of that part, we choose an injective stable $(\mathbb{A}^1, \tau)$-fibrant replacement $f : \mathbf{E} \to \mathbf{E}'$. f is a sectionwise quasi-isomorphism at each level. Since pullback along a smooth morphism, and pushforward along a closed embedding both preserve sectionwise quasi-isomorphisms, we see that $K_l(f) : K_l(\mathbf{E}_n) \to K_l(\mathbf{E}'_n)$ is sectionwise a quasi-isomorphism for each l hence so is $K(f) : K(\mathbf{E}_n) \to K(\mathbf{E}'_n)$ for each n. It follows from what we proved in the first part of the theorem that $K(\mathbf{E})$ is projective stable $(\mathbb{A}^1, \tau)$-fibrant hence the same computation as above shows that $\underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E})$ provides a model for $\mathrm{R}\pi_* j_! \mathbf{E}|_U$. □

We will need a similar result in the analytic setting. Thus let X be a complex manifold, and Z a closed subset which is the union of finitely many complex submanifolds. We call this an *almost smooth analytic pair*, and as before, we denote by $Z_1, \ldots, Z_p$ the "components" of Z, namely the connected components of the normalization of Z. We can then define, analogously, $\Lambda(X, Z_\bullet)$, $\Lambda(X, Z)$ and $\underline{\mathrm{hom}}((X, Z), \bullet)$ (cf. [5, §2.2.1]).

Theorem 7.9 *Let* $\mathbf{E}$ *be a projective stable* $(\mathbb{D}^1, \mathrm{usu})$*-fibrant presheaf of complexes on* $\mathrm{Man}_{\mathbb{C}}$. *Then* $\underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E})$ *provides a model for* $\mathrm{R}\pi_* j_! \mathbf{E}|_U$ *in* **AnDA**. *Moreover, this identification is functorial in* $\mathbf{E}$.

Proof. The proof is very similar to the one of the last theorem and we omit the details. (Also, the other parts of the previous theorem are equally true in the analytic setting, with almost identical proofs.) □

Later on, we will use the following relation between the two descriptions of motives we just gave. Choose a projective stable $(\mathbb{A}^1, \tau)$-fibrant replacement $\mathbf{E}$ of the unit spectrum $\mathbb{1}$. Also choose a projective stable $(\mathbb{D}^1, \mathrm{usu})$-fibrant replacement $\mathbf{E}'$ of $\mathbb{1}$. $\mathrm{An}^*\mathbb{1} \cong \mathbb{1} \to \mathbf{E}'$ induces, by adjunction, $\mathbb{1} \to \mathrm{An}_*\mathbf{E}'$ and the latter is projective stable $(\mathbb{A}^1, \tau)$-fibrant. It follows that there is a morphism $\mathbf{E} \to \mathrm{An}_*\mathbf{E}'$ which induces $\mathrm{An}^*\mathbf{E} \to \mathbf{E}'$ rendering the triangle

$$
\begin{tikzcd}
\mathbb{1} \arrow[r] \arrow[dr] & \mathrm{An}^*\mathbf{E} \arrow[d] \\
& \mathbf{E}'
\end{tikzcd}
$$

commutative. Notice that the morphism $\mathrm{An}^*\mathbf{E} \to \mathbf{E}'$ is necessarily a stable $(\mathbb{D}^1, \mathrm{usu})$-local equivalence.

Lemma 7.10 *Under the identifications of the two previous theorems, the following square commutes ((X,Z) is an almost smooth pair):*

$$\begin{tikzcd} \mathrm{An}^* \mathrm{R}\pi_* j_! \mathbb{1} \arrow[r, equal] \arrow[d] & \mathrm{An}^* \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E}) \arrow[d] \\ \mathrm{R}\pi^{\mathrm{an}}_* j^{\mathrm{an}}_! \mathbb{1} \arrow[r, equal] & \underline{\mathrm{Hom}}(\Lambda(X^{\mathrm{an}}, Z^{\mathrm{an}}_\bullet), \mathbf{E}') \end{tikzcd}$$

Proof. By adjunction, this is equivalent to the commutativity of the following outer diagram:

$$\begin{tikzcd} \mathrm{R}\pi_* j_! \mathbf{E}|_U \arrow[r, equal] \arrow[d, "\mathrm{adj}"'] & \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{E}) \arrow[d, "\mathrm{adj}"] \\ \mathrm{R}\pi_* j_! (\mathrm{An}_* \mathrm{An}^* \mathbf{E})|_U \arrow[d] & \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathrm{An}_* \mathrm{An}^* \mathbf{E}) \arrow[d] \\ \mathrm{R}\pi_* j_! (\mathrm{An}_* \mathbf{E}')|_U \arrow[r, equal] \arrow[d, dash, "\sim"'] & \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathrm{An}_* \mathbf{E}') \arrow[d, dash, "\sim"] \\ \mathrm{R}\mathrm{An}_* \mathrm{R}\pi^{\mathrm{an}}_* j^{\mathrm{an}}_! \mathbf{E}'|_{U^{\mathrm{an}}} \arrow[r, equal] & \mathrm{An}_* \underline{\mathrm{Hom}}(\Lambda(X^{\mathrm{an}}, Z^{\mathrm{an}}_\bullet), \mathbf{E}') \end{tikzcd}$$

Commutativity of the upper part follows from the functoriality statement in Theorem 7.5. For the lower square, one checks that An_* commutes with the relevant equalities in (7.8). The main point is that $\mathrm{An}_* K(\mathbf{E}') = K(\mathrm{An}_* \mathbf{E}')$. □

This lemma states that the analytification functor is compatible with our choices of models for the relative motives. We now want to prove the analogous statement for the Betti realization functor. Factoring the latter as $\Gamma \mathrm{Ev}_o \mathrm{An}^*$ (where Γ is the global sections functor, cf. section 2), we reduce to showing this compatibility for the composed functor $\Gamma \mathrm{Ev}_o$. Thus let (X,Z) be an almost smooth analytic pair. By what we saw in section 4 (or rather appendix B, specifically Fact B.3), the object $\Gamma \mathrm{Ev}_o \mathrm{R}\pi_* j_! \mathbb{1}$ is modeled by the complex of relative singular cochains on (X,Z). Now suppose that in the situation of the previous lemma, we choose $\mathbf{E}'$ to be $\mathbf{Sg}^\vee$ of Remark 2.4. Then we find a canonical quasi-isomorphism

$$\begin{aligned} \mathrm{Sg}(X,Z)^\vee &\xrightarrow{\sim} \mathrm{Tot}(\mathrm{Sg}(X)^\vee \to \oplus_i \mathrm{Sg}(Z_i)^\vee \to \cdots) \\ &= \Gamma\, \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathrm{Sg}^\vee) \\ &= \Gamma \mathrm{Ev}_o\, \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{Sg}^\vee). \end{aligned} \tag{7.11}$$

Lemma 7.12 *The following square commutes:*

$$\begin{tikzcd} \Gamma \mathrm{Ev}_o \mathrm{R}\pi_* j_! \mathbb{1} \arrow[r, equal] \arrow[d, dash, "\sim"'] & \Gamma \mathrm{Ev}_o\, \underline{\mathrm{Hom}}(\Lambda(X, Z_\bullet), \mathbf{Sg}^\vee) \\ \mathrm{R}\tilde{\pi}_* \tilde{j}_! \Lambda_{\mathrm{cst}} \arrow[r, equal] & \mathrm{Sg}(X,Z)^\vee \arrow[u, "(7.11)", "\sim"'] \end{tikzcd}$$

Here, we temporarily decorate the functors operating on sheaves on locally compact topological spaces with a tilde to distinguish them from their counterparts in the complex analytic world.

Proof. Since the identification of Theorem 7.9 inducing the top horizontal arrow is levelwise, we may prove the lemma staying completely on the effective level, thus decomposing the square as follows:

$$\begin{tikzcd}
\mathrm{R}\Gamma\mathrm{R}\pi_* j_!\Lambda \arrow[r, "\sim"] \arrow[d, dash, "\sim"'] & \Gamma\pi_* K(\mathrm{Sg}^\vee) \arrow[rr, equal] \arrow[d, equal] & & \mathrm{Sg}^\vee(X, Z_\bullet) \arrow[dl, "\sim"'] \\
\mathrm{R}\tilde\pi_*\mathrm{R}\iota_{X*} j_!\Lambda \arrow[r, "\sim"] & \tilde\pi_*\iota_{X*}K(\mathrm{Sg}^\vee) \arrow[r, "\sim"] & \tilde\pi_*\mathrm{Tot}(\mathcal{S}_X \to \oplus_i i_{i*}\mathcal{S}_{Z_i} \to \cdots) & \\
\mathrm{R}\tilde\pi_*\tilde{j}_!\Lambda_{\mathrm{cst}} \arrow[r, "\sim"] \arrow[u, "\sim"] & \tilde\pi_*(\mathcal{S}_X \otimes \Lambda_U) \arrow[r, "\sim", "\alpha"'] & \tilde\pi_*\mathcal{K}_{(X,Z)} \arrow[u, "\sim"'] & \mathrm{Sg}(X,Z)^\vee \arrow[l, "\sim"', "\beta"] \arrow[uu, "\sim"']
\end{tikzcd}$$

Recall (from appendix B) that $\mathcal{S}_X$ is the sheaf of singular cochains on the topological space X, $U = X\backslash Z$, and $\mathcal{K}_{(X,Z)}$ is the kernel of the canonical morphism $\mathcal{S}_X \to i_*\mathcal{S}_Z$. $K(\mathrm{Sg}^\vee)$ is defined as in the proof of Theorem 7.5, the maps α and β are also defined in appendix B.

Everything except possibly the lower left inner diagram clearly commutes. Commutativity of this remaining diagram can be proved before applying $\mathrm{R}\tilde\pi_*$. We replace the constant presheaf Λ by $\mathrm{Sg}^\vee$, and the constant sheaf Λ_{cst} by $\mathcal{S}_U$. Then the lemma follows from commutativity of the following diagram, which is obvious.

$$\begin{tikzcd}
\mathrm{R}\iota_{X*} j_!\mathrm{Sg}^\vee|_U \arrow[r] & \iota_{X*}\mathrm{Tot}(\mathrm{Sg}^\vee|_X \to \oplus_i i_{i*}\mathrm{Sg}^\vee|_{Z_i} \to \cdot) \arrow[r] & \mathrm{Tot}(\mathcal{S}_X \to \oplus_i i_{i*}\mathcal{S}_{Z_i} \to \cdot) \\
\tilde{j}_!\iota_{U*}\mathrm{Sg}^\vee|_U \arrow[u] \arrow[ur] \arrow[r] & \tilde{j}_!\mathcal{S}_U \arrow[ur] \arrow[r, equal] & \mathcal{S}_X \otimes \Lambda_U \arrow[u]
\end{tikzcd}$$

□

We end this section with the following result expressing a duality between relative Morel-Voevodsky motives associated to complements of two different divisors in a smooth projective scheme. We will make essential use of it in the following section.

Lemma 7.13 (cf. [59, p. 13], [39, Lem. 1.13], [48, Lem. I.IV.2.3.5]) *Let W be a smooth projective scheme of dimension d, $W_0 \cup W_\infty$ a simple normal crossings divisor. Then there is a canonical isomorphism*

$$\mathcal{R}_A(W - W_\infty, W_0 - W_\infty, n)^\vee \cong \mathcal{R}_A(W - W_0, W_\infty - W_0, 2d - n)(-d)$$

in **DA**.

Proof. Fix the notation as in the following diagram:

$$\begin{tikzcd}
W - W_\infty \arrow[r, "j_\infty"] & W \\
W - (W_\infty \cup W_0) \arrow[u, "j'_0"] \arrow[r, "j'_\infty"'] & W - W_0 \arrow[u, "j_0"']
\end{tikzcd}$$

The left hand side of the equality to establish is

$$\pi_{W*} j_{\infty*} j'_{0!} j'^{!}_{0} j^*_\infty \pi^*_W \mathbb{1}[-n] \cong \pi_{W!} j_{\infty*} j'_{0!} j'^{*}_{\infty} j^{!}_{0} \pi^{!}_W \mathbb{1}(-d)[2d-n]$$

by relative purity, hence to prove the lemma it suffices to provide a canonical isomorphism $j_{0!}j'_{\infty *}\mathbb{1} \cong j_{\infty *}j'_{0!}\mathbb{1}$. The candidate morphism is obtained by adjunction from the composition

$$j'_{\infty *} \xrightarrow[\sim]{\text{adj}} j'_{\infty *}j'^{!}_{0}j'_{0!} \cong j^{!}_{0}j'_{\infty *}j'_{0!}.$$

It is clear that the candidate morphism is invertible on $W - W_0$, hence by localization, it remains to prove the same on W_0. Denote by $i_\bullet$ the closed immersion complementary to $j_\bullet$. We add a second subscript 0 (resp. ∞) to denote the pullback of a morphism along i_0 (resp. i_∞).

Note that $i_0^* j_{0!} = 0$ hence it suffices to prove $i_0^* j_{\infty *} j'_{0!}\mathbb{1} = 0$. By one of the localization triangles for the couple $(W - W_\infty, W - (W_\infty \cup W_0))$ we can equivalently prove that the morphism

$$\text{adj} : i_0^* j_{\infty *}\mathbb{1} \to i_0^* j_{\infty *} i'_{0*}\mathbb{1} \tag{7.14}$$

is invertible. The codomain of this morphism is isomorphic to

$$i_0^* j_{\infty *} i'_{0*}\mathbb{1} \cong i_0^* i_{0*} j_{\infty 0 *}\mathbb{1} \cong j_{\infty 0 *}\mathbb{1},$$

and under this identification, (7.14) corresponds to the morphism

$$i_0^* j_{\infty *}\mathbb{1} \xrightarrow{\text{bc}} j_{\infty 0 *} i'^{*}_{0}\mathbb{1} \cong j_{\infty 0 *}\mathbb{1}. \tag{7.15}$$

Here, as in the rest of the proof, bc denotes the canonical base change morphism of the functors involved. Consider now the following diagram:

$$\begin{tikzcd}
i_0^* i_{\infty !} i_\infty^! \mathbb{1} \arrow[r] \arrow[d, "\alpha"'] & i_0^* \mathbb{1} \arrow[r] \arrow[d, "\sim"'] & i_0^* j_{\infty *}\mathbb{1} \arrow[r] \arrow[d, "(7.15)"'] & i_0^* i_{\infty !} i_\infty^! \mathbb{1}[-1] \arrow[d, "\alpha"] \\
i_{\infty 0 !} i^!_{\infty 0}\mathbb{1} \arrow[r] & \mathbb{1} \arrow[r] & j_{\infty 0 *}\mathbb{1} \arrow[r] & i_{\infty 0 !} i^!_{\infty 0}\mathbb{1}[-1]
\end{tikzcd}$$

The bottom row is a localization triangle, the top row arises from such by application of i_0^*. It is clear that the middle square commutes. α is defined as the composition

$$i_0^* i_{\infty !} i^!_\infty \mathbb{1} \xrightarrow[\sim]{\text{bc}} i_{\infty 0 !} i^*_{0\infty} i^!_\infty \mathbb{1} \xrightarrow{\text{bc}} i_{\infty 0 !} i^!_{\infty 0} i_0^* \mathbb{1} \cong i_{\infty 0 !} i^!_{\infty 0}\mathbb{1},$$

and it is again easy to see that the left square commutes. Since there is only the zero morphism from $i_0^* i_{\infty !} i^!_\infty \mathbb{1}[-1]$ to $j_{\infty 0 *}\mathbb{1}$, this implies commutativity of the whole diagram. Now we have a morphism of distinguished triangles, and to prove invertibility of (7.15) (and therefore (7.14)) it suffices to prove invertibility of α. Only the middle arrow in its definition needs to be considered, and for this we note that $\text{bc} : i^*_{0\infty} i^!_\infty \mathbb{1} \to i^!_{\infty 0} i^*_0 \mathbb{1}$ is invertible by purity. □

8. Main result

The goal of this section is to prove the following theorem. The two main inputs are Proposition 8.2 and Theorem 8.3 which we prove subsequently.

Theorem 8.1 *Assume that Λ is a principal ideal domain. Then φ_A and φ_N are isomorphisms of Hopf algebras $\mathbf{H}_A \cong \mathbf{H}_N$, inverse to each other. In particular, there is an isomorphism of affine pro-group schemes over* $\mathrm{Spec}(\Lambda)$*:*

$$\mathcal{G}_A \cong \mathcal{G}_N.$$

Proof. Consider the following triangle:

$$\begin{tikzcd} \mathscr{D}^{\mathrm{g}}_{\mathrm{N}} \arrow[r, "\overline{\varphi_{\mathrm{N}}}\mathrm{H}_0\widetilde{\mathrm{Bti}}^*\mathcal{R}_{\mathrm{A}}"] \arrow[d, "\tilde{\mathrm{H}}_\bullet"'] & \mathbf{coMod}^{\mathrm{f}}(\mathbf{H}_{\mathrm{N}}) \\ \mathbf{coMod}^{\mathrm{f}}(\mathbf{H}^{\mathrm{eff}}_{\mathrm{N}}) \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

By construction, the triangle commutes (up to u. g. m. isomorphism) for $\varepsilon = \overline{\varphi_{\mathrm{N}}\varphi_{\mathrm{A}}}$. By Proposition 8.2 below it also commutes (up to u. g. m. isomorphism) for $\varepsilon = \bar{\iota}$, where $\iota : \mathbf{H}^{\mathrm{eff}}_{\mathrm{N}} \to \mathbf{H}_{\mathrm{N}}$ is the canonical localization map. By universality of Nori's category (Corollary 1.1), we must therefore have a monoidal isomorphism of functors $\overline{\varphi_{\mathrm{N}}\varphi_{\mathrm{A}}} \cong \bar{\iota}$. By [64, II, 3.3.1], we must then have $\varphi_{\mathrm{N}}\varphi_{\mathrm{A}} = \iota$. Hence $\varphi_{\mathrm{N}}\varphi_{\mathrm{A}} : \mathbf{H}_{\mathrm{N}} \to \mathbf{H}_{\mathrm{N}}$ is the identity. In particular, φ_{A} is injective. If we prove surjectivity of φ_{A} then it will follow that φ_{A} and φ_{N} are bialgebra isomorphisms, inverse to each other. And since an antipode of a Hopf algebra is unique, they are automatically isomorphisms of Hopf algebras.

We use Theorem 8.3 below to deduce that the comultiplication $\mathrm{cm} : \mathbf{H}_{\mathrm{A}} \to \mathbf{H}_{\mathrm{A}} \otimes \mathbf{H}_{\mathrm{A}}$ is equal to

$$\mathbf{H}_{\mathrm{A}} \xrightarrow{\mathrm{ca}} \mathbf{H}_{\mathrm{N}} \otimes \mathbf{H}_{\mathrm{A}} \xrightarrow{\varphi_{\mathrm{A}}\otimes\mathrm{id}} \mathbf{H}_{\mathrm{A}} \otimes \mathbf{H}_{\mathrm{A}}$$

for some coaction ca of $\mathbf{H}_{\mathrm{N}}$ on $\mathbf{H}_{\mathrm{A}}$. Composing cm with the counit $\mathrm{id} \otimes \mathrm{cu} : \mathbf{H}_{\mathrm{A}} \otimes \mathbf{H}_{\mathrm{A}} \to \mathbf{H}_{\mathrm{A}} \otimes \Lambda \cong \mathbf{H}_{\mathrm{A}}$ yields the identity, therefore also

$$\mathrm{id} = (\mathrm{id} \otimes \mathrm{cu}) \circ (\varphi_{\mathrm{A}} \otimes \mathrm{id}) \circ \mathrm{ca} = \varphi_{\mathrm{A}} \circ (\mathrm{id} \otimes \mathrm{cu}) \circ \mathrm{ca}.$$

It follows that φ_{A} is surjective. ☐

Proposition 8.2 *Assume that Λ is a principal ideal domain. The following square commutes up to an isomorphism of u. g. m. representations:*

$$\begin{tikzcd} \mathscr{D}^{\mathrm{g}}_{N} \arrow[r, "\mathcal{R}_A"] \arrow[d, "\tilde{\mathrm{H}}_\bullet"'] & \mathbf{DA} \arrow[d, "\mathrm{H}_0\circ\widetilde{\mathrm{Bti}}^*"] \\ \mathbf{coMod}(\mathbf{H}_N) & \mathbf{coMod}(\mathbf{H}_A) \arrow[l, "\overline{\varphi_N}"] \end{tikzcd}$$

Proof. By Proposition 4.7, we already know that after composition with the forgetful functor $\mathbf{coMod}(\mathbf{H}_{\mathrm{N}}) \to \mathbf{Mod}(\Lambda)$, the two u. g. m. representations are naturally isomorphic. Call the isomorphism η. It therefore suffices to prove that the components of η are compatible with the $\mathbf{H}_{\mathrm{N}}$-comodule structure.

Let $v = (X, Z, n)$ be an arbitrary vertex in $\mathscr{D}^{\mathrm{g}}_{\mathrm{N}}$. We find by resolution of singularities a vertex $v' = (X', Z', n)$ and an edge $p : v' \to v$ such that (X', Z') is almost smooth and $\mathrm{H}_\bullet(p)$ is an isomorphism. Consider the following commutative square in $\mathbf{Mod}(\Lambda)$:

$$\begin{tikzcd} \mathrm{H}_\bullet(v') \arrow[r, "\eta_{v'}"] \arrow[d, "\mathrm{H}_\bullet(p)"'] & \overline{\varphi_{\mathrm{N}}}\mathrm{H}_0\widetilde{\mathrm{Bti}}^*\mathcal{R}_{\mathrm{A}}(v') \arrow[d, "\overline{\varphi_{\mathrm{N}}}\mathrm{H}_0\widetilde{\mathrm{Bti}}^*\mathcal{R}_{\mathrm{A}}(p)"] \\ \mathrm{H}_\bullet(v) \arrow[r, "\eta_v"'] & \overline{\varphi_{\mathrm{N}}}\mathrm{H}_0\widetilde{\mathrm{Bti}}^*\mathcal{R}_{\mathrm{A}}(v) \end{tikzcd}$$

All arrows are invertible and both vertical arrows are $\mathbf{coMod}(\mathbf{H}_{\mathrm{N}})$-morphisms. We may therefore assume that (X, Z) is almost smooth.

Now consider the following diagram in $\mathbf{Mod}(\Lambda)$:

$$\begin{tikzcd}
\mathrm{H}_0\mathrm{Bti}^*\mathcal{R}_{\mathrm{A}}(X,Z,n) \arrow[r, "\sim", "\text{Thm. 7.5}"'] \arrow[d, "\sim"', "\eta"] & \mathrm{H}_0\mathrm{Bti}^{\mathrm{eff},*}\Lambda(X,Z)[n] \arrow[d, "\text{Cor. 6.6}"', "\sim"] \\
\mathrm{H}_n(X,Z) & \mathrm{H}_0\mathcal{R}_{\mathrm{N}}\Lambda(X,Z)[n] \arrow[l, "\sim", "(7.2)"']
\end{tikzcd}$$

By Lemma 7.3, the bottom horizontal arrow is compatible with the $\mathbf{H}_{\mathrm{N}}^{\mathrm{eff}}$-coaction; the same is clearly true for the top horizontal and the right vertical one. We are thus reduced to show commutativity of this square.

For this, it suffices to prove it before applying H_n. Modulo the identification of $\mathrm{Bti}^{\mathrm{eff},*}$ with $\mathrm{LSg}^*\circ\mathrm{An}^*$ of Proposition 2.2, the composition of the right vertical and the bottom horizontal arrow can be equivalently described as the composition

$$\mathrm{LSg}^*\mathrm{An}^*\Lambda(X,Z)\to\mathrm{LSg}^*\Lambda(X^{\mathrm{an}},Z^{\mathrm{an}})\to\mathrm{Sg}(X^{\mathrm{an}},Z^{\mathrm{an}}).$$

Thus the square above will commute if the following diagram does:

$$\begin{tikzcd}
\mathrm{Bti}^*\mathrm{L}\pi_!\mathrm{R}j_*j^*\pi^!\mathbb{1} \arrow[r, dash, "\sim"] \arrow[d, dash, "\sim"'] & \mathrm{Bti}^*\mathrm{LSus}^0_T\Lambda(X,Z_\bullet) \arrow[r, dash, "\sim"] \arrow[d, dash, "\sim"'] & \mathrm{LSg}^*\mathrm{An}^*\Lambda(X,Z_\bullet) \arrow[d, "\sim"] \\
(\mathrm{Bti}^*\mathrm{R}\pi_*j_!\mathbb{1})^\vee \arrow[r, equal] \arrow[d, dash, "\sim"'] & (\mathrm{Bti}^*\,\underline{\mathrm{Hom}}(\Lambda(X,Z_\bullet),\mathbf{E}))^\vee \arrow[d, dash, "\sim"'] & \mathrm{LSg}^*\Lambda(X^{\mathrm{an}},Z^{\mathrm{an}}_\bullet) \arrow[d, "\sim"] \\
(\mathrm{R}\tilde\pi^{\mathrm{an}}_*\tilde j^{\mathrm{an}}_!\Lambda_{\mathrm{cst}})^\vee \arrow[r, equal] & \mathrm{Sg}(X^{\mathrm{an}},Z^{\mathrm{an}})^{\vee\vee} & \mathrm{Sg}(X^{\mathrm{an}},Z^{\mathrm{an}}) \arrow[l, "\sim"]
\end{tikzcd}$$

The arrows in the top left square are induced by the identifications in Theorem 7.5 ($\mathbf{E}$ is a projective stable $(\mathbb{A}^1,\tau)$-fibrant replacement of the unit spectrum) and duality which makes the square clearly commutative. Commutativity of the lower left square is Lemma 7.10 and 7.12. For the right half, consider the following diagram (all "arrows" are isomorphisms, either canonical or introduced before):

$$\begin{tikzcd}[column sep=small]
(\mathrm{Bti}^*\mathrm{LSus}^0\Lambda(X,Z_\bullet))^\vee \arrow[r, dash] \arrow[d, dash] & (\mathrm{Bti}^{\mathrm{eff},*}\Lambda(X,Z_\bullet))^\vee \arrow[r, dash] & (\mathrm{LSg}^*\mathrm{An}^*\Lambda(X,Z_\bullet))^\vee \arrow[r, dash] & (\mathrm{LSg}^*\Lambda(X^{\mathrm{an}},Z^{\mathrm{an}}_\bullet))^\vee \arrow[d, dash, dotted] \\
\mathrm{R\Gamma REv}_0(\mathrm{An}^*\mathrm{LSus}^0\Lambda(X,Z_\bullet))^\vee \arrow[r, dash] \arrow[d, dash] & \mathrm{R\Gamma REv}_0(\mathrm{LSus}^0\Lambda(X^{\mathrm{an}},Z^{\mathrm{an}}_\bullet))^\vee \arrow[r, dash] \arrow[d, dash] & \mathrm{R\Gamma REv}_0\mathrm{LSus}^0\Lambda(X^{\mathrm{an}},Z^{\mathrm{an}}_\bullet)^\vee \arrow[r, dash] \arrow[d, dash] & \mathrm{LSg}^*\Lambda(X^{\mathrm{an}},Z^{\mathrm{an}}_\bullet)^\vee \arrow[d, dash, dotted] \\
\mathrm{Bti}^*\,\underline{\mathrm{Hom}}(\Lambda(X,Z_\bullet),\mathbf{E}) \arrow[r, dash] & \mathrm{R\Gamma REv}_0\,\underline{\mathrm{Hom}}(\Lambda(X^{\mathrm{an}},Z^{\mathrm{an}}_\bullet),\mathbf{Sg}^\vee) \arrow[r, dash] & \mathrm{R\Gamma}\,\underline{\mathrm{Hom}}(\Lambda(X^{\mathrm{an}},Z^{\mathrm{an}}_\bullet),\mathrm{Sg}^\vee) \arrow[r, dash, dotted] & \mathrm{LSg}^*\,\underline{\mathrm{Hom}}(\Lambda(X^{\mathrm{an}},Z^{\mathrm{an}}_\bullet),\mathrm{Sg}^\vee)
\end{tikzcd}$$

The upper part clearly commutes as does the lower right square. For the lower left square we need to prove commutative

$$\begin{tikzcd}
\mathrm{An}^*\,\underline{\mathrm{Hom}}(\Lambda(X,Z_\bullet),\mathbf{E}) \arrow[r, equal] \arrow[d] & \mathrm{An}^*\mathrm{R}\,\underline{\mathrm{Hom}}(\mathrm{LSus}^0\Lambda(X,Z_\bullet),\mathbb{1}) \arrow[d] \\
\underline{\mathrm{Hom}}(\Lambda(X^{\mathrm{an}},Z^{\mathrm{an}}_\bullet),\mathbf{Sg}^\vee) \arrow[r, equal] & \mathrm{R}\,\underline{\mathrm{Hom}}(\mathrm{LSus}^0\Lambda(X^{\mathrm{an}},Z^{\mathrm{an}}_\bullet),\mathbb{1})
\end{tikzcd}$$

and this is done as in Lemma 7.10. The lower middle square is easily seen to commute hence, using duality, it only remains to prove that the composition of the dotted arrows is equal to

$$\mathrm{R\Gamma}\,\underline{\mathrm{Hom}}(\Lambda(X^{\mathrm{an}},Z^{\mathrm{an}}_\bullet),\mathrm{Sg}^\vee)\leftarrow\mathrm{Sg}^\vee(X^{\mathrm{an}},Z^{\mathrm{an}})\to(\mathrm{LSg}^*\Lambda(X^{\mathrm{an}},Z^{\mathrm{an}}_\bullet))^\vee.$$

For this notice that $\mathrm{Sg}^\vee = \mathrm{Sg}_*\Lambda$. Then, writing B for $\Lambda(X^{\mathrm{an}}, Z_\bullet^{\mathrm{an}})$, we reduce to prove commutative:

$$\begin{tikzcd}
\Gamma\,\underline{\mathrm{Hom}}(B, \mathrm{Sg}_*\Lambda) \arrow[r, dash] \arrow[d] & \mathrm{LSg}^*\,\underline{\mathrm{Hom}}(B, \mathrm{Sg}_*\Lambda) \arrow[r, equal] \arrow[d] & \mathrm{LSg}^*\mathrm{R}\,\underline{\mathrm{Hom}}(B, \Lambda) \arrow[d] \\
\Gamma\mathrm{Sg}_*\,\underline{\mathrm{Hom}}(\mathrm{Sg}^*B, \Lambda) \arrow[r, dash] \arrow[dr, dash] & \mathrm{LSg}^*\mathrm{Sg}_*\,\underline{\mathrm{Hom}}(\mathrm{Sg}^*B, \Lambda) \arrow[d] & \mathrm{R}\,\underline{\mathrm{Hom}}(\mathrm{LSg}^*B, \Lambda) \\
& \underline{\mathrm{Hom}}(\mathrm{Sg}^*B, \Lambda) \arrow[ur] &
\end{tikzcd}$$

Here we used that $\mathrm{Sg}\circ\iota_\star : \star \to \mathbf{Cpl}(\Lambda)$ takes the value $\mathrm{Sg}(\star) \simeq \Lambda$ hence $\Gamma\circ\mathrm{Sg}_*$ is canonically quasi-isomorphic to the identity. Since the undecorated functors Sg^* and Sg_* appearing are only applied to cofibrant, respectively fibrant, objects they can be identified with their derived counterparts and the diagram is easily seen to commute. ☐

Theorem 8.3 *Assume that Λ is a principal ideal domain. The bialgebra $\mathbf{H}_A$ considered as a comodule over itself lies in the essential image of*

$$\mathbf{coMod}(\mathbf{H}_N) \xrightarrow{\overline{\varphi_A}} \mathbf{coMod}(\mathbf{H}_A).$$

Proof. We may prove this statement for the Nisnevich topology. Ayoub gives in [5, Thm. 2.67] an explicit model for the symmetric T-spectrum $\mathrm{Bti}_*^{\mathrm{Nis}}\Lambda$ which we are now going to describe at a level of detail appropriate for our proof.

Recall the category $\mathcal{V}_{\text{ét}}(\overline{\mathbb{D}}^n/\mathbb{A}^n)$ $(n \geq 0)$ whose objects are étale neighborhoods of the closed polydisk $\overline{\mathbb{D}}^n$ inside affine space $\mathbb{A}^n$ (for the precise definition see [5, §2.2.4]). It is a cofiltered category. Forgetting the presentation as a scheme over $\mathbb{A}^n$ defines a canonical functor $\overline{\mathbb{D}}^n_{\text{ét}} : \mathcal{V}_{\text{ét}}(\overline{\mathbb{D}}^n/\mathbb{A}^n) \to \mathrm{Sm}$. In other words we obtain a pro-smooth scheme. We write $\overline{\mathbb{D}}_{\text{ét}}$ for the associated cocubical object in pro-smooth schemes where the faces $d_{i,\varepsilon}$ are induced from the faces in $\mathbb{A}^n$ (the "coordinate hyperplanes" through 0 and 1). For $n \in \mathbb{N}$ and $\varepsilon = 0, 1$ write $\partial_\varepsilon\overline{\mathbb{D}}^n_{\text{ét}}$ for the union of the faces $d_{i,\varepsilon}(\overline{\mathbb{D}}^{n-1}_{\text{ét}})$, where i runs through $1, \ldots, n$. Also write $\partial\overline{\mathbb{D}}^n_{\text{ét}}$ for the union $\partial_0\overline{\mathbb{D}}^n_{\text{ét}} \cup \partial_1\overline{\mathbb{D}}^n_{\text{ét}}$, and $\partial_{1,1}\overline{\mathbb{D}}^n_{\text{ét}}$ for the union of all $d_{i,\varepsilon}(\overline{\mathbb{D}}^{n-1}_{\text{ét}})$ except $(i, \varepsilon) = (1, 1)$.

We obtain the bicomplex $N(\underline{\mathrm{hom}}(\overline{\mathbb{D}}_{\text{ét}}, K))$ which in degree n (in the direction of the cocubical dimension) is given by $\underline{\mathrm{hom}}((\overline{\mathbb{D}}^n_{\text{ét}}, \partial_{1,1}\overline{\mathbb{D}}^n_{\text{ét}}), K)$, and whose differential in degree $n > 0$ is $d_{1,1}$.[6] In particular, the cycles in degree $n > 0$ are given by $\underline{\mathrm{hom}}((\overline{\mathbb{D}}^n_{\text{ét}}, \partial\overline{\mathbb{D}}^n_{\text{ét}}), K)$. Thus we obtain a canonical morphism of bicomplexes

$$\underline{\mathrm{hom}}((\overline{\mathbb{D}}^n_{\text{ét}}, \partial\overline{\mathbb{D}}^n_{\text{ét}}), K)[-n] \to N^{\leq n}(\underline{\mathrm{hom}}(\overline{\mathbb{D}}_{\text{ét}}, K))$$

where the right hand side denotes the bicomplex truncated at degree n from above. One can check that this induces a quasi-isomorphism on the associated total complexes whenever K is injective fibrant.

Taking the total complex of the bicomplex $N(\underline{\mathrm{hom}}(\overline{\mathbb{D}}_{\text{ét}}, K))$ (resp. $N^{\leq n}(\underline{\mathrm{hom}}(\overline{\mathbb{D}}_{\text{ét}}, K))$) we obtain an endofunctor ${}^{\mathrm{n}}\underline{\mathrm{Sg}}^{\mathbb{D}}_{\text{ét}}$ (resp. ${}^{\mathrm{n}}\underline{\mathrm{Sg}}^{\mathbb{D}^{\leq n}}_{\text{ét}}$) of presheaves of complexes on smooth schemes.

---

[6] Given a pro-object $(X_i, Z_i)_{i\in I}$ of almost smooth pairs, $\underline{\mathrm{hom}}((X_i, Z_i)_i, K)$ takes a smooth scheme Y to

$$\varinjlim_{i\in I} K(Y \times X_i, Y \times Z_i).$$

It extends canonically to an endofunctor on symmetric T-spectra. Let $\mathbf{E}$ be an injective stable $(\mathbb{A}^1, \mathrm{Nis})$-fibrant replacement of the unit spectrum $\mathbb{1}$. [5, Thm. 2.67] states that $\mathrm{Bti}^{\mathrm{Nis}}_*\mathbb{1}$ is given explicitly by the symmetric T-spectrum

$$\mathbf{Sing}^{\mathbb{D},\infty}_{\text{ét}}(\mathbf{E}) := \varinjlim_r s_-^r\, {}^{\mathrm{n}}\underline{\mathrm{Sg}}^{\mathbb{D}}_{\text{ét}}(\mathbf{E})[2r],$$

where s_- denotes the "shift down" functor (so that $s_-(\mathbf{E})_m = \mathbf{E}_{m+1}$; see [2, Déf. 4.3.13]). As we will not need a description of the transition morphisms in the sequential colimit above, we content ourselves with referring to [5, Déf. 2.65]. Let $Q : \mathbf{Spt}^{\Sigma}_T\mathrm{USm} \to \mathbf{DA}^{\mathrm{Nis}}$ denote the canonical localization functor, and consider the following canonical morphisms:

$$\varinjlim_{r,n} \mathrm{H}_0\widetilde{\mathrm{Bti}}^{\mathrm{Nis},*}Q(s_-^r\,\underline{\mathrm{hom}}((\overline{\mathbb{D}}^n_{\text{ét}}, \partial\overline{\mathbb{D}}^n_{\text{ét}}), \mathbf{E})[2r-n]) \to \varinjlim_{r,n} \mathrm{H}_0\widetilde{\mathrm{Bti}}^{\mathrm{Nis},*}Q(s_-^r\,{}^{\mathrm{n}}\underline{\mathrm{Sg}}^{\mathbb{D}^{\leq n}}_{\text{ét}}(\mathbf{E})[2r])$$
$$\to \mathrm{H}_0\widetilde{\mathrm{Bti}}^{\mathrm{Nis},*}Q(\mathbf{Sing}^{\mathbb{D},\infty}_{\text{ét}}(\mathbf{E})) \qquad (8.4)$$

in $\mathbf{coMod}(\mathbf{H}_{\mathrm{A}})$. The last term is the bialgebra $\mathbf{H}_{\mathrm{A}}$ considered as a comodule over itself. We are going to show first that the composition in (8.4) is invertible, and then that the comodules in the filtered system on the left hand side are in the essential image of $\overline{\varphi_{\mathrm{A}}}$. This is enough since, as seen in the proof of Theorem 8.1, the previous proposition implies that $\overline{\varphi_{\mathrm{N}}\varphi_{\mathrm{A}}} \cong \mathrm{id}$ hence the essential image of $\overline{\varphi_{\mathrm{A}}}$ is a full subcategory of $\mathbf{coMod}(\mathbf{H}_{\mathrm{A}})$ (since both $\overline{\varphi_{\mathrm{A}}}$ and $\overline{\varphi_{\mathrm{N}}}$ are faithful) closed under small colimits (by Fact C.1).

In order to show invertibility of (8.4), we can do so after forgetting the comodule structure. Just as in the case of the étale singular complex there is an endofunctor $\mathbf{Sing}^{\mathbb{D},\infty}$ on symmetric $\mathrm{An}^*(T)$-spectra defined using $\mathrm{Sg}^{\mathbb{D}}$ instead of ${}^{\mathrm{n}}\underline{\mathrm{Sg}}^{\mathbb{D}}_{\text{ét}}$ (cf. [5, Déf. 2.45]). Denote by $F : \mathbf{Spt}^{\Sigma}_T\mathrm{USm} \to \mathbf{Mod}(\Lambda)$ the composition of functors $\mathrm{H}_0\Gamma\mathrm{Ev}_0\mathbf{Sing}^{\mathbb{D},\infty}\mathrm{An}^*$ and notice that

(a) F commutes with filtered colimits, by construction;
(b) F takes levelwise quasi-isomorphisms of symmetric T-spectra to isomorphisms of modules, as follows essentially from [5, Lem. 2.55];
(c) F applied to a projective stable $(\mathbb{A}^1, \mathrm{Nis})$-fibrant spectrum $\mathbf{K}$ is a model for $\mathrm{H}_0\mathrm{Bti}^{\mathrm{Nis},*}\mathbf{K}$, by [5, Lem. 2.72 and Thm. 2.48].

We claim that the morphism of Λ-modules underlying (8.4) can be identified with the composition

$$\varinjlim_{r,n} F(s_-^r\,\underline{\mathrm{hom}}((\overline{\mathbb{D}}^n_{\text{ét}}, \partial\overline{\mathbb{D}}^n_{\text{ét}}), \mathbf{E})[2r-n]) \to \varinjlim_{r,n} F(s_-^r\,{}^{\mathrm{n}}\underline{\mathrm{Sg}}^{\mathbb{D}^{\leq n}}_{\text{ét}}(\mathbf{E})[2r])$$
$$\to F(\mathbf{Sing}^{\mathbb{D},\infty}_{\text{ét}}(\mathbf{E})). \qquad (8.5)$$

This follows from (c) because both $\mathbf{Sing}^{\mathbb{D},\infty}_{\text{et}}(\mathbf{E})$ and $\underline{\mathrm{hom}}((\overline{\mathbb{D}}^n_{\text{ét}}, \partial\overline{\mathbb{D}}^n_{\text{ét}}), \mathbf{E}))$ are projective stable $(\mathbb{A}^1, \mathrm{Nis})$-fibrant, as follows from [5, Thm. 2.67] for the first, and from our proof of Theorem 7.5 together with [5, Lem. 2.69] for the second. But the first arrow in (8.5) is invertible by (b), and the second one by (a) so we conclude that (8.4) is invertible.

Next we fix $(r, n) \in \mathbb{N}^2$ and consider the canonical morphism

$$\varinjlim_{(X,x)\in\mathcal{V}_{\text{ét}}(\overline{\mathbb{D}}^n/\mathbb{A}^n)} \mathrm{H}_0\widetilde{\mathrm{Bti}}^{\mathrm{Nis},*}Q(s_-^r\,\underline{\mathrm{hom}}((X, \partial X), \mathbf{E})[2r-n]) \to$$
$$\mathrm{H}_0\widetilde{\mathrm{Bti}}^{\mathrm{Nis},*}Q(s_-^r\,\underline{\mathrm{hom}}((\overline{\mathbb{D}}^n_{\text{ét}}, \partial\overline{\mathbb{D}}^n_{\text{ét}}), \mathbf{E})[2r-n]).$$

The same argument as above establishes invertibility of this arrow and reduces us to show that the comodules in the filtered system on the left hand side lie in the essential image of $\overline{\varphi_{\mathrm{A}}}$. Hence fix $(X,x) \in \mathcal{V}_{\text{ét}}(\overline{\mathbb{D}}^n/\mathbb{A}^n)$. By resolution of singularities there is a smooth projective scheme W and a simple normal crossings divisor $W_0 \cup W_\infty$ on W together with a projective surjective morphism $p : W - W_\infty \to X$ such that $p^{-1}(\partial X) = W_0 - W_\infty$ and $p|_{W-p^{-1}(\partial X)} : W - p^{-1}(\partial X) \to X - \partial X$ is an isomorphism. Therefore, canonically, $\mathcal{R}_{\mathrm{A}}(X, \partial X, 0) \cong \mathcal{R}_{\mathrm{A}}(W - W_\infty, W_0 - W_\infty, 0)$, and we obtain in $\mathbf{DA}^{\mathrm{Nis}}$:

$$\begin{aligned} s_-^r\, \underline{\mathrm{hom}}((X,\partial X),\mathbf{E})[2r-n] &\cong \underline{\mathrm{hom}}((X,\partial X),\mathbf{E})[-n](r) \\ &\cong \mathcal{R}_{\mathrm{A}}(X,\partial X,0)^\vee[-n](r) \\ &\cong \mathcal{R}_{\mathrm{A}}(W-W_\infty, W_0-W_\infty,0)^\vee[-n](r) \\ &\cong \mathcal{R}_{\mathrm{A}}(W-W_0, W_\infty-W_0,n)(r-n) \\ &\cong \mathcal{R}_{\mathrm{A}}(W-W_0, W_\infty-W_0,n) \otimes^{\mathrm{L}} \mathcal{R}_{\mathrm{A}}(\mathbb{G}_m,\{1\},1)^{\otimes^{\mathrm{L}}(r-n)}, \end{aligned}$$

where we used [2, Thm. 4.3.38] for the first, Theorem 7.5 for the second, and Lemma 7.13 for the penultimate isomorphism. Applying $\mathrm{H}_0\widetilde{\mathrm{Bti}}^*$ to these isomorphisms, and using (4.8) as well as (4.9) we obtain the following sequence of isomorphisms

$$\begin{aligned} &\mathrm{H}_0\widetilde{\mathrm{Bti}}^{\mathrm{Nis},*} s_-^r\, \underline{\mathrm{hom}}((X,\partial X),\mathbf{E})[2r-n] \\ &\cong \mathrm{H}_0\widetilde{\mathrm{Bti}}^{\mathrm{Nis},*}\left(\mathcal{R}_{\mathrm{A}}(W-W_0, W_\infty-W_0,n) \otimes^{\mathrm{L}} \mathcal{R}_{\mathrm{A}}(\mathbb{G}_m,\{1\},1)^{\otimes^{\mathrm{L}}(r-n)}\right) \\ &\cong \overline{\varphi_{\mathrm{A}}}\left(\tilde{\mathrm{H}}_\bullet(W-W_0, W_\infty-W_0,n) \otimes^{\mathrm{L}} \tilde{\mathrm{H}}_\bullet(\mathbb{G}_m,\{1\},1)^{\otimes^{\mathrm{L}}(r-n)}\right), \end{aligned}$$

which concludes the proof. □

A. Nori's Tannakian formalism in the monoidal setting

In this section we indicate briefly which modifications to [39, App. B] have to be made in order to justify our arguments in the main body of the text regarding Nori's Tannakian formalism. Most importantly we seek to obtain a universality statement for Nori's construction in the monoidal setting. Something similar was undertaken by Bruguières in [12], and for the main proof below we follow his ideas. However the results there on monoidal representations do not seem to apply directly to Nori's construction since there is no obvious monoidal structure (in the sense of [12]) on Nori's diagrams.[7]

A *graded diagram* and a *commutative product structure* on such a graded diagram are defined as in [39, Def. B.14]. From now on, fix such a graded diagram $\mathscr{D}$ with a commutative product structure. Let $(\mathcal{C},\otimes)$ be an additive (symmetric, unitary) monoidal category. A *graded multiplicative representation* $T : \mathscr{D} \to \mathcal{C}$ is a representation of $\mathscr{D}$ in $\mathcal{C}$ together with a choice of isomorphisms

$$\tau_{(f,g)} : T(f \times g) \to T(f) \otimes T(g)$$

for any vertices f and g of $\mathscr{D}$, satisfying (1)-(5) of [39, Def. B.14]. *Unital graded multiplicative (u. g. m.) representations* are then defined as in [39, Def. B.14]. A *u. g. m. transformation* $\eta : T \to U$ between two unital graded multiplicative representations $T, U : \mathscr{D} \to \mathcal{C}$ is a family of morphisms in $\mathcal{C}$:

$$\eta_f : T(f) \to U(f),$$

compatible with edges in $\mathscr{D}$ and the choices of isomorphisms τ, and such that $\eta_{\mathrm{id}} = \mathrm{id}$. η is a *u. g. m. isomorphism* if all its components are invertible.

From now on, fix also a u. g. m. representation $T : \mathscr{D} \to \mathbf{Mod}^{\mathrm{f}}(\Lambda)$ taking values in projective modules (we assume Λ to be of global dimension at most 2; see [12, §5.3]). By

[7] This is related to the problem discussed in [39, Rem. B.13].

Nori's theorem ([59, Thm. 1.6], [39, Pro. B.8]), there is a universal abelian Λ-linear category $\mathcal{C}(T)$ with a representation $\tilde{T} : \mathscr{D} \to \mathcal{C}(T)$, through which T factors via a faithful exact Λ-linear functor $o_T : \mathcal{C}(T) \to \mathbf{Mod}^{\mathrm{f}}(\Lambda)$. Nori also showed (see [39, Pro. B.16]) that in this case $\mathcal{C}(T)$ carries naturally a (right exact) monoidal structure such that o_T is a monoidal functor. It is obvious from the construction of this monoidal structure that $\tilde{T}$ is a u. g. m. representation. The following theorem states that these data are *universal*.

Theorem A.1 *Given a right exact monoidal abelian Λ-linear category $\mathcal{C}$ and a factorization of T into*

$$\mathscr{D} \xrightarrow{S} \mathcal{C} \xrightarrow{o_S} \mathbf{Mod}^{\mathrm{f}}(\Lambda)$$

with S u. g. m. and o_S a faithful exact Λ-linear monoidal functor, there exists a monoidal functor (unique up to unique monoidal isomorphism) $F : \mathcal{C}(T) \to \mathcal{C}$ making the following diagram commutative (up to monoidal isomorphism).

$$\begin{tikzcd}
\mathscr{D} \arrow[r, "S"] \arrow[d, "\tilde{T}"'] & \mathcal{C} \arrow[d, "o_S"] \\
\mathcal{C}(T) \arrow[r, "o_T"'] \arrow[ur, dotted, "F"] & \mathbf{Mod}^{\mathrm{f}}(\Lambda)
\end{tikzcd}$$

Moreover, F is faithful exact Λ-linear.

Explicitly, there exists a monoidal functor $F : \mathcal{C}(T) \to \mathcal{C}$, a u. g. m. isomorphism $\alpha : S\tilde{T} \xrightarrow{\sim} F$, and a monoidal isomorphism $\beta : o_T \xrightarrow{\sim} o_S F$ such that $o_S\alpha = \beta\tilde{T}$. Moreover given another triple (F', α', β') satisfying these conditions, there exists a unique monoidal isomorphism $\gamma : F' \xrightarrow{\sim} F$ transforming α to α' and β to β'.

Proof. Given [39, Pro. B.8 and B.16], the only thing left to prove is that the functor and transformations whose existence is asserted are monoidal. This could be proven by going through the construction of these and checking monoidality directly. Alternatively, one can deduce the monoidal structure from the existence of the functor and transformations alone without referring to their construction. We sketch the latter proof which is due to Bruguières. For the details we refer the reader to [12].

Let $\Psi_{X,Y}$ be the composition

$$o_S(FX \otimes FY) = o_S FX \otimes o_S FY \cong o_S F(X \otimes Y),$$

for any $X, Y \in \mathcal{C}(T)$. This defines a natural isomorphism of functors. Since o_S is faithful, it suffices to construct morphisms $\Phi_{X,Y} : FX \otimes FY \to F(X \otimes Y)$ which realize $\Psi_{X,Y}$. Thus consider the class

$$L = \{(X, Y) \in \mathcal{C}(T) \times \mathcal{C}(T) \mid \exists \Phi_{X,Y} : o_S\Phi_{X,Y} = \Psi_{X,Y}\}.$$

Notice that L contains all pairs in the image of $\tilde{T}$:

$$F\tilde{T}f \otimes F\tilde{T}g \cong Sf \otimes Sg \to S(f \times g) \cong F\tilde{T}(f \times g) \to F(\tilde{T}f \otimes \tilde{T}g)$$

can (and has to) be taken as $\Phi_{\tilde{T}f,\tilde{T}g}$. Now for fixed f the functors $F\tilde{T}f \otimes F(\bullet)$ and $F(\tilde{T}f \otimes \bullet)$ are exact hence one can define $\Phi_{\tilde{T}f,\bullet}$ on the subcategory of $\mathcal{C}(T)$ containing the image of $\tilde{T}$ and closed under kernels, cokernels and direct sums. But this is all of $\mathcal{C}(T)$. By symmetry one sees that L contains all pairs (X, Y) where one of X or Y is contained in the image of $\tilde{T}$. Now a similar argument shows that L also contains all pairs (X, Y) where one of $o_T X$ or $o_T Y$ is projective (since then the functors considered above are still exact). Finally, one uses that every object in $\mathcal{C}(T)$ is a quotient of an object with underlying projective Λ-module to conclude that L consists of all pairs of objects in $\mathcal{C}(T)$.

It is obvious from the definition of $\Phi_{\tilde{T}f,\tilde{T}g}$ that α is monoidal, and from the definition of $\Psi_{X,Y}$ that β is as well. It is an easy exercise to prove that γ is monoidal as well. $\square$

B. Relative cohomology

It is well-known that singular and sheaf cohomology agree on locally contractible topological spaces. The same is true for pairs of such spaces. However, we have not been able to find in the literature the statements in the form we need them in the main body of the chapter (in particular in section 4) although the book of Bredon [11] comes close. We will freely use the results of [11, §III.1]. Λ is a fixed principal ideal domain. All topological spaces are assumed locally contractible and paracompact.

B.1. Model. For a topological space X, denote by $\mathcal{S}_X$ the complex of sheaves of singular cochains on X with values in Λ. This is a flabby resolution of the constant sheaf Λ. Moreover, the canonical map $\mathrm{Sg}(X)^\vee \to \mathcal{S}_X(X)$ is a quasi-isomorphism.

Now let $i : Z \hookrightarrow X$ a closed subset with open complement $j : U \hookrightarrow X$. We denote by Λ_U (respectively Λ_Z) the constant sheaf Λ supported at U (respectively Z), i. e. $\Lambda_U = j_! j^* \Lambda_X$ (resp. $\Lambda_Z = i_* i^* \Lambda_X$). The canonical morphism $\mathcal{S}_X \otimes \Lambda_Z \to i_* \mathcal{S}_Z$ induces the diagram of solid arrows in the category of complexes of sheaves on X with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_{(X,Z)} \arrow[r] & \mathcal{S}_X \arrow[r] & i_*\mathcal{S}_Z \\ 0 \arrow[r] & \mathcal{S}_X \otimes \Lambda_U \arrow[r] \arrow[u, dotted, "\alpha"] & \mathcal{S}_X \arrow[r] \arrow[u, equal] & \mathcal{S}_X \otimes \Lambda_Z \arrow[u] \end{tikzcd} \tag{B.1}$$

We obtain a unique morphism α rendering the diagram commutative. It induces a quasi-isomorphism after taking global sections.

Similarly, β is the unique morphism of complexes making the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_{(X,Z)}(X) \arrow[r] & \mathcal{S}_X(X) \arrow[r] & i_*\mathcal{S}_Z(X) \\ 0 \arrow[r] & \mathrm{Sg}(X,Z)^\vee \arrow[r] \arrow[u, dotted, "\beta"] & \mathrm{Sg}(X)^\vee \arrow[r] \arrow[u] & \mathrm{Sg}(Z)^\vee \arrow[u] \end{tikzcd} \tag{B.2}$$

Again, it is a quasi-isomorphism.

Now, $\mathcal{S}_X \otimes \Lambda_U$ is a resolution of Λ_U which computes derived global sections hence we deduce the following result.

Fact B.3 *The zigzag of α and β exhibits $\mathrm{Sg}(X,Z)^\vee$ as a model for $\mathrm{R}\Gamma(X, \Lambda_U) = \mathrm{R}\pi_* j_! \Lambda$ in $\mathbf{D}(\Lambda)$, where $\pi : X \to \star$.*

B.2. Functoriality. We now turn to functoriality of these constructions. Suppose we are given a morphism of pairs of topological spaces $f : (X, Z) \to (X', Z')$. We keep the notation from above, decorating the symbols with a prime when the objects are associated to the second pair.

Lemma B.4 *The following diagram commutes in* $\mathbf{D}(\Lambda)$:

$$\begin{tikzcd}
\mathrm{R}\pi'_* j'_! \Lambda \arrow[r] \arrow[d, "\beta^{-1}\alpha"', "\sim"] & \mathrm{R}\pi_* j_! \Lambda \arrow[d, "\beta^{-1}\alpha"', "\sim"] \\
\mathrm{Sg}(X', Z')^\vee \arrow[r, "\mathrm{Sg}(f)^\vee"'] & \mathrm{Sg}(X, Z)^\vee
\end{tikzcd}$$

Here the top horizontal arrow is defined as $\mathrm{R}\pi'_*$ *applied to*

$$j'_!\Lambda \xrightarrow{\text{adj}} \mathrm{R}f_* f^* j'_!\Lambda \to \mathrm{R}f_* j_! f^*\Lambda \xrightarrow{\sim} \mathrm{R}f_* j_!\Lambda.$$

Proof. We will construct the two middle horizontal arrows below, and then prove that they make each square in the following diagram commute:

$$\begin{tikzcd}
\mathrm{R}\pi'_* j'_! \Lambda \arrow[r] \arrow[d, equal] & \mathrm{R}\pi_* j_! \Lambda \arrow[d, equal] \\
\mathcal{S}_{X'} \otimes \Lambda_{U'}(X') \arrow[r, "(B.5)"] \arrow[d, "\alpha"', "\sim"] & \mathcal{S}_X \otimes \Lambda_U(X) \arrow[d, "\alpha"', "\sim"] \\
\mathcal{K}_{(X',Z')}(X') \arrow[r, "\mathcal{K}_f"] & \mathcal{K}_{(X,Z)}(X) \\
\mathrm{Sg}(X', Z')^\vee \arrow[r, "\mathrm{Sg}(f)^\vee"] \arrow[u, "\beta", "\sim"'] & \mathrm{Sg}(X, Z)^\vee \arrow[u, "\beta", "\sim"']
\end{tikzcd}$$

From the inclusion $f^{-1}(U') \subset U$ we obtain a canonical morphism of sheaves on X:

$$f^*\Lambda_{U'} \xrightarrow{\sim} \Lambda_{f^{-1}(U')} \to \Lambda_U.$$

Composition with f induces a morphism $\mathcal{S}_f : \mathcal{S}_{X'} \to f_*\mathcal{S}_X$ and thus by adjunction also $f^*\mathcal{S}_{X'} \to \mathcal{S}_X$. Together we obtain a morphism

$$f^*(\mathcal{S}_{X'} \otimes \Lambda_{U'}) \xrightarrow{\sim} f^*\mathcal{S}_{X'} \otimes f^*\Lambda_{U'} \to \mathcal{S}_X \otimes \Lambda_U. \tag{B.5}$$

Similarly, we define morphisms

$$f^*(\mathcal{S}_{X'} \otimes \Lambda_{Z'}) \xrightarrow{\sim} f^*\mathcal{S}_{X'} \otimes f^*\Lambda_{Z'} \to \mathcal{S}_X \otimes \Lambda_Z$$

and

$$f^* i'_*\mathcal{S}_{Z'} \to i_* f^*\mathcal{S}_{Z'} \to i_*\mathcal{S}_Z.$$

It is then clear that the following diagram commutes

$$\begin{tikzcd}
f^*(\mathcal{S}_{X'} \otimes \Lambda_{U'}) \arrow[r] \arrow[d] & \mathcal{S}_X \otimes \Lambda_U \arrow[d] \\
f^*\mathcal{S}_{X'} \arrow[r] \arrow[d] & \mathcal{S}_X \arrow[d] \\
f^*(\mathcal{S}_{X'} \otimes \Lambda_{Z'}) \arrow[r] \arrow[d] & \mathcal{S}_X \otimes \Lambda_Z \arrow[d] \\
f^* i'_* \mathcal{S}_{Z'} \arrow[r] & i_* \mathcal{S}_Z
\end{tikzcd}$$

so that, in particular, we deduce the existence of a morphism $f^*\mathcal{K}_{(X',Z')} \to \mathcal{K}_{(X,Z)}$ rendering the following two squares commutative:

$$\begin{tikzcd}
f^*\mathcal{K}_{(X',Z')} \arrow[r] \arrow[d] & \mathcal{K}_{(X,Z)} \arrow[d] \\
f^*\mathcal{S}_{X'} \arrow[r] & \mathcal{S}_X
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*\mathcal{K}_{(X',Z')} \arrow[r] & \mathcal{K}_{(X,Z)} \\
f^*(\mathcal{S}_{X'} \otimes \Lambda_{U'}) \arrow[r] \arrow[u, "\alpha"] & \mathcal{S}_X \otimes \Lambda_U \arrow[u, "\alpha"']
\end{tikzcd}$$

Denote by $\mathcal{K}_f : \mathcal{K}_{(X',Z')} \to f_*\mathcal{K}_{(X,Z)}$ the morphism obtained by adjunction. We now claim that also the following square of complexes commutes:

$$\begin{tikzcd}
\mathcal{K}_{(X',Z')}(X') \arrow[r, "\mathcal{K}_f"] & \mathcal{K}_{(X,Z)}(X) \\
\mathrm{Sg}(X',Z')^\vee \arrow[r, "\mathrm{Sg}(f)^\vee"'] \arrow[u, "\beta"] & \mathrm{Sg}(X,Z)^\vee \arrow[u, "\beta"']
\end{tikzcd}$$

Indeed, using the injection $\mathcal{K}_{(X,Z)}(X) \hookrightarrow \mathcal{S}_X(X)$ one reduces to prove commutativity of

$$\begin{tikzcd}
\mathcal{S}_{X'}(X') \arrow[r, "\mathcal{S}_f"] & \mathcal{S}_X(X) \\
\mathrm{Sg}(X')^\vee \arrow[r, "\mathrm{Sg}(f)^\vee"'] \arrow[u] & \mathrm{Sg}(X)^\vee \arrow[u]
\end{tikzcd}$$

which is clear.

Finally, notice that (B.5) is compatible with the coaugmentations $\Lambda \to \mathcal{S}_{X'}$ and $\Lambda \to \mathcal{S}_X$, thus the lemma. ☐

Lemma B.6 *The following defines a morphism of distinguished triangles in* $\mathbf{D}(\Lambda)$*:*

$$\begin{tikzcd}
\mathrm{R}\pi_* j_!\Lambda \arrow[r] & \mathrm{R}\pi_*\Lambda \arrow[r] & \mathrm{R}\pi_* i_*\Lambda \arrow[r] & \mathrm{R}\pi_* j_!\Lambda[-1] \\
\mathrm{Sg}(X,Z)^\vee \arrow[r] \arrow[u, "\sim", "\alpha^{-1}\beta"'] & \mathrm{Sg}(X)^\vee \arrow[r] \arrow[u, "\sim"'] & \mathrm{Sg}(Z)^\vee \arrow[r] \arrow[u, "\sim"] & \mathrm{Sg}(X,Z)^\vee[-1] \arrow[u, "\sim", "\alpha^{-1}\beta"']
\end{tikzcd} \tag{B.7}$$

Here, the first row is induced by the localization triangle while the second row is the distinguished triangle associated to the short exact sequence consisting of the first three terms.

Proof. It is clear that the first two squares commute. We only need to prove this for the third one.

Extend the first square in (B.1) to a morphism of triangles in $\mathbf{Cpl}(\mathbf{Sh}(X))$

$$\begin{tikzcd}
\mathcal{K}_{(X,Z)} \arrow[r, "a"] & \mathcal{S}_X \arrow[r] & \mathrm{cone}(a) \arrow[r] & \mathcal{K}_{(X,Z)}[-1] \\
\mathcal{S}_X \otimes \Lambda_U \arrow[u, "\alpha"] \arrow[r, "j"'] & \mathcal{S}_X \arrow[u, equal] \arrow[r] & \mathrm{cone}(j) \arrow[u] \arrow[r] & \mathcal{S}_X \otimes \Lambda_U[-1] \arrow[u, "\alpha"]
\end{tikzcd}$$

using the mapping cones. Since $\Gamma(X, \mathrm{cone}(a)) = \mathrm{cone}(a_X)$ the first square in (B.2) extends to a morphism of triangles in $\mathbf{Cpl}(\Lambda)$:

$$\begin{tikzcd}
\mathcal{K}_{(X,Z)}(X) \arrow[r, "a_X"] & \mathcal{S}_X(X) \arrow[r] & \mathrm{cone}(a)(X) \arrow[r] & \mathcal{K}_{(X,Z)}(X)[-1] \\
\mathrm{Sg}(X,Z)^\vee \arrow[u, "\beta"] \arrow[r, "b"'] & \mathrm{Sg}(X)^\vee \arrow[u] \arrow[r] & \mathrm{cone}(b) \arrow[u] \arrow[r] & \mathrm{Sg}(X,Z)^\vee[-1] \arrow[u, "\beta"]
\end{tikzcd}$$

Notice that under the canonical identification $\mathrm{cone}(b) \xrightarrow{\sim} \mathrm{Sg}(Z)^\vee$, the bottom row is precisely the bottom row of (B.7), while modulo the canonical identification $\mathrm{cone}(a)(X) \xrightarrow{\sim} \mathcal{S}_Z(Z)$, the top row of the first diagram induces the top row of (B.7) (taking global sections). Indeed, the latter contention follows from the fact that in $\mathbf{D}(\mathbf{Sh}(X))$ there is a *unique* morphism δ making the following candidate triangle distinguished:

$$j_!\Lambda \longrightarrow \Lambda \longrightarrow i_*\Lambda \xrightarrow{\delta} j_!\Lambda[-1].$$

The lemma now follows from the commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{S}_X \otimes \Lambda_Z(X) \arrow[d] & \mathrm{cone}(j)(X) \arrow[l] \arrow[d] \\
i_*\mathcal{S}_Z(X) & \mathrm{cone}(a)(X) \arrow[l] \\
\mathrm{Sg}(Z)^\vee \arrow[u] & \mathrm{cone}(b) \arrow[l] \arrow[u]
\end{tikzcd}$$

The first square commutes since the second square in (B.2) does, while the second square does since the second square in (B.1) does. ☐

B.3. Monoidality. We come to the last compatibility of the model, namely with the cup product. For this we fix a topological space X and two closed subspaces Z_1 and Z_2 of X. We write $Z = Z_1 \cup Z_2$, and we assume that there exist open neighborhoods V_i of Z_i in X such that V_i deformation retracts onto Z_i and $V_1 \cap V_2$ deformation retracts onto $Z_1 \cap Z_2$. This is satisfied e. g. if X is a CW-complex and the Z_i are subcomplexes. The cup product in cohomology is denoted by $\smile$, and $\mathrm{Sg}(X, Z_1 + Z_2)$ is the free Λ-module on simplices in X which are neither contained in Z_1 nor in Z_2.

Lemma B.8 *The following diagram commutes in* $\mathbf{D}(\Lambda)$*:*

$$\begin{tikzcd}
R\pi_* j_{1!}\Lambda \otimes^{L} R\pi_* j_{2!}\Lambda \arrow[rr, "\smile"] \arrow[d, "\beta^{-1}\alpha"', "\sim"] & & R\pi_* j_!\Lambda \arrow[d, "\beta^{-1}\alpha"', "\sim"] \\
\mathrm{Sg}(X, Z_1)^\vee \otimes \mathrm{Sg}(X, Z_2)^\vee \arrow[r, "\smile"'] & \mathrm{Sg}(X, Z_1 + Z_2)^\vee & \mathrm{Sg}(X, Z)^\vee \arrow[l, "\sim"]
\end{tikzcd}$$

Here the top horizontal arrow is defined as the composition

$$R\pi_* j_{1!}\Lambda \otimes^{L} R\pi_* j_{2!}\Lambda \to R\pi_*(j_{1!}\Lambda \otimes^{L} j_{2!}\Lambda) \xrightarrow{\sim} R\pi_* j_!\Lambda. \tag{B.9}$$

Proof. Notice that the composition

$$\mathrm{Sg}(X, V_i)^\vee \to \mathrm{Sg}(X, Z_i)^\vee \xrightarrow{\beta} \mathcal{K}_{(X,Z_i)}(X)$$

factors through $\alpha : \mathcal{S}_X \otimes \Lambda_{U_i}(X) \to \mathcal{K}_{(X,Z_i)}(X)$ because V_i is open in X and $\mathcal{S}_X \otimes \Lambda_{U_i}(X)$ consists of sections of $\mathcal{S}_X$ whose support is contained in U_i. It follows that the left vertical arrow in the lemma is equal to the composition of the left vertical arrows in the following diagram.

$$\begin{tikzcd}
(\mathcal{S}_X \otimes \Lambda_{U_1})(X) \otimes^{L} (\mathcal{S}_X \otimes \Lambda_{U_2})(X) \arrow[rr, "\smile"] & & (\mathcal{S}_X \otimes \Lambda_U)(X) \arrow[d, "\sim"', "\alpha"] \\
\mathrm{Sg}(X, V_1)^\vee \otimes \mathrm{Sg}(X, V_2)^\vee \arrow[u, "\sim"] \arrow[r, "\smile"] \arrow[d, "\sim"'] & \mathrm{Sg}(X, V_1 + V_2)^\vee \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathcal{K}_{(X,Z)}(X) \\
\mathrm{Sg}(X, Z_1)^\vee \otimes \mathrm{Sg}(X, Z_2)^\vee \arrow[r, "\smile"'] & \mathrm{Sg}(X, Z_1 + Z_2)^\vee & \mathrm{Sg}(X, Z)^\vee \arrow[l, "\sim"] \arrow[u, "\sim", "\beta"']
\end{tikzcd} \tag{B.10}$$

Recall that the sheaf $\mathcal{S}_Z$ is the quotient of the presheaf $V \mapsto \mathrm{Sg}(V)^\vee$ where a section $f \in \mathrm{Sg}(V)^\vee$ becomes 0 in $\mathcal{S}_Z(V)$ if there exists an open cover $(W_i)_i$ of V such that $f|_{W_i} = 0$ for all i. Now, start with $f \in \mathrm{Sg}(X)^\vee$ vanishing on both V_1 and V_2, i. e. an element of $\mathrm{Sg}(X, V_1 + V_2)^\vee$. These two open subsets of X cover Z, and by the description of $\mathcal{S}_Z$ just given, we see that f defines the zero class in $i_*\mathcal{S}_Z(X)$ hence lands in $\mathcal{K}_{(X,Z)}(X)$. This yields the right horizontal arrow in the middle row. It follows that the upper half of the diagram commutes. Evidently the lower left square does as well. For the lower right square denote by V the union of V_1 and V_2. Then we may decompose this square as follows:

$$\begin{tikzcd}
\mathrm{Sg}(X, V_1 + V_2)^\vee \arrow[rr, bend left, "\sim"] \arrow[d, "\sim"'] & \mathrm{Sg}(X, V)^\vee \arrow[l, "\sim"'] \arrow[d] & \mathcal{K}_{(X,Z)}(X) \\
\mathrm{Sg}(X, Z_1 + Z_2)^\vee & \mathrm{Sg}(X, Z)^\vee \arrow[l, "\sim"] \arrow[ur, "\sim", "\beta"']
\end{tikzcd}$$

Commutativity is now clear.

It remains to prove that the top horizontal arrow in (B.10) is a model for (B.9). This follows from the fact that the resolution $\Lambda \xrightarrow{\sim} \mathcal{S}_X$ of the constant sheaf on X is *multiplicative*. Namely, this makes the right square of the following diagram commutative; the left one clearly commutes.

$$\begin{tikzcd}
\mathrm{R}\pi_*\Lambda_{U_1} \otimes^{\mathrm{L}} \mathrm{R}\pi_*\Lambda_{U_2} \arrow[r] \arrow[d, equal] & \mathrm{R}\pi_*(\Lambda_{U_1} \otimes \Lambda_{U_2}) \arrow[r, "\sim"] \arrow[d, equal] & \mathrm{R}\pi_*\Lambda_U \arrow[d, equal] \\
\pi_*(\mathcal{S}_X \otimes \Lambda_{U_1}) \otimes^{\mathrm{L}} \pi_*(\mathcal{S}_X \otimes \Lambda_{U_2}) \arrow[r] & \pi_*(\mathcal{S}_X \otimes \Lambda_{U_1} \otimes \mathcal{S}_X \otimes \Lambda_{U_2}) \arrow[r, "\sim"'] & \pi_*(\mathcal{S}_X \otimes \Lambda_U)
\end{tikzcd}$$

□

C. Comodule categories

In this section we recall some facts about categories of (complexes of) comodules used in the main body of the text. Throughout we fix a ring Λ and a *flat* Λ-coalgebra C. By a C-comodule we mean a counitary left C-comodule. $\mathbf{coMod}(C)$ (respectively, $\mathbf{coMod}^{\mathrm{f}}(C)$) denotes the category of C-comodules (respectively, C-comodules finitely generated as Λ-modules).

The starting point is really the following result.

Fact C.1

(1) $\mathbf{coMod}^{\mathrm{f}}(C)$ *and* $\mathbf{coMod}(C)$ *are abelian Λ-linear categories, and there is a canonical equivalence of abelian Λ-linear categories* $\mathbf{coMod}^{\mathrm{f}}(C)_{\oplus} \simeq \mathbf{coMod}(C)$.

(2) The forgetful functor $o : \mathbf{coMod}(C) \to \mathbf{Mod}(\Lambda)$ *is exact Λ-linear and creates colimits and finite limits.*

(3) $\mathbf{coMod}(C)$ *is a Grothendieck category, copowered over* $\mathbf{Mod}(\Lambda)$. *In particular, it is bicomplete.*

Proof. The first statement follows from [64, II, 2.0.6 and 2.2.3]. The rest is proved in [69], see [69, Cor. 3 and 9, Pro. 38, Cor. 26]. Explicitly, the copower of a Λ-module m and a C-comodule c is given by the tensor product (as Λ-modules) $m \otimes c$ with the comodule coaction on c. □

Next, we are interested in different models for the derived category of $\mathbf{coMod}(C)$. The following result is true more generally for any Grothendieck category.

Fact C.2 ([15, Thm. 1.2]) $\mathbf{Cpl}(\mathbf{coMod}(C))$ *is a proper cellular model category with quasi-isomorphisms as weak equivalences and monomorphisms as cofibrations.*

The model structure in the statement is called the *injective model structure*.

From now on assume that C is a (commutative) bialgebra. $\mathbf{coMod}(C)$ then becomes a monoidal Λ-linear category with C coacting on the tensor product (as Λ-modules) $c \otimes d$ by

$$c \otimes d \xrightarrow{\mathrm{ca}\otimes\mathrm{ca}} (c \otimes C) \otimes (d \otimes C) \xrightarrow{\sim} (c \otimes d) \otimes (C \otimes C) \to (c \otimes d) \otimes C,$$

the last arrow being induced by the multiplication of C. In particular, the forgetful functor $o : \mathbf{coMod}(C) \to \mathbf{Mod}(\Lambda)$ is monoidal. The category $\mathbf{Cpl}(\mathbf{coMod}(C))$ inherits a monoidal structure in the usual way.

Proposition C.3 *Let T be a flat object in* $\mathbf{Cpl}(\mathbf{coMod}(C))$. *Then there is a proper cellular model structure on* $\mathbf{Spt}^{\Sigma}_{T}\,\mathbf{Cpl}(\mathbf{coMod}(C))$ *with stable equivalences as weak equivalences and monomorphisms as cofibrations.*

Proof. The stable equivalences are described in [37, Def. 8.7], and the proof in [15, Pro. 6.31] applies. □

The model structure of the proposition is called the *injective stable model structure*.

Unfortunately, the monoidal structure does not, in general, interact well with the injective model structures. In the cases of interest in the main body of the text (namely, when Λ is a principal ideal domain) we have the following result, essentially due to Serre.

Lemma C.4 *Let Λ be a Dedekind domain. In* $\mathbf{Cpl}(\mathbf{coMod}(C))$ *there exist functorial flat resolutions. In particular,* $\mathbf{D}(\mathbf{coMod}(C))$ *admits naturally a monoidal structure.*

Proof. We follow [66, Pro. 3]. Let E be a comodule and consider the morphism of comodules $\mathrm{ca}_E : E \to C \otimes E$ given by the coaction of C on E, where the target has a comodule structure induced by the comultiplication on C (sometimes called the "extended comodule associated to E"). In fact, the coaction $\mathrm{ca}_\bullet$ defines a natural transformation from the identity functor on $\mathbf{coMod}(C)$ to the "extended comodule"-functor (this is the unit of an adjunction whose left adjoint is the forgetful functor $o : \mathbf{coMod}(C) \to \mathbf{Mod}(\Lambda)$; cf. [5, Lem. 1.53]). Since E is counitary, this natural transformation is objectwise injective. Let $F : \mathbf{Mod}(\Lambda) \to \mathbf{Mod}(\Lambda)$ be the functor which associates to a Λ-module M the free Λ-module $\oplus_{m\in M}\Lambda$. It comes with a natural transformation $\eta : F \to \mathrm{Id}$ which is objectwise an epimorphism. We obtain a diagram

$$\begin{tikzcd} E \arrow[r, "\mathrm{ca}_E"] & C \otimes E \\ & C \otimes F(E) \arrow[u, "1\otimes\eta_E"'] \end{tikzcd}$$

in the category of C-comodules (the module in the bottom row is again an extended comodule). Since the forgetful functor from $\mathbf{coMod}(C)$ to $\mathbf{Mod}(\Lambda)$ commutes with finite limits, we see that the pullback of this diagram is a comodule E' which both maps surjectively onto E, and embeds into $C \otimes F(E)$. By assumption, $C \otimes F(E)$ is torsion-free thus so is E'. It is clear from the construction that the association $E \mapsto E'$ defines a functor together with a natural transformation η' from it to the identity functor.

Using that $\mathbf{coMod}(C)$ is a Grothendieck category, the usual procedure leads to functorial flat resolutions. □

If Λ is a field then we can do better.

Lemma C.5 *Let Λ be a field. The injective model structure on* $\mathbf{Cpl}(\mathbf{coMod}(C))$ *is a monoidal model structure.*

Proof. Indeed, since the forgetful functor is monoidal, exact and creates colimits, the conditions for the injective model structure to be monoidal can be checked in $\mathbf{Cpl}(\Lambda)$. □

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