∞-categories: a first course

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Abstract

These are lecture notes for a course delivered through the Taught Courses Center (Bath, Bristol, Imperial, Oxford, Warwick) in Fall 2023. The goal is to introduce the language of ∞-categories (as quasi-categories), hopefully enabling the audience to tackle more advanced texts and read some of the literature employing this language. Many proofs are omitted, some of which are difficult and would require developing substantial extra material. On the other hand, this omission allows us to talk about a number of subjects, including constructing ∞-categories, (co)limits and presentability. We end with spectra and the K-theory of stable ∞-categories.

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Contents

Preface

Goals I would have liked to call this course *∞-categories for the working mathematician*. However, Mac Lane's book (titled *Categories for the working mathematician*) not only treats an impressive amount of category theory but does so with essentially full details on proofs. I will not be able to do that here but the spirit of the title hopefully remains. By the end you should be able to pick up a random article using ∞-categories and read along without getting too distracted by the jargon. And if you want or need to, having such a 'light' first exposure hopefully makes it easier to tackle the foundational texts on the subject. The main references are Lurie's *Higher Topos Theory* and *Higher Algebra* (with further developments in *Spectral Algebraic Geometry* and the Kerodon project). An alternative perspective on much of HTT is Cisinski's *Higher Categories and Homotopical Algebra*, while Land's *Introduction to ∞-categories* is a more introductory but still formidable book.

Acknowledgment Given the scope of this document I hope I will be excused for the lack of attributions in the main body of the text. Let me do this here in brief instead. Most of the material covered is due to Joyal and Lurie and I've learnt it from the latter's books. In addition to the references mentioned above, there are several other short accounts, some with a similar goal as these lectures. I've been influenced in my exposition by Groth's *A short course on ∞-categories* and lectures on *Higher Algebra* by Krause and Nikolaus. It goes without saying that I claim no originality for any of the material below.

Prerequisites The only strict prerequisite is a solid foundation in basic (1-)category theory. Some familiarity with simplicial sets (simplicial homotopy theory) is an advantage but I will introduce everything as we go along. As usual, examples will be more useful to those that have seen these before in other contexts. So, later on in the course, when we construct the ∞-category of spectra, you will probably be less impressed if you haven't encountered their 'classical' counterparts. I will try to account for that with some commentary.

Exercises There are exercises interspersed in the text and I *highly* recommend that you do them. Given that relatively few proofs will be given in the course, these exercises are some of my few invitations to you to see what's going on 'behind the curtains'.

1 Definition

The goal of this section is to 'explain' the following definition: what it means and why it has at least some plausibility of capturing what one would like it to. Of course, the rest of this course is about turning plausibility into something stronger. (I'll let you be the judge.)

Definition 1.1. An ∞-*category* is a simplicial set C such that each inner horn admits a filler. That is, for all $0 < i < n$:

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\forall"] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

We will start by explaining what the terms mean. Feel free to skip the following subsection if you can parse Definition 1.1.

Commentary 1.2. If you think about the definition of an ordinary 1-category (objects, morphisms, composition, associativity, ...) you might be struck by the simplicity of Definition 1.1. Shouldn't the latter be infinitely more long and complicated? The reason it isn't is that the *structure* of composition is turned into a *property* instead. See Commentary 1.23 for elaboration.

1.1 Simplicial sets

Let $\mathbf{\Delta}$ denote the category of finite ordinals $[n] = (0 < \cdots < n)$, $n \geq 0$, and (weakly) order-preserving maps[1]. It is called the *simplex category*. Accordingly:

Definition 1.3. A *simplicial set* is a presheaf on $\mathbf{\Delta}$. In other words, $\mathsf{sSet} := \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathsf{Set})$.

Remark 1.4. The image of $[n]$ under the Yoneda embedding $\mathbf{\Delta} \hookrightarrow \mathsf{sSet}$ will be denoted by Δ^n. The Yoneda lemma says that for any simplicial set C, we have a natural bijection

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, C) \cong C_n$$

that we will use throughout without explicit mention. This is the set of *n-simplices* of C. For example, the n-simplices of Δ^m are the order-preserving maps $[n] \to [m]$.

Remark 1.5. In other words, a simplicial set C consists of n-simplices C_n of arbitrary dimensions $n \geq 0$ together with maps $C(\alpha)\colon C_m \to C_n$ for every order-preserving $\alpha\colon [n] \to [m]$ (suitably functorially). Each such map is the composite of two special kinds of maps:

- the *face* maps $d_i := d_i^{(n)}\colon C_n \to C_{n-1}$ for $0 \leq i \leq n$, induced by $d^i\colon [n-1] \to [n]$ that is injective and only misses i; and

[1] Explicitly, $\alpha : [n] \to [m]$ is order-preserving if $\alpha(i) \leq \alpha(j)$ whenever $i \leq j$.

- the *degeneracy* maps $s_i := s_i^{(n)} \colon C_n \to C_{n+1}$ for $0 \leq i \leq n$, induced by $s^i \colon [n+1] \to [n]$ which is surjective and hits i twice.

Therefore, C is uniquely determined by the sets C_m and the face and degeneracy maps that need to satisfy the *simplicial identities*:

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i & i &< j \\
d_i s_j &= s_{j-1} d_i & i &< j \\
d_j s_j &= 1 = d_{j+1} s_j & & \\
d_i s_j &= s_j d_{i-1} & i &> j+1 \\
s_i s_j &= s_{j+1} s_i & i &\leq j
\end{aligned}$$

Accordingly, you might see a simplicial set sometimes depicted as follows:

$$\begin{tikzcd}
\cdots \arrow[r, shift left=6] \arrow[r, leftarrow, shift left=4] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[r, leftarrow, shift right=4] \arrow[r, shift right=6] & C_2 \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] & C_1 \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & C_0
\end{tikzcd}$$

where the arrows are the face (going right) and degeneracy (going left) maps.

Convention 1.6. An n-simplex in C is called *degenerate* if it is in the image of one of the $s_i \colon C_{n-1} \to C_n$. Otherwise it is called *non-degenerate*.

Example 1.7.

1. For each $m \geq 0$, there is a unique order-preserving map $[m] \to [0]$. That is, the simplicial set Δ^0 is constant of value a point. It is the terminal object in sSet. It has a unique non-degenerate simplex, namely the identity map $\mathrm{id}_0 \colon [0] \to [0]$ in dimension 0.
2. More generally, the simplicial set Δ^n has non-degenerate simplices in dimensions $0, \ldots, n$ only. Indeed, for $m > n$ any order-preserving map $[m] \to [n]$ must hit an element of $[n]$ twice. The only non-degenerate n-simplex of Δ^n is the identity map $\mathrm{id}_n \colon [n] \to [n]$. (In general, the non-degenerate simplices of Δ^n are precisely the injective maps.)

Commentary 1.8. Geometrically, you can think of a simplicial set as a bunch of non-degenerate simplices, with the face maps telling you how to glue them together. The underpinning of that is given by the *geometric realization* of a simplicial set C, namely the topological space

$$|C| := \left(\coprod_{n \geq 0} C_n \times |\Delta^n| \right) / {\sim},$$

where each C_n is given the discrete topology, $|\Delta^n| = \{(t_0, \ldots, t_n) \mid 0 \leq t_i \leq 1, \sum t_i = 1\} \subseteq \mathbb{R}^{n+1}$ denotes the standard n-simplex, and the equivalence relation is generated by:

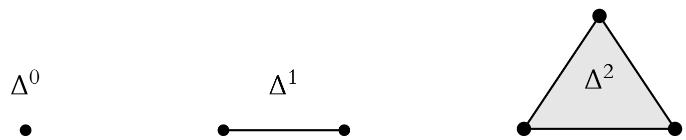
(i) the ith face of $\{x\} \times |\Delta^n|$ is identified with $\{d_i x\} \times |\Delta^{n-1}|$ (by the linear homeomorphism preserving the order of the vertices);

(ii) $\{s_i x\} \times |\Delta^n|$ is collapsed onto $\{x\} \times |\Delta^{n-1}|$ via the linear projection parallel to the line connecting the ith and $(i+1)$st vertex.

You can convince yourself that $|C|$ is the (set-theoretic disjoint) union of the interiors of standard simplices, one for each non-degenerate simplex of C. In particular, it is a CW-complex.

Warning 1.9. A general simplicial set is not determined by its geometric realization. We will see later that a special class of simplicial sets (Kan complexes) *are*, at least up to homotopy equivalence.

Example 1.10. We often picture some basic simplicial sets in a form akin to their geometric realization, for example:



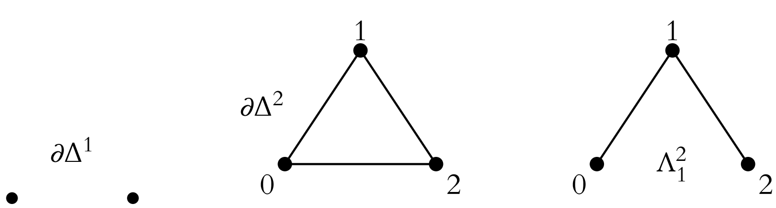
Convention 1.11. The intersection of simplicial sets is again a simplicial set hence it makes sense to speak of the simplicial subset generated by a family of simplices:

1. Given $0 \leq i \leq n$ we let $\partial_i \Delta^n \subset \Delta^n$ denote the sub-simplicial set generated by the ith face
$$d_i(\mathrm{id}_n : [n] \to [n]) = d^i : [n-1] \to [n].$$
2. Let $\partial \Delta^n \subset \Delta^n$ denote the sub-simplicial set generated by all its faces $\partial_i \Delta^n$. We set $\partial \Delta^0 = \emptyset$, the constant simplicial set with value the empty set. It is the initial object in sSet.
3. The *ith horn* $\Lambda^n_i \subseteq \Delta^n$ is generated by all faces except the ith one. It is called *inner* if $0 < i < n$ and *outer* else.

Exercise 1.12. Show that

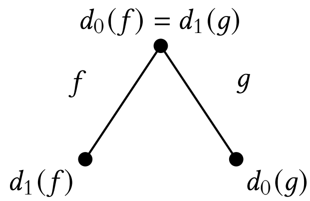
1. $\partial \Delta^n = \cup_{i=0}^n \partial_i \Delta^n$ as simplicial subsets of Δ^n.
2. $\partial \Delta^n \subset \Delta^n$ is the sub-simplicial set of non-surjective maps.
3. $\Lambda^n_i \subset \Delta^n$ is the sub-simplicial set of maps whose image doesn't cover $[n] \backslash \{i\}$.

Example 1.13. If you're not yet convinced that these definitions capture what they should please take a moment to verify that the following depict the non-degenerate simplices each time:



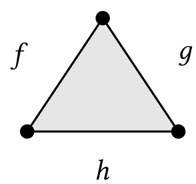
where i denotes the 0-simplex $i : [0] \to [2]$. The last picture should also explain why these are called horns.

Commentary 1.14. Let us stay with this last picture for a moment. A map of simplicial sets $\Lambda^2_1 \to C$ consists of two 1-simplices f, g in C such that $d_0(f) = d_1(g)$:



We also write this as $(g, \bullet, f)\colon \Lambda^2_1 \to C$.

An extension of $(g, \bullet, f)\colon \Lambda^2_1 \to C$ to $\Delta^2 \to C$ consists of a 2-simplex σ in C:



with $d_2(\sigma) = f$, $d_0(\sigma) = g$. Again, it should now be clear why this is called a *filler* in Definition 1.1.

Exercise 1.15. Show directly from the definition that Δ^n is an ∞-category for all $n \geq 0$.

1.2 Categories as ∞-categories

At this point, all the terms in Definition 1.1 have been explained. But why is that a sensible definition? Categories, be it 1-categories, ∞-categories or other variants, are fundamentally about morphisms and ways of *composing* these. And composition is something that turns (finite, ordered) sequences of morphisms into other sequences of morphisms. It should therefore not come as a surprise that the simplex category Δ and simplicial sets could be useful in describing such structures. In this subsection and the next we'll see two fundamental examples of ∞-categories that should give you some intuition about how this structure is encoded. Along the way, some terminology will be introduced to aid in this.

Construction 1.16. Let C be an ordinary category. We define its *nerve* $N(C)$ as the following simplicial set. The n-simplices are paths of length n in C:

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} c_n \tag{1.17}$$

The face map d_i takes this to

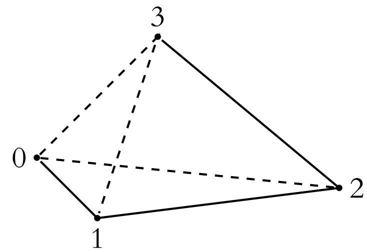
$$c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} c_{i-1} \xrightarrow{f_{i+1} \circ f_i} c_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} c_n$$

if $0 < i < n$ and discards f_0 (resp. f_n) if $i = 0$ (resp. $i = n$). The degeneracy map s_i takes it to

$$c_0 \xrightarrow{f_1} \cdots \xrightarrow{f_i} c_i \xrightarrow{\mathrm{id}_{c_i}} c_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} c_n.$$

Lemma 1.18. *Every inner horn $\Lambda^n_i \to N(C)$ admits a* unique *filler $\Delta^n \to N(C)$. In particular, the nerve $N(C)$ is an ∞-category.*

Proof. (Note that since $0 < i < n$, n must be at least 2.) Let $\mathrm{Sp}^n \subseteq \Delta^n$ be the sub-simplicial set generated by the adjacent edges $\Delta^{\{j<j+1\}}$, $0 \leq j < n$. (It is called the *spine*.) So, for example, $\mathrm{Sp}^2 = \Lambda^2_1$ and Sp^3 is pictured below in solid.



The first point to observe is that $\mathrm{Sp}^n \subseteq \Lambda^n_i$ for all $0 < i < n$. Indeed, for $n = 2$ we just saw that they are equal, and for $n \geq 3$, Λ^n_i contains all edges. Also, note that the restriction of $f\colon \Lambda^n_i \to N(C)$ to Sp^n describes nothing but a path of length n as in (1.17), that we may therefore identify with an n-simplex $\bar{f}\colon \Delta^n \to N(C)$. In particular, if $\bar{f}|_{\Lambda^n_i} = f$ then it must be the unique such n-simplex. It remains to check this identity.

If $n = 2$ there is nothing to prove so let us assume $n \geq 3$. It suffices to show $\bar{f}|_{\partial_j\Delta^n} = f|_{\partial_j\Delta^n}$ for all $j \neq i$, and by the observation above we may further restrict to $\mathrm{Sp}^{n-1} \subseteq \Delta^{n-1} \cong \partial_j\Delta^n$ to check the identity. You might think that $g|_{\partial_j\Delta^n}|_{\mathrm{Sp}^{n-1}}$ is just a restriction of $g|_{\mathrm{Sp}^n}$ (and therefore we're done), and that's almost true. The only problem is the jth edge in Sp^{n-1} which is really the restriction $g|_{\Delta^{j-1<j+1}}$. (This problem does not occur if $j = 0$ or $j = n$.) But $\bar{f}|_{\Delta^{j-1<j+1}} = f_{j+1} \circ f_j$. We have $\Delta^{j-1<j<j+1} \subseteq \Lambda^n_i$ (use Exercise 1.12.3) hence we see that also $f|_{\Delta^{j-1<j+1}} = f_{j+1} \circ f_j$, concluding the proof. □

Remark 1.19. The existence of fillers in $N(C)$ really uses that we start with an *inner* horn. (Take a moment to check where this was used in the proof above.) For example, let us be given a map $\Lambda^2_2 \to N(C)$ which we may identify with two morphisms $f_0\colon c_0 \to c_2$, $f_1\colon c_1 \to c_2$ with the same target. A filler $\Delta^2 \to N(C)$ of that outer horn would amount to a morphism $g\colon c_0 \to c_1$ such that $f_1 \circ g = f_0$. It is clear that this does not exist for general 1-categories C.

I recommend that you prove the following proposition. It isn't hard. (Of course, Lemma 1.18 already gives one part of the statement.)

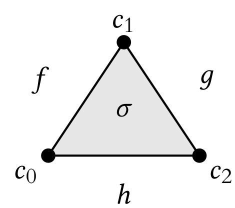
Proposition 1.20. *The nerve functor $N\colon \mathsf{Cat} \to \mathsf{sSet}$ is fully faithful. The essential image consists of those simplicial sets that admit unique fillers for inner horns.*

Remark 1.21. In particular, the nerve of a 1-category is a complete invariant. From now on we will not always distinguish between a 1-category and its nerve.

Having seen how a category is encoded in its nerve we generalize some categorical notions to arbitrary simplicial sets.

Definition 1.22. Let C be a simplicial set, for example an ∞-category.

1. C_0 is the set of *objects* of C. We sometimes write $c \in C$ instead of $c \in C_0$.
2. C_1 is the set of *morphisms* in C. Given a morphism $f \in C_1$ we write $f\colon c_0 \to c_1$ if $c_0 = d_1(f), c_1 = d_0(f) \in C_0$. These are called *source* and *target* of f.
3. Given an object $c \in C_0$, the *identity* of c is the morphism $\mathrm{id}_c = s_0(c) \in C_1$.
4. Let $\sigma \in C_2$ be a 2-simplex

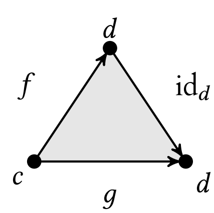


with $d_2(\sigma) = f$, $d_0(\sigma) = g$, $d_1(\sigma) = h$. In this situation we say that h is a *composite* of f and g and we write $h \simeq g \circ f$.

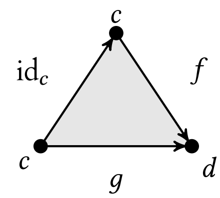
Commentary 1.23. In view of Commentary 1.14 we can then say that every *composable pair* $(g, \bullet, f)\colon \Lambda^2_1 \to C$ in an ∞-category C admits a composite $h \simeq g \circ f$. For the nerve of a 1-category this composite is unique but this need not be so for other ∞-categories. However, we will see later that the 'space of composites' is contractible and this is the right infinite-dimensional analogue of uniqueness. See Corollary 2.15.

Construction 1.24. Recall the geometric realization from Remark 1.34. There is another 'realization' of a simplicial set, in Cat, the 1-category of (small) 1-categories. This is called the *homotopy category* $\mathrm{h}\colon \mathsf{sSet} \to \mathsf{Cat}$. It sends $[n]$ to the totally ordered set $\{0 < 1 < \cdots < n\}$ viewed as a category (and sometimes still denoted $[n]$), with the obvious functoriality, and it preserves colimits. It is easy to see that it is left adjoint to the nerve functor $N\colon \mathsf{Cat} \to \mathsf{sSet}$ from Construction 1.16. In the case of an ∞-category C we can describe $\mathrm{h}(C)$ more explicitly.

Definition 1.25. Let C be an ∞-category. Two morphisms $f, g\colon c \to d$ are *homotopic* or *equivalent* if there exists a 2-simplex like so:



This is equivalent to the existence of a 2-simplex like so:

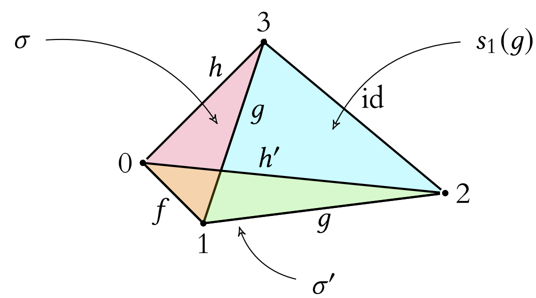


This turns out to be an equivalence relation. (Check for yourself!) We write $f \simeq g$.

Remark 1.26. At this point you should ask how the notions we have introduced before behave with respect to homotopy. The interesting one is that of a composite. For example, we can show that any two composites are homotopic. Indeed, let $(g, \bullet, f)$ be a composable pair and σ, σ' two 2-simplices as in Definition 1.22. Let $h = d_1(\sigma)$ and $h' = d_1(\sigma')$ so that $h \simeq g \circ f$ as well as $h' \simeq g \circ f$. We now define a horn

$$(s_1(g), \bullet, \sigma, \sigma') \colon \Lambda^3_1 \to C,$$

or in pictures:



(You can check that these faces indeed glue together, that is, coincide on the common intersections.) Since C is an ∞-category we obtain a filler $\mu \colon \Delta^3 \to C$ whose first face $\partial_1(\mu)$ defines a homotopy between h and h'.

Similarly, one shows that if $f \simeq f'$ then $g \circ f \simeq g \circ f'$ etc. Many of these arguments are quite similar and we trust that you can easily fill them in. (Obviously, you should do it a few times until you feel confident.)

Proposition 1.27. *Let C be an ∞-category. Then the homotopy category* $\mathrm{h}(C)$ *can be described as follows:*

1. *Its objects are those of C and morphisms are equivalence classes of morphisms in C, written $[f]$.*
2. *The composition and identities are given by*

$$[g] \circ [f] = [g \circ f], \qquad \mathrm{id}_c = [\mathrm{id}_c].$$

Exercise 1.28. Check that this really does define a category.

Example 1.29. It follows from Proposition 1.20 that the counit of the adjunction

$$\mathrm{h}(N(C)) \xrightarrow{\sim} C$$

is an equivalence for any category C. From this (or directly) you see that if $f \simeq g$ in $N(C)$ then $f = g$.

Commentary 1.30. Let us temporarily denote the category (Exercise 1.28) described in Proposition 1.27 by $\mathrm{h}'(C)$. If you think about what a map of simplicial sets $C \to N(D)$ (for some given ordinary category D) amounts to, it should be quite plausible that this is the same as defining a functor $\mathrm{h}'(C) \to D$. And after all, that's what Proposition 1.27 is essentially saying. The proof is straightforward.

Definition 1.31. Let C be an ∞-category, and $f\colon c \to d$ a morphism. Then f is an *isomorphism* if there exists $g\colon d \to c$ such that $g \circ f \simeq \mathrm{id}_c$, $f \circ g \simeq \mathrm{id}_d$.

Note that for the nerve of a category this recovers the notion of an isomorphism. More generally, we have the following easy result.

Exercise 1.32. Let C be an ∞-category, $f\colon c \to d$. The following are equivalent:

1. f is an isomorphism in C.
2. $[f]$ is an isomorphism in $\mathrm{h}(C)$.

Commentary 1.33. An ∞-category C is an ∞-*groupoid* if all its morphisms are isomorphisms. By Exercise 1.32, this is equivalent to $\mathrm{h}(C)$ being a groupoid.

1.3 Spaces as ∞-categories

We now turn to the second fundamental example.

Remark 1.34. For formal reasons, the geometric realization of Commentary 1.8 has a right adjoint $\mathrm{Sing}\colon \mathsf{Top} \to \mathsf{sSet}$ given by

$$\mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathsf{Top}}(|\Delta^n|, X)$$

and face and degeneracies induced by the corresponding maps on the topological standard simplices. The simplicial set $\mathrm{Sing}(X)$ is called the *singular simplicial complex* of X. If you know about singular homology, this will look familiar.

Lemma 1.35. *Let $X \in \mathsf{Top}$ be a topological space. Every horn (inner or outer) $\Lambda^n_i \to \mathrm{Sing}(X)$ admits a filler. In particular, $\mathrm{Sing}(X)$ is an ∞-category.*

Proof. Using the adjunction of Remark 1.34 our extension problem translates into one about topological spaces:

$$\begin{tikzcd} {|\Lambda^n_i|} \arrow[r] \arrow[d, hook] & X \\ {|\Delta^n|} \arrow[ur, dotted] & \end{tikzcd}$$

This problem is solvable because $|\Lambda^n_i|$ is a retract of $|\Delta^n|$. □

Exercise 1.36. Let X be a topological space. Show that $\mathrm{h}(\mathrm{Sing}(X)) = \Pi_{\leq 1}(X)$ is the *fundamental groupoid* of X.

Definition 1.37. A *Kan complex* is a simplicial set C for which every horn $\Lambda^n_i \to C$, $0 \leq i \leq n$, admits a filler $\Delta^n \to C$.

Hence we have shown that $\mathrm{Sing}(X)$ is a Kan complex. Just as for ordinary categories and their nerves we would now like to say that topological spaces are subsumed by the theory of Kan complexes. This turns out to be true, but in a somewhat more complicated sense. To express it we need to recall the notion of weak homotopy equivalences.

Commentary 1.38. Let (X, x) be a pointed topological space. For $n \geq 1$ we consider pointed maps from the n-ball $(D^n, \partial D^n) \to (X, x)$, that is, continuous maps $f\colon D^n \to X$ such that $f(\partial D^n) = x$. Recall that two pointed maps f, g are *homotopic* if they can be continuously deformed into one another. That is, if there exists a pointed map $H\colon (D^n \times [0,1], \partial D^n \times [0,1]) \to (X, x)$ such that $H|_0 = f$, $H|_1 = g$. This is an equivalence relation and the equivalence classes of pointed maps are denoted $\pi_n(X, x)$. For $n = 1$ one obtains the fundamental group. For $n > 1$ these are the higher homotopy groups of (X, x). (The group structure on these generalizes the composition of paths in the case $n = 1$.) One also extends the notion to $n = 0$ by setting $\pi_0(X)$ the set of path-connected components of X.

A map of topological spaces $f\colon X \to Y$ is a *weak homotopy equivalence* if it induces a bijection $\pi_0(X) \to \pi_0(Y)$ and an isomorphism $\pi_n(X, x) \to \pi_n(Y, f(x))$ for every choice of base point $x \in X$. Whitehead's theorem implies that if X and Y are CW-complexes then a weak homotopy equivalence $f\colon X \to Y$ is automatically a homotopy equivalence. (That is, there exists a *homotopy inverse* $g\colon Y \to X$ and homotopies $g \circ f \simeq \mathrm{id}$, $g \circ f \simeq \mathrm{id}$.) Of course, the converse is true for every topological space.

Exercise 1.39. Repeat the definitions of Commentary 1.38 for Kan complexes, by replacing Δ^n for D^n. These are called *simplicial homotopy groups*. (The group structure is probably non-obvious but we won't seriously need it in the sequel.)

Note that for every pointed space (X, x), there is a natural isomorphism $\pi_n(\mathrm{Sing}(X), x) \cong \pi_n(X, x)$ (resp. a bijection $\pi_0(\mathrm{Sing}(X)) \cong \pi_0(X)$).

Commentary 1.40. In fact, the counit of the adjunction $|\mathrm{Sing}(X)| \to X$ is a weak homotopy equivalence for all topological spaces X, as is the unit $Y \to \mathrm{Sing}(|Y|)$ for all Kan complexes Y. (You should be able to prove this directly for π_0 but the higher homotopy groups require some more work.) These two facts are at the heart of an equivalence (due to Quillen) between topological spaces and Kan complexes at the level of homotopy theory (more precisely a Quillen equivalence of model categories).

The upshot for us will be: As long as we care about topological spaces up to weak homotopy equivalence, we may think of them as completely subsumed by the ∞-categorical world (see also Remark 2.12). Starting from the next section, spaces for us will therefore mean Kan complexes.

Commentary 1.41. (Feel free to skip this.) Given a topological space X, its *fundamental ∞-groupoid* $\Pi_{<\infty}(X)$ is thought to be an infinite-dimensional object encoding the homotopy theory of X, that is, the homotopy type of X. Intuitively, one might think of it as having objects the points of X, morphisms the paths in X, 2-morphisms the homotopies between paths, etc. This does not apriori define a simplicial set nor, a fortiori, an ∞-category in our sense. In line with the previous commentary, for us the singular simplicial set $\mathrm{Sing}(X)$ is a good way of making this intuition precise. It is an actual Kan complex and records exactly the kind of information one would like. For example, its objects and morphisms are indeed precisely the points of X and paths between them. (See Exercise 1.36.) If $H\colon [0,1]^2 \to X$ is a homotopy between two paths $H_0, H_1\colon x \to y$ one may subdivide the square $[0,1]^2$ along the diagonal into two copies of $|\Delta^2|$ to obtain 2-simplices in $\mathrm{Sing}(X)$. Conversely, if $\Delta^2 \to \mathrm{Sing}(X)$ provides a homotopy between two morphisms $f, g\colon \Delta^1 \to \mathrm{Sing}(X)$ then an appropriate projection $[0,1]^2 \to |\Delta^2|$ yields a homotopy between f and g. Similar techniques can be used in higher dimensions, justifying the view that $\mathrm{Sing}(X)$ is a good stand-in for $\Pi_{<\infty}(X)$.

Proposition 1.42. *For $C \in \mathsf{sSet}$ the following are equivalent:*

1. *C is a Kan complex.*
2. *C is an ∞-groupoid.*

"Proof". If C is a Kan complex then it is an ∞-category, a fortiori. And given a morphism $f\colon c \to d$ in C, the outer horn $(\bullet, \mathrm{id}_c, f)\colon \Lambda^2_0 \to C$ admits a filler σ with $\partial_0(\sigma) = g\colon d \to c$ so that $[g] \circ [f] = \mathrm{id}_c$. Similarly, the outer horn $(f, \mathrm{id}_d, \bullet)\colon \Lambda^2_2 \to C$ admits a filler σ' with $\partial_2(\sigma') = g'$ so that $[f] \circ [g'] \simeq \mathrm{id}_d$. It follows that $[f]$ is an isomorphism with inverse $[g] = [g']$.

If C is an ∞-groupoid and we are given a horn $\Lambda^n_i \to C$ we should find a filler. This is clear if $0 < i < n$ (since C is an ∞-category). Only outer horns are an issue. Now, for example, let $n = 2$ and $i = 0$. So, we are given an outer horn $\alpha\colon \Lambda^2_0 \to C$:

$$\begin{tikzcd} & d & \\ c \arrow[ur, "f"] \arrow[rr, "h"'] & & e \end{tikzcd}$$

The idea is pretty simple: Since f is an equivalence, we may choose an inverse g and consider the following *inner* horn instead:

$$\begin{tikzcd} & d \arrow[dl, "g"'] & \\ c \arrow[rr, "h"'] & & e \end{tikzcd}$$

This admits a filler

$$\begin{tikzcd} & d \arrow[dl, "g"'] \arrow[dr, "k"] \arrow[d, phantom, "\rho"] & \\ c \arrow[rr, "h"'] & \, & e \end{tikzcd}$$

which we would like to turn into a filler for α. This is not too difficult. Let $\sigma\colon \Delta^2 \to C$ be a witness of $g \circ f \simeq \mathrm{id}_c$ and consider the inner horn

$$(\rho, s_0(h), \bullet, \sigma)\colon \Lambda^3_2 \to C.$$

You should check that the second face of any filler is a filler for α.

Of course, a similar argument works for outer horns $\Lambda^2_2 \to C$. However, in higher dimensions, the combinatorial task becomes more and more intractable and one needs an extrinsic idea. In fact, this is arguably the first 'non-trivial' (in this sense) result in these notes. It relies on the study of lifting properties among maps of simplicial sets (see for example Commentary 2.8 below), and we omit the proof. □

Commentary 1.43. Grothendieck's homotopy hypothesis asserts that in higher category theory, spaces should be the 'same' as ∞-groupoids. Given our identification of spaces with Kan complexes (see Commentary 1.40) we may view Proposition 1.42 as a precise version of this hypothesis.

1.4 Summary

Commentary 1.44. Looking at Lemmata 1.18 and 1.35 you will observe that the definition of an ∞-category is a very natural generalization of both 1-categories and Kan complexes: from the former we drop uniqueness of fillers, from the latter we drop fillers for outer horns.

Remark 1.45. The connection between 1-categories and their nerves on the one hand, and between topological spaces and their singular simplicial complexes on the other, is quite close as we have mentioned. Let us summarize it here, starting with the adjunctions discussed above:

$$\begin{tikzcd} & \mathsf{sSet} \arrow[dl, shift right, "|\cdot|"'] \arrow[dr, shift left, "\mathrm{h}"'] & \\ \mathsf{Top} \arrow[ur, shift right, "\mathrm{Sing}"'] & & \mathsf{Cat} \arrow[ul, shift left, "N"'] \end{tikzcd}$$

We collect the pertinent facts.

Theorem 1.46. *1. The counit*

$$\mathrm{h}(N(C)) \xrightarrow{\sim} C$$

is an isomorphism for every category C. In particular, the nerve functor $N\colon \mathsf{Cat} \to \mathsf{sSet}$ is fully faithful. The essential image consists of those simplicial sets that admit unique fillers for inner horns.

2. *The counit*
$$|\operatorname{Sing}(X)| \to X$$
is a weak homotopy equivalence, for every topological space X. In particular, the singular simplicial complex functor $\operatorname{Sing}\colon \mathsf{Top} \to \mathsf{sSet}$ *is fully faithful after inverting weak homotopy equivalences on both sides. Its essential image consists of those simplicial sets that admit fillers for all horns, that is, are Kan complexes.*

Commentary 1.47. The terminology around ∞-categories is a bit confusing. On the one hand ∞-*category* is really a short-hand for an $(\infty, 1)$-category: a 'category' that has morphisms of arbitrary dimensions, with morphisms of dimensions $2, 3, 4, \ldots$ being invertible. On the other hand, it's really only *one* way of modeling such $(\infty, 1)$-categories. To distinguish it from others, the literature also uses the terms *quasi-category* or *weak Kan complex*. So, Definition 1.1 is a misnomer in two different ways. Nevertheless, it is becoming more and more established terminology. Here is a table to put some of the terms into perspective.

conceptually	**a.k.a.**	**model in this course** = simplicial set having…	**a.k.a.**
$(\infty, 1)$-category	∞-category	fillers for inner horns	quasi-category, weak Kan complex
$(\infty, 0)$-category	∞-groupoid	fillers for all horns	Kan complex
$(1, 1)$-category	(1-)category	unique fillers for inner horns	(nerve of a category)
$(1, 0)$-category	(1-)groupoid	unique fillers for all horns	(nerve of a groupoid)

2 The ∞-category of spaces

Commentary 2.1. The category Set of sets plays a distinguished role in ordinary category theory. For example,

1. it is the free cocompletion of a singleton;
2. morphisms between objects in a category C form sets, and this is reflected in the fully faithful Yoneda embedding
$$C \hookrightarrow \operatorname{Fun}(C^{\mathrm{op}}, \mathsf{Set})$$

(These two reasons are closely related, of course.) In the theory of ∞-categories, this distinguished role is played by the ∞-category Spc of *spaces*. We will first explain why this should be so and then proceed with the construction of this ∞-category. In this section we will see the analogue of property 2. The analogue of property 1 has to wait until Section 4.1.

2.1 Functors

So far we have defined ∞-categories but not functors between them. This is easily remedied: A *functor* $C \to D$ between ∞-categories is simply a map of simplicial sets. As we now explain, functors naturally form an ∞-category.

Remark 2.2. As a presheaf category, the category of simplicial sets has all colimits and limits and these are computed pointwise.

Exercise 2.3. Let $(C_i)_i$ be a set of ∞-categories. Show that $\coprod_i C_i$ and $\prod_i C_i$ are ∞-categories as well (where the (co)product is computed in sSet).

Exercise 2.4. List all non-degenerate simplices of $\Delta^1 \times \Delta^1$.

Definition 2.5. Given a simplicial set K and an ∞-category C, we define a new simplicial set by

$$\mathrm{Fun}(K, C)_n = \mathrm{Hom}_{\mathsf{sSet}}(K \times \Delta^n, C),$$

with face and degeneracy maps induced by the $d^i \colon \Delta^{n-1} \to \Delta^n$ and $s^i \colon \Delta^{n+1} \to \Delta^n$. In other words, this is just the internal hom object in sSet that we also denote by $\underline{\mathrm{Hom}}_{\mathsf{sSet}}(K, C)$. The simplicial set $\mathrm{Fun}(K, C)$ is called the *∞-category of functors from K to C* because it is indeed an ∞-category. This is another result that isn't easy to prove directly and relies on the lifting properties for maps of simplicial sets (see Commentary 2.8 below).

Objects of $\mathrm{Fun}(K, C)$ are called *functors*, and morphisms of $\mathrm{Fun}(K, C)$ are called *natural transformations*. A *natural isomorphism* is a natural transformation that is an isomorphism in $\mathrm{Fun}(K, C)$.

Example 2.6.

1. The ∞-category $\mathrm{Fun}(\Delta^0, C)$ is isomorphic as a simplicial set to C itself.
2. The *arrow ∞-category* of C is the functor ∞-category $\mathrm{Fun}(\Delta^1, C)$. Its objects are morphisms in C.
3. The ∞-category $\mathrm{Fun}(\Delta^1 \times \Delta^1, C)$ has as objects 'commutative squares' in C. If you've done Exercise 2.4 you should be able to identify these with pairs of 2-simplices (σ, τ) in C such that $d_1(\sigma) = d_1(\tau)$ (the 'diagonal' of the square).

Exercise 2.7. Let C, D be ordinary categories. Exhibit a natural isomorphism of simplicial sets

$$\mathrm{Fun}(N(C), N(D)) \cong N(\mathrm{Fun}(C, D)).$$

Commentary 2.8. While we are not about to give a proof that $\mathrm{Fun}(K, C)$ is an ∞-category, let us explain how fibrations appear naturally in the proof. We want to show that the filling problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & \mathrm{Fun}(K, C) \\
\Delta^n \arrow[ur, dotted] &
\end{tikzcd}$$

admits a solution. By adjunction, this is equivalent to another problem:

$$\begin{tikzcd} \Lambda^n_i \times K \arrow[r] \arrow[d, hook] & C \\ \Delta^n \times K \arrow[ur, dotted] & \end{tikzcd}$$

We can turn this into a more symmetric problem by replacing the object C by the unique map $C \to \Delta^0$:

$$\begin{tikzcd} \Lambda^n_i \times K \arrow[r] \arrow[d, hook] & C \arrow[d] \\ \Delta^n \times K \arrow[r] \arrow[ur, dotted] & \Delta^0 \end{tikzcd}$$

Define an *inner fibration* $f\colon X \to Y$ of simplicial sets to have the right-lifting property (RLP) with respect to all inner horn inclusions, that is, every lifting problem (with $0 < i < n$)

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

admits a solution. Note that C is an ∞-category iff $C \to \Delta^0$ is an inner fibration so this is an obvious relativization. The sought-for property therefore follows from the fact that $\mathrm{Fun}(K, -)$ preserves inner fibrations, or, as we explained above, if the class of maps having the left-lifting property with respect to inner fibrations is closed under $- \times K$. This is what is typically proved in the literature. The maps having this left-lifting property are called *inner anodyne*.

Example 2.9. Any functor $f\colon C \to N(D)$ from an ∞-category to the nerve of an ordinary category is an inner fibration. This follows directly from the uniqueness in Lemma 1.18.

Remark 2.10. A natural transformation $\eta\colon f \to g$ of functors $f, g\colon C \to D$ of ordinary categories is an isomorphism if and only if each component $\eta_c\colon f(c) \to g(c)$ is. The same statement is true for ∞-categories: A natural transformation $\eta\colon C \times \Delta^1 \to D$ is a natural isomorphism if and only if for each $c \in C$, the morphism $\eta_c := \eta(\mathrm{id}_c, \mathrm{id}_{\Delta^1})$ is an isomorphism in D. Equivalently, the obvious functor $\mathrm{Fun}(C, D) \to \mathrm{Fun}(N(C_0), D)$ is conservative[2]. However, the proof is non-trivial. One possibility is a dévissage argument (using the lifting properties again) to reduce to the case where $C = \Delta^1$. This is a rather concrete combinatorial problem (think about what exactly you need to show!) that can be solved using the same kind of results alluded to in the proof of Proposition 1.42.

Definition 2.11. A functor $f\colon C \to D$ between ∞-categories is an *equivalence* if there exists a functor $g\colon D \to C$ and natural isomorphisms $f \circ g \simeq \mathrm{id}$, $g \circ f \simeq \mathrm{id}$.

[2]This is defined just as for ordinary categories: A functor $f\colon X \to Y$ of ∞-categories is *conservative* if $f(\alpha) \in Y_1$ an isomorphism implies $\alpha \in X_1$ an isomorphism.

Remark 2.12. Of course, an isomorphism of simplicial sets $f: C \to D$ is an equivalence. But an equivalence is a much weaker notion. For example, for nerves of ordinary categories, an equivalence is the same as an equivalence of ordinary categories. This follows immediately from Exercise 2.7. And a morphism of spaces (= Kan complexes) $f: X \to Y$ is an equivalence iff it is a weak homotopy equivalence.

Later, we will see that equivalences between ∞-categories are precisely the 'fully faithful' and 'essentially surjective' functors.

Commentary 2.13. Finally, the internal hom of simplicial sets allows to give a very pleasing characterization of ∞-categories themselves. Another way of expressing the condition (Definition 1.1) is that every inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ (for $0 < i < n$) induces a surjection

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, C) \to \mathrm{Hom}_{\mathsf{sSet}}(\Lambda^n_i, C).$$

In the discussion of the ∞-category of functors we have already seen that in fact something stronger holds: The map induced on internal homs is an *epimorphism* of simplicial sets:

$$\underline{\mathrm{Hom}}_{\mathsf{sSet}}(\Delta^n, C) \to \underline{\mathrm{Hom}}_{\mathsf{sSet}}(\Lambda^n_i, C).$$

We finish this subsection by discussing a further strengthening. For this, let us define a *trivial Kan fibration* $p: X \to Y$ of simplicial sets to have the RLP with respect to all inclusions of simplicial sets $S \hookrightarrow S'$. Note that a trivial Kan fibration $p: X \to Y$

- is in particular an inner fibration, and
- admits a section (hence is an epimorphism).

Finally, given $y: \Delta^0 \to Y$, the pullback (the 'fiber of p over y)

$$\begin{tikzcd} X_y \arrow[r] \arrow[d, "p'"] & X \arrow[d, "p"] \\ \Delta^0 \arrow[r, "y"] & Y \end{tikzcd}$$

is a contractible space. This is because p' is again a trivial Kan fibration. In particular, it has the RLP with respect to all horn inclusions so that the domain is a Kan complex. It also has the RLP with respect to the inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ which implies that it is a weak homotopy equivalence of spaces. That is, the domain is contractible.

Lemma 2.14. *Let C be a simplicial set. The following are equivalent:*

1. *C is an ∞-category.*
2. *For every inner anodyne map $i: X \to Y$, the induced map of simplicial sets*

$$\underline{\mathrm{Hom}}_{\mathsf{sSet}}(Y, C) \to \underline{\mathrm{Hom}}_{\mathsf{sSet}}(X, C)$$

   is a trivial Kan fibration.

3. *The canonical map $\underline{\mathrm{Hom}}_{\mathsf{sSet}}(\Delta^2, C) \to \underline{\mathrm{Hom}}_{\mathsf{sSet}}(\Lambda^2_1, C)$ is a trivial Kan fibration.*

This result exhibits a fundamental relation between said classes of maps (inner anodyne, inner horn inclusions, trivial Kan fibrations). Although conceptually maybe not a difficult result, the combinatorics involved are well beyond the scope of these lectures. (Note that you can deduce that $\mathrm{Fun}(K, C)$ is an ∞-category (if C is) from 1.⇒3. You only need to know that trivial Kan fibrations are stable under $\underline{\mathrm{Hom}}_{\mathsf{sSet}}(K, -)$.)

This result has many consequences, for example:

Corollary 2.15. *Let $(g, \bullet, f)\colon \Lambda^2_1 \to C$ be a composable pair in an ∞-category C. The 'simplicial set of compositions (or fillers)'*

$$\underline{\mathrm{Hom}}_{\mathsf{sSet}}(\Delta^2, C) \times_{\underline{\mathrm{Hom}}_{\mathsf{sSet}}(\Lambda^2_1, C)} \{(g, \bullet, f)\}$$

is a contractible space. (That is, a Kan complex (weakly) homotopy equivalent to D^0.)

Proof. This is exactly the fiber of the trivial Kan fibration

$$\underline{\mathrm{Hom}}_{\mathsf{sSet}}(\Delta^2, C) \to \underline{\mathrm{Hom}}_{\mathsf{sSet}}(\Lambda^2_1, C)$$

over $(g, \bullet, f)$. □

Remark 2.16. There is a direct generalization of this result. Namely, given a path of composable morphisms $\alpha\colon \mathrm{Sp}^n \to C$ in an ∞-category (see the proof of Lemma 1.18), the 'simplicial set of compositions'

$$\underline{\mathrm{Hom}}_{\mathsf{sSet}}(\Delta^n, C) \times_{\underline{\mathrm{Hom}}_{\mathsf{sSet}}(\mathrm{Sp}^n, C)} \{\alpha\}$$

is a again a contractible space. (This is again because the inclusion $\mathrm{Sp}^n \hookrightarrow \Delta^n$ is inner anodyne.)

In other words, while compositions are not unique (as in an ordinary category), any two compositions are homotopic. And the homotopies are not unique but themselves homotopic. And the homotopies between homotopies are not unique but themselves homotopic. And so on *ad infinitum*.

2.2 Mapping spaces

Commentary 2.17. Let C be an ordinary 1-category, and let $c, d \in C$ be two objects. Consider the following pullback square of categories:

$$\begin{tikzcd}
\mathrm{Hom}_C(c,d) \arrow[r] \arrow[d] & \mathrm{Fun}([1], C) \arrow[d, "{(0,1)}"] \\
\Delta^0 \arrow[r, "{(c,d)}"] & C \times C
\end{tikzcd}$$

In other words, the pullback turns out to be a 0-category, and identifies with the set of morphisms from c to d.

For an ∞-category we will define a simplicial set of morphisms in the same way. And this will similarly decrease the categorical level by 1, resulting in an $(\infty, 0)$-category, or ∞-groupoid. By Proposition 1.42, we identify these with spaces thus the name 'mapping space'.

Definition 2.18. Let C be an ∞-category and $c, d \in C$ two objects. The *mapping space* $\mathrm{Map}_C(c, d)$ is the pullback in **sSet** in the following diagram:

$$\begin{tikzcd} \mathrm{Map}_C(c,d) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, C) \arrow[d] \\ \Delta^0 \arrow[r, "{(c,d)}"] & \mathrm{Fun}(\partial\Delta^1, C) \end{tikzcd} \tag{2.19}$$

Here, the right vertical map is restriction along the inclusion $\partial\Delta^1 \hookrightarrow \Delta^1$. We may identify $\mathrm{Fun}(\partial\Delta^1, C) \cong C \times C$ and the bottom horizontal map picks the pair $(c, d) \in C \times C$.

Remark 2.20. As the name suggests, the mapping space is indeed a Kan complex. Indeed, it is another consequence of the stability properties of fibrations that the right vertical arrow in (2.19) is an inner fibration. By Remark 2.10, it is also conservative. And it is true that conservative inner fibrations are stable under pullbacks (along functors of ∞-categories) hence the left vertical arrow is also a conservative inner fibration. But this just means that the mapping space is an ∞-groupoid.

Exercise 2.21. Show that the objects of $\mathrm{Map}_C(c, d)$ are precisely morphisms $f : c \to d$ in C, and there is a morphism $f \to g$ iff $f \simeq g$ as morphisms in C. In particular, $\pi_0 \mathrm{Map}_C(c, d) = \mathrm{Hom}_{\mathrm{h}(C)}(c, d)$.

Deduce (or prove otherwise) that if C is an ordinary category then $\mathrm{Map}_{N(C)}(c, d)$ is a discrete space that can be identified with $\mathrm{Hom}_C(c, d)$.

Example 2.22. Let X be a space and $x \in X$ a point. Then the mapping space

$$\Omega(X) = \Omega(X, x) = \mathrm{Map}_X(x, x)$$

is the *based loop space* of X. As you might be familiar with from algebraic topology, one has a canonical homotopy equivalence

$$\pi_n(\Omega(X), \mathrm{id}_x) \simeq \pi_{n+1}(X, x), \qquad n \geq 0.$$

From Exercise 2.21, we see that the following definition generalizes fully faithfulness from ordinary categories to ∞-categories.

Definition 2.23. Let $f : C \to D$ be a functor between ∞-categories. By naturality of the construction, it induces maps of spaces

$$\mathrm{Map}_C(c, d) \to \mathrm{Map}_D(f(c), f(d))$$

for all objects $c, d \in C$. We say that f is *fully faithful* if these maps are all homotopy equivalences.

Commentary 2.24. By Exercise 2.21, a fully faithful functor $f : C \to D$ induces a fully faithful functor $\mathrm{h}(f): \mathrm{h}(C) \to \mathrm{h}(D)$ on homotopy categories. However, the converse is not true in general. (For example, let X be a connected, simply connected space that is not contractible. The canonical map $X \to \Delta^0$ induces an equivalence on homotopy categories (that is, on fundamental groupoids; see Exercise 1.36) but is not fully faithful.)

Construction 2.25. Let C be an ∞-category and $\iota\colon \mathrm{h}(C)' \hookrightarrow \mathrm{h}(C)$ a subcategory of its homotopy category. (Not necessarily full.) We define a simplicial set C' by the following pullback diagram:

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[d] \\ N(\mathrm{h}(C)') \arrow[r, "\iota"] & N(\mathrm{h}(C)) \end{tikzcd}$$

Here, the right vertical functor is the unit of the adjunction $\mathrm{h} \dashv N$.

Exercise 2.26. Show that C' is an ∞-category. It is the *subcategory spanned by* $\mathrm{h}(C)'$. (Despite the name, it is not an ordinary category in general.)

Remark 2.27. If $\mathrm{h}(C)' \subseteq \mathrm{h}(C)$ is a *full* subcategory then the induced functor $C' \to C$ is fully faithful. In fact, it induces not just homotopy equivalences on mapping spaces but isomorphisms of simplicial sets. This follows readily from the definitions.

Example 2.28. Let C be an ∞-category and $\mathrm{h}(C)^{\simeq} \subseteq \mathrm{h}(C)$ the core of $\mathrm{h}(C)$. That is, it is spanned by all isomorphisms in $\mathrm{h}(C)$. We define the *core of* C, denoted $C^{\simeq}$ to be the subcategory spanned by $\mathrm{h}(C)^{\simeq}$. Note that it is an ∞-groupoid, in fact, it is (obviously) the maximal ∞-groupoid contained in C.

Definition 2.29. Let $f\colon C \to D$ be a functor between ∞-categories. It is called *essentially surjective* if every object $d \in D$ is isomorphic to an object in the image of f, that is, $f(c) \cong d$ for some $c \in C$.

Commentary 2.30. In contrast to fully faithfulness, essentiall surjectivity is detected at the level of homotopy categories: The functor f is essentially surjective iff $\mathrm{h}(f)\colon \mathrm{h}(C) \to \mathrm{h}(D)$ is essentially surjective (in the sense of ordinary 1-categories).

Exercise 2.31. Let $f\colon X \to Y$ be a map of Kan complexes. Show that f is essentially surjective iff $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is surjective.

Remark 2.32. Combining Example 2.22 and Exercise 2.31, one sees that a map of spaces is fully faithfulness and essentially surjective if and only if it is a homotopy equivalence. (One also uses Whitehead's theorem (Remark 2.12): It suffices to show it is a weak homotopy equivalence.) This is a special case of the following result.

Proposition 2.33. *Let $f\colon C \to D$ be a functor between ∞-categories. Then f is an equivalence iff it is fully faithful and essentially surjective.*

Sketch of proof. The forward direction being easy let us focus on the converse. Note first that f is an equivalence iff $f_*\colon \mathrm{Fun}(K,C)^{\simeq} \to \mathrm{Fun}(K,D)^{\simeq}$ is an equivalence for all simplicial sets K. (This is an easy exercise. You only need $K = C$ and $K = D$.) By a dévissage argument one reduces to $K = \Delta^1$. Now, passing to cores preserves fully faithfulness and essential surjectivity.

This implies that the bottom horizontal arrow in

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, C)^{\simeq} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, D)^{\simeq} \arrow[d] \\
\mathrm{Fun}(\partial\Delta^1, C)^{\simeq} \arrow[r] & \mathrm{Fun}(\partial\Delta^1, D)^{\simeq}
\end{tikzcd}$$

is fully faithful and essentially surjective and thus a homotopy equivalence, by Remark 2.32. Now, the vertical arrows turn out to be Kan fibrations hence one reduces to show homotopy equivalences on the fibers. But the fibers are just the mapping spaces in C and D, respectively, so this follows from fully faithfulness of f. □

Exercise 2.34. Show that for $f\colon X \to Y$ a map between Kan complexes, fully faithfulness amounts to f being the inclusion of a summand. That is, f induces an equivalence $Y = X \coprod Y'$ for some Kan complex Y'.

Exercise 2.35. Let $f\colon C \to D$ be a functor of ∞-categories. Its *essential image* is the full subcategory of D spanned by the essential image of $\mathrm{h}(f)\colon \mathrm{h}(C) \to \mathrm{h}(D)$. Show that if f is fully faithful then it factors as an equivalence followed by the inclusion of a full subcategory.

Commentary 2.36. We finish this subsection with a more sophisticated version of Commentary 1.23. Let $c, d, e \in C$ be objects in an ∞-category and define the following simplicial set as a pullback:

$$\begin{tikzcd}
\mathrm{Map}_C(c, d, e) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, C) \arrow[d] \\
\Delta^0 \arrow[r, "{(c,d,e)}"] & C \times C \times C
\end{tikzcd}$$

where the right vertical arrow is restriction to the three 0-simplices of Δ^2. It turns out (see Exercise 2.37 below) that $\mathrm{Map}_C(c, d, e)$ is also a Kan complex and the canonical restriction map

$$\mathrm{Map}_C(c, d, e) \xrightarrow{\partial_0 \times \partial_2} \mathrm{Map}_C(d, e) \times \mathrm{Map}_C(c, d)$$

is a trivial Kan fibration. Choosing a homotopy inverse one obtains a map

$$\mathrm{Map}_C(d, e) \times \mathrm{Map}_C(c, d) \to \mathrm{Map}_C(c, d, e) \xrightarrow{\partial_1} \mathrm{Map}_C(c, e)$$

that one may interpret as a composition. It depends on the choice of the homotopy inverse but only in the mildest manner possible: The choices form a contractible space. (Again, see Exercise 2.37 below.)

Exercise 2.37.

1. Let $p\colon X \to Y$ be a trivial Kan fibration (Commentary 2.13). We already observed that it admits a section. Show that the simplicial set of sections naturally forms a contractible space. (Of course, this involves defining this simplicial set in the first place.)
2. Using Lemma 2.14, establish the claims made in Commentary 2.36.

2.3 A construction

Commentary 2.38. So far, we have seen two classes of examples of ∞-categories, namely ordinary categories and spaces. We have also seen several constructions that produce new from old ones, such as functor categories or subcategories. Combining these one already gets a decent selection of ∞-categories. Nevertheless, one important class of examples is missing: Suppose C is an ordinary category with a class of morphisms $W \subseteq C_1$ that one would like to 'invert' (=localize at). Neglecting size issues this can always be done at the level of ordinary categories but the result is often too crude to be useful. (For example, even if C had all limits and colimits the localization typically has few.) The theory of $(\infty, 1)$-categories is supposed to provide a framework in which such localizations can be performed in a refined way. Our goal in this subsection is to explain this in the example of C the category of spaces (that is, Kan complexes) and W the class of homotopy equivalences. However, it should be noted that the same construction works in many other contexts of interest.

Commentary 2.39. The actual construction won't be that important in the sequel and you could in principle jump directly to Theorem 2.47. However, I want to convey that the resulting ∞-category Spc is nothing mysterious but has an explicit (although complicated) description. Moreover, it is worth comparing the construction with the nerve of an ordinary category.

Convention 2.40. Let $0 \le i \le j$ be two integers and denote by $P_{i,j}$ the set

$$P_{i,j} := \{I \subseteq \{i, \dots, j\} \mid \min(I) = i, \max(I) = j\}$$

partially ordered by inclusion. (If $i > j$ we agree that $P_{i,j} = \emptyset$.)

Definition 2.41. Let $n \ge 0$. We define a simplicial category $\mathfrak{C}[\Delta^n]$ with:

- objects $0, 1, \dots, n$;
- simplicial hom-sets $\underline{\mathrm{Hom}}_{\mathfrak{C}[\Delta^n]}(i, j) = N(P_{i,j})$;
- composition is induced by the union of subsets.

Commentary 2.42. Note that $\mathfrak{C}[\Delta^n]$ and the totally ordered set $\mathrm{h}(\Delta^n) = \{0 < 1 < \cdots < n\}$ have the same objects. But whereas there is a unique morphism $i \to j$ in $\mathrm{h}(\Delta^n)$ if $i \le j$, in $\mathfrak{C}[\Delta^n]$ there is—it turns out—a weakly contractible simplicial set of such morphisms.[3] The objects of this simplicial set are in bijection with all possible compositions

$$i = i_0 < i_1 < \cdots < i_m = j.$$

And, again, while the composition of $i < j$ and $j < k$ in $\mathrm{h}(\Delta^n)$ is just $i < k$, in $\mathfrak{C}[\Delta^n]$ it is given by $i < j < k$. One thinks of $\mathfrak{C}[\Delta^n]$ as a 'thickened' version of $\mathrm{h}(\Delta^n)$ in which associativity of composition on the nose is weakened to associativity up to coherent homotopy.

[3] A simplicial set is weakly contractible if its topological realization is (weakly) contractible.

Example 2.43.

- Let $n = 0$. In this case, $\mathfrak{C}[\Delta^0]$ is the terminal simplicial category with one object and a singleton simplicial mapping space. Therefore, $\mathfrak{C}[\Delta^0]$ and $\mathrm{h}(\Delta^0)$ are really the 'same'.
- Let $n = 1$. In this case, $\mathfrak{C}[\Delta^1]$ has two objects $0, 1$ and all simplicial mapping spaces are again trivial (empty or singletons). Hence we still see no thickening. (Of course, this is because there are no interesting compositions.)
- The first genuine thickening occurs when $n = 2$. In this case $\mathfrak{C}[\Delta^2]$ has three objects $0, 1, 2$. The only non-trivial simplicial hom-set is $\underline{\mathrm{Hom}}_{\mathfrak{C}[\Delta^2]}(0, 2) = N(P_{0,2})$. Here, the poset $P_{0,2}$ is actually an ordered set on two elements (namely, $\{0, 2\} \subset \{0, 1, 2\}$). In other words, $\underline{\mathrm{Hom}}_{\mathfrak{C}[\Delta^2]}(0, 2) \cong \Delta^1$.

Construction 2.44. Note that the construction in Definition 2.41 is functorial in n. Therefore it makes sense to define a simplicial set Spc by the following formula (for all $n \geq 0$):

$$\mathsf{Spc}_n = \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, \mathsf{Spc}) = \mathrm{hom}_{\mathsf{sCat}}(\mathfrak{C}[\Delta^n], \mathit{Kan}) \tag{2.45}$$

Here, Kan denotes the simplicial category of Kan complexes with simplicial hom-sets as in Definition 2.5.[4] And sCat denotes the category of simplicial categories, so morphisms are required to preserve the simplicial structure.

Example 2.46. From Example 2.43 we can determine the simplices of Spc in low dimension:

- The 0-simplices are just the objects of Kan, that is, the Kan complexes.
- The 1-simplices are the maps of Kan complexes.
- The 2-simplices are in bijection with maps of Kan complexes $f\colon X \to Y$, $g\colon Y \to Z$, $h\colon X \to Z$ together with a homotopy $g \circ f \simeq h$.

I recommend verifying these statements. In particular, the last one requires some unpacking of the definitions.

This description of the low-dimensional simplices might lend some plausibility to the following result.

Theorem 2.47. *The simplicial set* Spc *is an* ∞*-category and for any Kan complexes* X, Y *there is a canonical homotopy equivalence of spaces*

$$\underline{\mathrm{Hom}}_{\mathsf{sSet}}(X, Y) \xrightarrow{\sim} \mathrm{Map}_{\mathsf{Spc}}(X, Y).$$

[4]Let $X, Y, Z \in \mathit{Kan}$. The composition

$$\underline{\mathrm{Hom}}(Y, Z) \times \underline{\mathrm{Hom}}(X, Y) \to \underline{\mathrm{Hom}}(X, Z)$$

on n-simplices is given by

$$\mathrm{Hom}(Y \times \Delta^n, Z) \times \mathrm{Hom}(X \times \Delta^n, Y) \xrightarrow{\circ} \mathrm{Hom}(X \times \Delta^n \times \Delta^n, Z) \to \mathrm{Hom}(X \times \Delta^n, Z)$$

where the second map is restriction along the diagonal $\Delta^n \to \Delta^n \times \Delta^n$.

Commentary 2.48. The construction of the ∞-category Spc can be generalized. Given any simplicial category C such that the simplicial hom-sets are Kan complexes,[5] the *homotopy coherent nerve* $N(C)$ defined by the same formula as in (2.45),

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, N(C)) = \mathrm{hom}_{\mathsf{sCat}}(\mathfrak{C}[\Delta^n], C),$$

is an ∞-category, and its mapping spaces are the original simplicial mapping spaces (up to canonical homotopy equivalence).

Sketch of proof. To check that $N(C)$ is an ∞-category we need to show the inner horn filling condition. By definition (and as with the usual nerve), the homotopy coherent nerve is right adjoint to the functor $\mathfrak{C}[-]\colon \mathsf{sSet} \to \mathsf{sCat}$ constructed in the by now familiar way: You know what to do on the simplex category, namely $[n] \mapsto \mathfrak{C}[\Delta^n]$, and this forces what to do on general simplicial sets (left Kan extend). So, by adjunction, the filling problem becomes an analogous lifting problem in simplicial categories:

$$\begin{tikzcd} \mathfrak{C}[\Lambda^n_i] \arrow[r] \arrow[d, hook] & C \\ \mathfrak{C}[\Delta^n] \arrow[ur, dotted] & \end{tikzcd}$$

It turns out (as you can easily check) that the two simplicial categories on the left have the same objects and the same simplicial hom-sets except one, the one from 0 to n. Extending the horizontal functor on this simplicial hom-set is possible exactly because the target in C is a Kan complex.

There are relatively elementary proofs of the statement about the mapping spaces in the homotopy coherent nerve but these are rather long and I won't try to convey the combinatorics involved. $\square$

Exercise 2.49. If C is an ordinary category and $\underline{C}$ the canonical simplicial category associated to it (with same objects and discrete simplicial mapping spaces) then we have a canonical isomorphism of simplicial sets $N(C) \cong N(\underline{C})$. This isomorphism is obtained from the canonical simplicial functor $\mathfrak{C}[\Delta^n] \to \underline{[n]}$:

$$N(C)_n = \mathrm{hom}_{\mathsf{Cat}}([n], C) \cong \mathrm{hom}_{\mathsf{sCat}}(\underline{[n]}, \underline{C}) \to \mathrm{hom}_{\mathsf{sCat}}(\mathfrak{C}[\Delta^n], \underline{C}) = N(\underline{C})_n$$

In this sense, the homotopy coherent nerve can be seen as a generalization of the nerve discussed in Construction 1.16.

Commentary 2.50. Somebody is going to complain that Spc isn't a simplicial set and a fortiori not an ∞-category. Indeed, already the 0-simplices form a proper class and not a set. (This 'problem' already arose when considering the nerve of an ordinary category, if the latter had a proper class of objects or wasn't locally small.) As with ordinary categories there are

[5]We call these *locally Kan*. The simplicial category *Kan* is locally Kan, again, by the lifting properties of maps of simplicial sets, cf. Commentary 2.8.

different ways to deal with these issues, for example using Grothendieck universes and speaking of 'small' and 'large' ∞-categories. In the case at hand, if *Kan* is the category of *small* Kan complexes then the ∞-category Spc has a large set of objects but for each $X, Y \in \mathsf{Spc}$, the mapping space $\mathrm{Map}_{\mathsf{Spc}}(X, Y)$ is small. It would therefore be classified as a locally small ∞-category. (As with ordinary categories, this implies that $\mathrm{Fun}(C, \mathsf{Spc})$ is locally small whenever C is small. Hopefully that will put you at ease when we start considering presheaf categories.)

I have been cavalier about size issues as I think they would just cause distraction and are not really relevant at this point. When they become relevant I will highlight this.

Example 2.51. We define a simplicial category *qCat* as follows. (The simplicial category of 'quasi-categories', see Commentary 1.47.) Its objects are ∞-categories. The simplicial hom-sets between two ∞-categories C, D is given by $\mathrm{Fun}(C, D)^{\simeq}$, the maximal ∞-groupoid inside the functor ∞-category. You can check that the composition in sSet restricts to a composition in *qCat*, so that the latter is indeed a simplicial category. Note that by definition, the simplicial hom-sets are Kan complexes hence

$$\mathsf{Cat}_\infty := N(qCat)$$

is a (large) ∞-category, the ∞-*category of (small)* ∞-*categories*.

The objects of Cat_∞ are ∞-categories, the morphisms are functors, and the 2-simplices are homotopies (as in Example 2.46). Note how working with $(\infty, 1)$-categories here does not allow us to consider natural transformations that are not isomorphisms. But with this restriction in mind, the object Cat_∞ so defined really captures the 'theory of ∞-categories', as the following exercise also indicates.

Exercise 2.52. Let $f, g \colon C \to D$ be functors between ∞-categories. Show that:

1. $f \simeq g$ are equivalent in Cat_∞ iff there exists a natural isomorphism between them.
2. f is an isomorphism in Cat_∞ iff f is an equivalence of ∞-categories.

Example 2.53. Let C, D be two ordinary categories. We denote by $\mathrm{Fun}(C, D)^{\simeq}$ the core of the functor category, that is, the functors $C \to D$ together with invertible natural transformations. Let *oCat* be the simplicial category of ordinary 1-categories with simplicial hom-sets

$$\underline{\mathrm{Hom}}_{oCat}(C, D) = N(\mathrm{Fun}(C, D)^{\simeq}).$$

The composition is defined in the obvious way. (This is the simplicial category associated to the strict $(2, 1)$-category of 1-categories, functors and invertible transformations.) We define the ∞-category of 1-categories to be $\mathsf{Cat}_1 := N(oCat)$.

The following result refines parts of Theorem 1.46:

Corollary 2.54.
1. *The identity on objects extends to a canonical fully faithful embedding* $\mathsf{Spc} \hookrightarrow \mathsf{Cat}_\infty$.
2. *The nerve functor on objects extends to a canonical fully faithful embedding* $\mathsf{Cat}_1 \hookrightarrow \mathsf{Cat}_\infty$.

Proof. Every Kan complex is an ∞-category. We also already observed that for $X, Y \in \mathcal{K}an$, $\underline{\mathrm{Hom}}_{\mathsf{sSet}}(X, Y)$ is again a Kan complex hence equal to its maximal sub-∞-groupoid. It follows that the canonical simplicial functor $\mathcal{K}an \to q\mathcal{C}at$ induces isomorphisms on simplicial hom-sets. Applying the homotopy-coherent nerve we therefore obtain a fully faithful functor.

For the second statement, consider the simplicial functor $o\mathcal{C}at \to q\mathcal{C}at$ that sends an ordinary category C to its nerve, and on simplicial hom-sets is the isomorphism

$$N(\mathrm{Fun}(C, D)^{\simeq}) \cong \mathrm{Fun}(N(C), N(D))^{\simeq}$$

induced by Exercise 2.7. Applying the homotopy-coherent nerve we therefore obtain a fully faithful functor. □

2.4 Yoneda lemma

Commentary 2.55. Let $d \in C$ be an object in an ordinary category. One defines a functor $\mathrm{y}_d \colon C^{\mathrm{op}} \to \mathsf{Set}$ by the formula

$$\mathrm{y}_d(c) := \mathrm{Hom}_C(c, d)$$

and pre-composition of morphisms in C. On the other hand, post-composition of morphisms in C shows that the association $d \mapsto \mathrm{y}_d$ underlies a functor

$$\mathrm{y} \colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathsf{Set}), \tag{2.56}$$

called the Yoneda embedding.

Commentary 2.57. Now, let $c, d \in C$ be objects in an ∞-category and $f \colon c' \to c$ a morphism. We have already observed (Commentary 2.36) that composition with f induces a map of spaces

$$\mathrm{Map}_C(c, d) \xrightarrow{f^*} \mathrm{Map}_C(c', d),$$

which is well-defined up to contractible choice. Unfortunately, this indeterminacy makes it a non-trivial task to define an analogous functor $\mathrm{y}_d \colon C^{\mathrm{op}} \to \mathsf{Spc}$. And if we want to produce an analogue of (2.56) we have to cope with the same indeterminacy for *post*-composition.

It turns out that this is possible but non-trivial. This is a recurring theme in the theory of ∞-categories: It is impossible to solve infinite coherence issues 'by hand' so that the construction of ∞-categories and functors between them often is a major task.

In the case at hand, we will find a work-around by stepping outside the theory of ∞-categories, using the fact that Spc is the homotopy coherent nerve of a simplicial category, and that composition in simplicial categories *is* strictly functorial.

Convention 2.58. Let C be an ∞-category. The *opposite ∞-category* C^{op} is defined as the functor

$$C^{\mathrm{op}} \colon \Delta^{\mathrm{op}} \xrightarrow{\sim} \Delta^{\mathrm{op}} \to \mathsf{Set}$$

where the first functor (an automorphism of the simplex category) swaps the order of the elements of $[n]$. (Explicitly, it sends $[n]$ to $[n]$ and $d^i \colon [n-1] \to [n]$ (resp. $s^i \colon [n+1] \to [n]$) to d^{n-i} (resp. s^{n-i}).) It is clear that C^{op} is again an ∞-category.

Convention 2.59. Let C be a small ∞-category. We denote by $\mathcal{P}(C)$ the functor ∞-category $\mathrm{Fun}(C^{\mathrm{op}}, \mathsf{Spc})$. It is called the *$\infty$-category of presheaves on C*.

Theorem 2.60. *There is a fully faithful functor* $\mathrm{y}\colon C \to \mathcal{P}(C)$*, called the* Yoneda embedding*, such that for each $F \in \mathcal{P}(C)$ and $c \in C$, there is a homotopy equivalence of spaces*

$$\mathrm{Map}_{\mathcal{P}(C)}(\mathrm{y}_c, F) \simeq F(c). \tag{2.61}$$

Commentary 2.62. Note in particular that for each $d \in C$, the functor $\mathrm{y}_d\colon C^{\mathrm{op}} \to \mathsf{Spc}$ satisfies

$$\mathrm{y}_d(c) \simeq \mathrm{Map}_{\mathcal{P}(C)}(\mathrm{y}_c, \mathrm{y}_d) \simeq \mathrm{Map}_C(c, d)$$

as desired.

Construction 2.63. Let us explain one possible way to define the Yoneda functor. Consider the following composite of simplicial functors:

$$\mathfrak{C}[C^{\mathrm{op}} \times C] \to \mathfrak{C}[C]^{\mathrm{op}} \times \mathfrak{C}[C] \xrightarrow{\mathrm{Hom}_{\mathfrak{C}[C]}} \mathsf{sSet} \xrightarrow{\mathrm{Sing}\,|-|} \mathcal{K}an \tag{2.64}$$

Here, the first arrow is the canonical functor, the second is the hom simplicial-set in $\mathfrak{C}[C]$, and the composite $\mathrm{Sing} \circ |-|$ conveniently turns any simplicial set into a Kan complex without changing the weak homotopy type. By adjunction, (2.64) corresponds to a functor

$$C^{\mathrm{op}} \times C \to N(\mathcal{K}an) = \mathsf{Spc} \tag{2.65}$$

or, again by adjunction, to a functor

$$\mathrm{y}\colon C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathsf{Spc}) = \mathcal{P}(C). \tag{2.66}$$

This is the sought-for Yoneda functor.

Sketch of proof. It is not difficult to deduce fully faithfulness of the Yoneda functor (2.66) from fully faithfulness of its simplicial analogue. (Note that there is an enriched Yoneda lemma for simplicial categories. The proof is the same as in the classical case.) To prove the more precise (2.61), one similarly translates it to a statement about simplicial categories. However, the translation is more involved in this case. Among other things, one needs to express $\mathcal{P}(C)$ as the nerve of a simplicial category. We will not get involved in that here. □

Remark 2.67. Note that the functor $C^{\mathrm{op}} \times C \to \mathsf{Spc}$ constructed in (2.65) is pointwise *equivalent* to the mapping spaces in C but not *equal*. In other words, we have not really defined a functor $\mathrm{Map}_C(-,-)\colon C^{\mathrm{op}} \times C \to \mathsf{Spc}$.[6] Nevertheless, one is often sloppy and pretends that such a functor exists. In most situations that's okay. But keep in mind that the more precise statements should involve an actual functor, such as (2.65) (or another such functor, see for example Remark A.32). (In fact, many ∞-categories in practice come with a preferred choice of 'mapping space bifunctor', for example because they arise as the homotopy coherent nerve of a simplicial category.)

[6] Except if C is a 1-category.

3 (Co)limits

Commentary 3.1. In this section we will discuss limits and colimits in ∞-categories. We will adopt a definition which is close to the intuition from ordinary category theory, and which allows us to compute them in many examples quite explicitly. Developing the entire theory of limits and colimits and computing these in more involved situations, however, would require a different (but equivalent) approach, and more technology than we want to go into. For this reason, more results in this sections will be stated without proof. (But see Section 3.3.)

3.1 Limits

Commentary 3.2. Let $F\colon I \to C$ be a functor between categories. Recall that a limit is a universal cone over F. We can generalize this to ∞-categories in a straightforward way. In that context it is also useful to allow I to be an arbitrary simplicial set.

Convention 3.3. Throughout we fix a simplicial set I, an ∞-category C, and a map of simplicial sets $F\colon I \to C$. Given an object $\ell \in C$ we denote by

$$\underline{\ell}\colon I \to \Delta^0 \xrightarrow{\ell} C$$

the constant functor with value ℓ. To shorten the notation we sometimes write C^I instead of $\mathrm{Fun}(I, C)$.

Definition 3.4. A *cone over F* is a pair (ℓ, η) where $\ell \in C$ and $\eta\colon \underline{\ell} \to F$ is a natural transformation. It is a *limit cone* if for each $c \in C$ the following composite is a homotopy equivalence:

$$\mathrm{Map}_C(c, \ell) \xrightarrow{\underline{(-)}} \mathrm{Map}_{C^I}(\underline{c}, \underline{\ell}) \xrightarrow{\eta_*} \mathrm{Map}_{C^I}(\underline{c}, F) \tag{3.5}$$

In that case, we also say—abusively—that ℓ is the *limit of F* and we write $\ell = \lim_I F = \lim_{i\in I} F(i)$.

Commentary 3.6. The first arrow in (3.5) is induced by the diagonal map $\underline{(-)}\colon C \to C^I$ (which by adjunction corresponds to the canonical projection $C \times I \to C$). The second arrow is post-composition with η. As we discussed in Commentary 2.36, this is unique only up to contractible choice. Nevertheless, whether (3.5) is a homotopy equivalence or not is independent of this choice.

Exercise 3.7. Show that any two limits are isomorphic. (In fact, this can be improved. The simplicial set of limits is either empty or a contractible space. We will not prove this. But see Corollary 3.45.)

Exercise 3.8. Show that if I and C are (nerves of) ordinary categories, then Definition 3.4 recovers the notion of a limit in ordinary category theory.

Example 3.9. Let I be a discrete ∞-category. The functor ∞-category $C^I \cong \prod_I C$ is the product in simplicial sets so that $\mathrm{Map}_{C^I}(\underline{c}, F) \cong \prod_i \mathrm{Map}_C(c, F(i))$. We deduce that, up to equivalence, a morphism into the product $\prod_i F(i) := \lim F \in C$ amounts to a family of morphisms into each $F(i) \in C$, as in ordinary category theory.

Example 3.10. As a particular case of Example 3.9, let $I = \emptyset$ so that $C^I = \Delta^0$. We deduce that $\ell \in C$ is final if and only if every mapping space $\mathrm{Map}_C(c, \ell)$ is contractible.

Example 3.11. We now turn to an example where the ∞-categorical limit behaves differently than in ordinary categories. Let $I = \Lambda^2_2$ so that $F\colon I \to C$ can be depicted as follows:

$$\begin{tikzcd} & r \arrow[d, "f"] \\ s \arrow[r, "g"'] & t \end{tikzcd}$$

We are going to see that, up to equivalence, a map from $c \in C$ into the pullback $r \times_t s := \lim F$ (if it exists) amounts to a commutative square (Example 2.6):

$$\begin{tikzcd} c \arrow[r, "h"] \arrow[d, "i"'] & r \arrow[d, "f"] \\ s \arrow[r, "g"'] & t \end{tikzcd} \tag{3.12}$$

Similarly informally, one says, it amounts to maps $h\colon c \to r$, $i\colon c \to s$, together with an equivalence $g \circ i \simeq f \circ h$. Note how the equality $g \circ i = f \circ h$ for pullbacks in ordinary categories, is here replaced by additional data.

Proof. A natural transformation $\underline{c} \to F$ is, by definition, a map of simplicial sets $\eta\colon \Delta^1 \times \Lambda^2_2 \to C$ such that $\eta|_0 = \underline{c}$ and $\eta|_1 = F$. In other words, two commutative squares

$$\begin{tikzcd} c \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & c \arrow[d, "k"] & c \arrow[l, "\mathrm{id}"'] \arrow[d, "h"] \\ s \arrow[r, "g"'] & t & r \arrow[l, "f"] \end{tikzcd} \tag{3.13}$$

that share the vertical arrow k in the middle. Recall that this amounts to four 2-simplices filling the diagram like so:

$$\begin{tikzcd} c \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] \arrow[dr, "y"'] & c \arrow[d, "k"] & c \arrow[l, "\mathrm{id}"'] \arrow[d, "h"] \arrow[dl, "x"] \\ s \arrow[r, "g"'] & t & r \arrow[l, "f"] \end{tikzcd} \tag{3.14}$$

It follows from transitivity of the equivalence relation that $x \simeq y$ and therefore there exists a 2-simplex witnessing $g \circ i \simeq x$. From this 2-simplex together with the right-most 2-simplex in (3.14) we get a commutative square:

$$\begin{tikzcd} c \arrow[r, "h"] \arrow[d, "i"'] \arrow[dr, "x"] & r \arrow[d, "f"] \\ s \arrow[r, "g"'] & t \end{tikzcd}$$

Conversely, it is clear that every commutative square as in (3.12) gives rise to a diagram as in (3.13). One can check (although it is quite tedious) that these two assignments induce a bijection on equivalence classes of data as claimed. □

Remark 3.15. Assume $C = \mathsf{Spc}$ is the ∞-category of spaces. If you know something about the homotopy theory of spaces you can deduce from Example 3.11 that pullbacks in Spc can be computed as homotopy pullbacks of spaces (for example, in the Quillen model structure on simplicial sets). Exercise!

Commentary 3.16. The computation of limits in ordinary categories can be reduced to sets. Similarly, the computation of limits in ∞-categories can be reduced to spaces. For example, if you inspect the argument of Example 3.9 you'll see that it actually proves something stronger:

$$\mathrm{Map}_C(c, \prod_{i \in I} F(i)) \simeq \prod_{i \in I} \mathrm{Map}_C(c, F(i))$$

are (canonically) homotopy equivalent spaces. Similarly, Example 3.11 can be improved to show that

$$\mathrm{Map}_C(c, r \times_t s) \simeq \mathrm{Map}_C(c, r) \times_{\mathrm{Map}_C(c,t)} \mathrm{Map}_C(c, s),$$

where the right-hand side denotes the pullback in Spc (which, by Remark 3.15, is a homotopy pullback of spaces).

Both of these are instances of the following general statement.

Remark 3.17. Let $F\colon I \to C$ be a functor and (ℓ, η) a cone over F. For $c \in C$ we denote by $\mathrm{y}^c\colon C \to \mathcal{S}$ the corepresented functor (the image under the contravariant Yoneda embedding). Then (ℓ, η) is a limit cone if and only if for each $c \in C$, the induced cone $(\mathrm{y}^c(\ell), \mathrm{y}^c(\eta))$ is a limit cone over $\mathrm{y}^c \circ F$. Informally, we can summarize this by writing

$$\mathrm{Map}_C(c, \lim_i F(i)) \simeq \lim_i \mathrm{Map}_C(c, F(i)).$$

Note that this doesn't really make sense because we haven't constructed a functor $\mathrm{Map}_C(c, -)$ (but rather y^c). See Remark 2.67.

Remark 3.18. More formally, let $f\colon C \to D$ be a functor of ∞-categories.

1. Assume first that $F\colon I \to C$ admits a limit. We say that f *preserves* limits of F if given any limit cone (ℓ, η) over F, also the induced $(f(\ell), f(\eta))$ is a limit cone over $f \circ F$.
2. Now assume that $f \circ F\colon I \to D$ admits a limit. We say that f *reflects* limits of F if (ℓ, η) is a limit cone over F whenever $(f(\ell), f(\eta))$ is a limit cone over $f \circ F$.

The previous remark then says that the family of functors $\mathrm{y}^c\colon C \to \mathcal{S}$ (for $c \in C$) detects and reflects all limits that exist. This can be justified using a form of currying. For every space X there is an equivalence of spaces fitting into a commutative square

$$\begin{tikzcd}
\mathrm{Map}_{\mathsf{Spc}}(X, \mathrm{Map}_{C^I}(\underline{c}, F)) \arrow[r, "\sim"] & \mathrm{Map}_{\mathsf{Spc}^I}(\underline{X}, \mathrm{y}^c \circ F) \\
\mathrm{Map}_{\mathsf{Spc}}(X, \mathrm{Map}_C(c, \ell)) \arrow[r, "\sim"] \arrow[u] & \mathrm{Map}_{\mathsf{Spc}}(X, \mathrm{y}^c(\ell)) \arrow[u]
\end{tikzcd}$$

The left vertical arrow detects whether ℓ is a limit of F, and right vertical arrow detects whether $\mathrm{y}^c(\ell)$ is a limit of $\mathrm{y}^c \circ F$.

Commentary 3.19. Looking at Example 3.9 and Remark 3.15 you won't be surprised that limits in Spc can be computed as homotopy limits in the more classical setting of, for example, the Quillen model structure on simplicial sets. This is actually true more generally.

In any simplicial category C that is locally Kan there is a notion of homotopy limits and colimits. One can prove (with some work) that these notions recover limits (and colimits to be introduced below) in the associated ∞-category $N(C)$. This connects the discussion here with the more classical theory of homotopy (co)limits. It has two further consequences that we state separately:

Remark 3.20. Let C be a small ∞-category. Then the Yoneda embedding $\mathrm{y}\colon C \to \mathcal{P}(C)$ preserves all (small) limits that exist in C.

Proposition 3.21. *Both Spc and Cat_∞ admit all small limits and colimits. The inclusion $\mathsf{Spc} \hookrightarrow \mathsf{Cat}_\infty$ preserves all small limits and colimits.*

3.2 Colimits

Definition 3.22. The definition of a colimit is dual to the one of limit (Definition 3.4). Thus a *cone under* $F\colon I \to C$ is a pair (ℓ, η) where $\ell \in C$ and $\eta\colon F \to \underline{\ell}$ a natural transformation. It is a *colimit cone* if the induced map

$$\mathrm{Map}_C(\ell, c) \to \mathrm{Map}_{C^I}(F, \underline{c})$$

is a homotopy equivalence for each $c \in C$. In this case, we again write $\ell = \mathrm{colim}_I F = \mathrm{colim}_{i \in I} F(i)$. We also use familiar notation for coproducts, pushouts, initial objects etc.

The following examples are justified just as in the case of limits.

Example 3.23. An object $\ell \in C$ is initial iff $\mathrm{Map}_C(\ell, c)$ is contractible for all $c \in C$. More generally, $\mathrm{Map}_C(\coprod_i F(i), c) \simeq \prod_i \mathrm{Map}_C(F(i), c)$.

Example 3.24. For pushouts we have the formula

$$\mathrm{Map}_C(y \amalg_x z, c) \simeq \mathrm{Map}_C(y, c) \times_{\mathrm{Map}_C(x,c)} \mathrm{Map}_C(z, c)$$

where the right-hand side denotes the pullback in Spc. (In this generality, this follows from Remark 3.17 applied to pushouts in C^{op}. You should be able to justify the statement directly on π_0.)

Example 3.25. Let I be an ∞-category with final object $\infty \in I$. Then any functor $F\colon I \to C$ admits a colimit, namely $F(\infty)$.

This is because $\infty \in I$ is final iff the inclusion $\Delta^0 \xrightarrow{\infty} I$ is right-anodyne.[7] It follows that $\mathrm{Fun}(I, C) \xrightarrow{\mathrm{ev}_\infty} \mathrm{Fun}(\Delta^0, C) = C$ is a trivial Kan fibration hence

$$\mathrm{Map}_{C^I}(F, \underline{c}) \simeq \mathrm{Map}_C(F(\infty), c).$$

Exercise 3.26. Let $I = \mathrm{colim}_{j\in J} I_j$ be a colimit of simplicial sets. Show that the mapping spaces in C^I are limits, computed in sSet, of mapping spaces in the C^{I_j}. (These won't be *homotopy* limits in general, and will not therefore present limits in Spc.)

Exercise 3.27. Let $I \subseteq N(\mathbb{N})$ be the 1-skeleton, that is, the sub-simplicial set generated by the edges in $N(\mathbb{N})$. Equivalently,

$$I = \Delta^1 \amalg_{\Delta^0} \Delta^1 \amalg_{\Delta^0} \Delta^1 \cdots$$

so that a functor $F\colon I \to C$ is given by a countable family of composable morphisms in C,

$$d_0 \xrightarrow{f_1} d_1 \xrightarrow{f_2} d_2 \to \cdots .$$

Show that the sequential colimit, $\mathrm{colim}\, F$, can be computed as the coequalizer of the diagram

$$\amalg_n d_n \underset{\amalg f_n}{\overset{\mathrm{id}}{\rightrightarrows}} \amalg_n d_n \ .$$

Hint: One possibility is to use Exercise 3.26 to understand $\mathrm{Map}_{C^I}(-,-)$. Similarly, the coequalizer shape $\bullet \rightrightarrows \bullet$ is a quotient in simplicial sets of $\Delta^1 \amalg \Delta^1$. Again, one can then use Exercise 3.26.

Commentary 3.28. Our next goal is to generalize a criterion familiar from ordinary categories for when a category is *cocomplete*, that is, admits all (small, always) colimits. It is also useful to have a finite version of that. For this we say that the simplicial set I is *finite* if it has finitely many non-degenerate simplices. A finite (co)limit in C is a (co)limit of a functor $F\colon I \to C$ where I is finite.

Exercise 3.29. Characterize the 1-categories whose nerve is a finite simplicial set.

Proposition 3.30. *For an ∞-category C the following are equivalent:*

1. *C admits (finite) colimits.*
2. *C admits coequalizers and (finite) coproducts.*

[7] This is defined exactly as inner-anodyne, starting from *right* horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$, $0 < i \leq n$ instead of inner horn inclusions. This statement is not obvious but should be quite plausible. For example, show that the inclusion $2\colon \Delta^0 \to \Delta^2$ is right-anodyne.

3. *C admits pushouts and (finite) coproducts.*

Sketch of proof. It is clear that 1. implies 2. (and 3.) (in both the finite and the infinite version).

For 2.⇒3. you can translate a pushout

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"] & b \\ c & \end{tikzcd}$$

into a coequalizer

$$\begin{tikzcd} a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \amalg c \end{tikzcd}$$

assuming that finite coproducts exist. Conversely, a coequalizer of

$$\begin{tikzcd} x \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & y \end{tikzcd}$$

is translated into a pushout of

$$\begin{tikzcd} x \amalg y \arrow[r, "f \amalg \mathrm{id}"] \arrow[d, "g \amalg \mathrm{id}"'] & y \\ y & \end{tikzcd}$$

using finite coproducts. This shows 2.⇔3.

Let us say something about 2.,3.⇒1. and let us start with the finite version. We do induction on the dimension n of the simplicial set I (that is, the maximal dimension of the non-degenerate simplices). If $n = 0$ we're done since I is finite discrete and we assume C admits finite coproducts. For $n > 0$ one may write a pushout diagram

$$\begin{tikzcd} \amalg \partial\Delta^n \arrow[r] \arrow[d, hook] & I_{n-1} \arrow[d, hook] \\ \amalg \Delta^n \arrow[r] & I \end{tikzcd}$$

expressing the fact that I is obtained from its $(n-1)$-skeleton (that is the sub-simplicial set generated by the simplices in dimensions $\leq n-1$) by attaching some non-degenerate n-simplices. By induction, F restricted to the top right vertex and the top left vertex admits colimits in C. By Example 3.25 (and the existence of finite coproducts), so it does when restricted to the bottom left vertex. One would like to conclude that F admits a colimit given by the pushout in C of these colimits. This almost follows from Exercise 3.26. In fact, in this particular case the limit in Exercise 3.26 *is* a homotopy limit so one concludes.

For the infinite version one may write $I = \cup_n I_n$ as union of its n-skeleta. Then Exercise 3.27 together with infinite coproducts reduces to finite-dimensional I. One can now repeat the induction argument using infinite coproducts instead of finite ones. □

Exercise 3.31. Show that admitting finite colimits is also equivalent to having an initial object and pushouts.

Remark 3.32. Of course, the dual statements about limits is equally true. This follows by considering the opposite ∞-category.

Commentary 3.33. Proposition 3.30 offers another approach to showing that Spc and Cat_∞ are complete and cocomplete than the one in Proposition 3.21. Namely, showing that they admit (co)products and pushouts/pullbacks.

Commentary 3.34. Proposition 3.30 is more than a criterion to check for (co)completeness of an ∞-category. Its proof provides a recipe for computing arbitrary (co)limits in terms of the basic ones we have already studied in detail.

Remark 3.35. There is a relative version of Proposition 3.30: A functor between ∞-categories preserves (finite) colimits iff it preserves coequalizers and (finite) coproducts iff it preserves pushouts and (finite) coproducts.

3.3 Alternative definitions

Commentary 3.36. If you read the literature on limits and colimits you will probably see other definitions than the ones of Definitions 3.4 and 3.22. Here I will try to explain the relation between these alternatives (and indeed of alternatives to earlier notions too). This also allows me to introduce some of the ingredients that go into proving many of the results in this section, and indeed, of earlier sections too. However, only little here will be used in the sequel and if you're happy with the preceding discussion you can safely skip this and refer to it later if needed.

Commentary 3.37. Let us be given a functor $F\colon I \to C$ of ordinary categories. Denote by $I^{\triangleleft} = [0] \star I$ the join (Commentary A.1). Then an extension to $\overline{F}\colon I^{\triangleleft} \to C$ amounts to providing a cone over F. (Similarly, an extension to $\overline{F}\colon I^{\triangleright} := I \star [0] \to C$ amounts to providing a cone under F.) Note that this is equivalent to a natural transformation $\eta\colon \underline{\ell} \to F$ from a constant diagram to F.

In the ∞-categorical world these two approaches are not quite the same although equivalent. I invite you at this point to consult the appendix (Appendices A.1 and A.2) for the join construction and the slice construction in the ∞-categorical world.

Remark 3.38. Let $F\colon I \to C$ be a diagram in an ∞-category C. We now have three different notions of a cone over F:

1. The one of Definition 3.22, that is, a natural transformation $\eta\colon I \times \Delta^1 \to C$ such that $\eta|_1 = F$, $\eta|_0$ is constant.
2. A map $\overline{F}\colon I^{\triangleleft} \to C$ extending F.
3. An object of $C_{/F}$.

Exercise 3.39. Observe that there is a pullback diagram in simplicial sets:

$$\begin{tikzcd} \mathrm{Map}_{C^I}(\underline{\ell}, F) \arrow[r] \arrow[d] & C^{/F} \arrow[d] \\ \Delta^0 \arrow[r, "\ell"] & C \end{tikzcd}$$

Construction 3.40. Start with a cone (ℓ, η) over F as in Remark 3.38. By definition, η defines an object of $\mathrm{Map}_{C^I}(\underline{\ell}, F)$ and, via the functor of Exercise 3.39, one of $C^{/F}$. We identify this with a map $\tilde{\eta}\colon \Delta^0 \to C_{/F}$ via the equivalence $C_{/F} \xrightarrow{\sim} C^{/F}$ of Lemma A.19. By adjunction, this corresponds to a map $\bar{F}\colon I^{\triangleleft} \to C$ extending F. Note that this process is invertible. Starting with $\overline{F}$ we obtain $\tilde{\eta}$ and then (ℓ, η).

Continuing with this notation we then have:

Proposition 3.41. *The following are equivalent:*

1. *(ℓ, η) is a limit cone over F.*
2. *Restriction along $I \hookrightarrow I^{\triangleleft}$ induces a homotopy equivalence of spaces for every $c \in C$:*

$$\mathrm{Map}_{C^{I^{\triangleleft}}}(\underline{c}, \bar{F}) \to \mathrm{Map}_{C^I}(\underline{c}, F) \tag{3.42}$$

3. *The object $\tilde{\eta} \in C_{/F}$ is final.*

Sketch of proof. By construction, $\bar{F}|_I = F$, $\bar{F}(-\infty) = \ell$, and η extends to a natural transformation $\bar{\eta}\colon \underline{\ell} \to \underline{F}$ such that $\bar{\eta}(-\infty) = \mathrm{id}_\ell$. We then have a commutative diagram:[8]

$$\begin{tikzcd} \mathrm{Map}_C(c, \ell) \arrow[r] \arrow[dr] & \mathrm{Map}_{C^{I^{\triangleleft}}}(\underline{c}, \underline{\ell}) \arrow[r, "\bar{\eta}"] \arrow[d] & \mathrm{Map}_{C^{I^{\triangleleft}}}(\underline{c}, \overline{F}) \arrow[r, "\mathrm{ev}_{-\infty}", "\sim"'] \arrow[d] & \mathrm{Map}_C(c, \bar{F}(-\infty)) \\ & \mathrm{Map}_{C^I}(\underline{c}, \underline{\ell}) \arrow[r, "\eta"] & \mathrm{Map}_{C^I}(\underline{c}, F) & \end{tikzcd}$$

where the vertical arrows are induced by the inclusion $I \hookrightarrow I^{\triangleleft}$. We already observed in Example 3.25 that the right-most horizontal arrow is an equivalence. By construction, the composition of the arrows in the top row is also an equivalence. It follows that η is a limit cone iff the right-most vertical arrow is an equivalence. This shows the equivalence between the first two conditions.

The last condition, that is, $\tilde{\eta} \in C_{/F}$ being final holds iff the canonical map $C_{/\bar{F}} \cong (C_{/F})_{/\bar{F}} \to C_{/F}$ is (a trivial fibration, or equivalently,) an equivalence. (This requires justification.) Moreover, this can be tested on the fibers of the right fibrations to C. By Exercise 3.39, we identify the fiber over $c \in C$ with the map (3.42). This completes the (sketch of) proof. □

Commentary 3.43. By Example 3.10, an object d in an ∞-category D is final iff $\mathrm{Map}_D(c, d)$ is a contractible space for all $c \in D$. As we observed in the proof of Proposition 3.41, this is

[8] The middle horizontal arrows were only defined up to contractible choice. The statement is that there is a choice of these making the square commute.

also equivalent to the canonical map $D_{/d} \to D$ being a trivial fibration. Hence a different but equivalent way of defining limits of F is as objects in $C_{/F}$ that satisfy one of these equivalent conditions. In fact, this translation can be used to prove most of the results on (co)limits (although some of them still require much effort). We will give only one example of this, strengthening Exercise 3.7.

Lemma 3.44. *Let D be an ∞-category and define D^{fin} as the full subcategory spanned by the final objects. Then D^{fin} is either empty or a contractible space.*

Proof. By Exercise 2.26, D^{fin} is an ∞-category. (It then follows from Exercise 3.7 that D^{fin} is a space but we won't need that.) If D^{fin} is not empty and we are given a map $f\colon \partial\Delta^n \to D^{\mathrm{fin}}$ we need to extend it to Δ^n. If $n = 0$ we can do so because of non-emptyness. For $n > 0$ consider now the following lifting problem:

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "f"] \arrow[d] & D \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

Every such lift must factor through D^{fin} since $\partial\Delta^n$ contains all vertices of Δ^n. It is therefore enough to solve this lifting problem.

Noting that $\partial\Delta^n = \Lambda^n_n \amalg_{\partial\Delta^{n-1}} \Delta^{n-1}$ this lifting problem translates to

$$\begin{tikzcd} & \Lambda^n_n \arrow[d] \arrow[dr] & \\ \partial\Delta^{n-1} \arrow[ur] \arrow[dr] & \Delta^n \arrow[r, dotted] & D \\ & \Delta^{n-1} \arrow[u] \arrow[ur] & \end{tikzcd}$$

Finally, we observe that $\Lambda^n_n = \partial\Delta^{n-1} \star \Delta^0$ while $\Delta^n = \Delta^{n-1} \star \Delta^0$ so that this translates to

$$\begin{tikzcd} \partial\Delta^{n-1} \arrow[r] \arrow[d] & D_{/f(n)} \arrow[d] \\ \Delta^{n-1} \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

Since $f(n) \in D$ is final the right vertical map is a trivial Kan fibration hence the lifting problem admits a solution. ◻

Let $F\colon I \to C$ be a diagram in an ∞-category C and denote by $\mathrm{Limits}(F)$ the ∞-category of limits of F, namely,

$$\mathrm{Limits}(F) := (C_{/F})^{\mathrm{fin}}.$$

Corollary 3.45. *The ∞-category* $\mathrm{Limits}(F)$ *is either empty or a contractible space.*

Remark 3.46. Let I be a simplicial set. If I is not an ∞-category we can add fillers for all inner horns to make it more like an ∞-category. This might create new inner horns that are not filled so we repeat the process, and again, and again.... After countably many iterations, however, the resulting simplicial set $\tilde{I}$ is an ∞-category. (Exercise!) Moreover, every functor $I \to C$ to an ∞-category extends to a functor $\tilde{I} \to C$ in an essentially unique way. Namely, the functor

$$\mathrm{Fun}(\tilde{I}, C) \to \mathrm{Fun}(I, C) \tag{3.47}$$

is an equivalence.[9] So, for the theory developed here we could have restricted to ∞-categories I as indexing diagrams. As mentioned at the top of this section it is useful to allow I to be an arbitrary simplicial set, however, because I can be (much) smaller than $\tilde{I}$.

Note that if you want to make the process $I \mapsto \tilde{I}$ functorial it's better to add fillers for *all* inner horns, whether they already possess fillers or not.

3.4 Adjunctions

Definition 3.48. Let C, D be ∞-categories. An *adjunction* between C and D is a pair of functors $f: C \to D$ and $g: D \to C$ together with unit $\eta: \mathrm{id}_C \to gf$ and counit transformations $\epsilon: fg \to \mathrm{id}_D$ satisfying the usual triangle identities. We write $f \dashv g$ and say that f is *left adjoint* to g, g is *right adjoint* to f.

Commentary 3.49. To be absolutely clear, the triangle identities amount to 2-simplices

$$\begin{tikzcd} f \arrow[rr, "\mathrm{id}"] \arrow[dr, "\eta"'] & & f \\ & fgf \arrow[ur, "\epsilon"'] & \end{tikzcd} \qquad \begin{tikzcd} g \arrow[rr, "\mathrm{id}"] \arrow[dr, "\eta"'] & & g \\ & gfg \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

in $\mathrm{Fun}(C, D)$ and $\mathrm{Fun}(D, C)$, respectively.

It is clear that this recovers the notion of an adjunction between ordinary categories.

Exercise 3.50. Show that an adjunction gives rise to a homotopy equivalence of spaces

$$\mathrm{Map}_D(f(c), d) \xrightarrow{g} \mathrm{Map}_C(gf(c), g(d)) \xrightarrow{\eta^*} \mathrm{Map}_C(c, g(d)) \tag{3.51}$$

for all $c \in C$, $d \in D$. (Of course, the dual with the counit instead of the unit is equally true.)

Remark 3.52. The converse of Exercise 3.50 is also true but harder to prove. In fact, even less data is needed to produce an adjunction.

More precisely, suppose we are given a functor $g: D \to C$. To produce an adjunction $f \dashv g$, it is enough to give, for each $c \in C$ an object $f(c) \in D$, and a morphism $c \to g(f(c))$ such that (3.51) is an equivalence for all $d \in D$. Informally, it is enough to give the left adjoint f and the natural transformation η on 0-simplices. More formally, the claim is that there is a

[9] In fact, these statements can be improved: $I \to \tilde{I}$ is an inner anodyne map hence (3.47) is a trivial fibration.

functor $f\colon C \to D$ sending c to $f(c)$ such that $f \dashv g$. Moreover, the unit natural transformation $\eta\colon \mathrm{id}_C \to gf$ of the adjunction can be chosen homotopic to the given maps.

We will not prove this as it requires straightening-unstraightening techniques. (For ordinary categories you can easily convince yourself of the validity of this criterion.)

Example 3.53. Given an ∞-category C, associating its core $C^{\simeq}$ assembles into a simplicial functor $qCat \to Kan$ (check!) and the inclusion $C^{\simeq} \hookrightarrow C$ provides the counit of an adjunction between simplicial categories:

$$\begin{tikzcd} Kan \arrow[r, bend left, "\mathrm{incl}"] \arrow[r, phantom, "\perp"] & qCat \arrow[l, bend left, "(-)^{\simeq}"] \end{tikzcd}$$

(The inclusion $Kan \hookrightarrow qCat$ is fully faithful so the unit transformation is the identity.) Since the triangle identities pass through the homotopy coherent nerve we obtain an adjunction of ∞-categories:

$$\begin{tikzcd} \mathsf{Spc} \arrow[r, bend left, "\mathrm{incl}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat}_\infty \arrow[l, bend left, "(-)^{\simeq}"] \end{tikzcd}$$

Example 3.54. Given an ∞-category C there is a functorial way to add inverses to all morphisms. One way to achieve this is as follows. Let J be the walking isomorphism (that is, the ordinary category with two objects, say 0 and 1, and singleton hom-sets, viewed as an ∞-category through its nerve) with the obvious inclusion $\Delta^1 \hookrightarrow J$. Consider then the following canonical pushout in sSet:

$$\begin{tikzcd} \amalg_{C_1}\Delta^1 \arrow[r] \arrow[d] & C \arrow[d] \\ \amalg_{C_1} J \arrow[r] & D \end{tikzcd}$$

D might not be an ∞-category yet, but we can make it into one by adding fillers $D \to \tilde{D}$ (see Remark 3.46). It remains to check that $\mathrm{h}(\tilde{D})$ is a groupoid. It is a fact that inner anodyne maps induce equivalences of homotopy categories so it suffices to show that $\mathrm{h}(D)$ is a groupoid. But the functor $\mathrm{h}\colon \mathsf{sSet} \to \mathsf{Cat}$ is a left adjoint and hence preserves pushouts. It is now easy to see from the pushout in Cat,

$$\begin{tikzcd} \amalg_{C_1}[1] \arrow[r] \arrow[d] & \mathrm{h}(C) \arrow[d] \\ \amalg_{C_1} J \arrow[r] & \mathrm{h}(D) \end{tikzcd}$$

that $\mathrm{h}(D)$ is a groupoid.

We now claim that $C \mapsto \tilde{D}$ extends to a left adjoint to the inclusion $\mathsf{Spc} \hookrightarrow \mathsf{Cat}_\infty$, and to prove this we apply Remark 3.52. The only thing left to prove is that the composite

$$\mathrm{Map}_{\mathsf{Spc}}(\tilde{D}, X) \to \mathrm{Map}_{\mathsf{Cat}_\infty}(\tilde{D}, X) \to \mathrm{Map}_{\mathsf{Cat}_\infty}(C, X)$$

is a homotopy equivalence for each $C \in \mathsf{Cat}_\infty$ and $X \in \mathsf{Spc}$. From Corollary 2.54 we know that the first map is an equivalence. Recall that the last two mapping spaces are given (up to canonical homotopy equivalence) by $\mathrm{Fun}(-,X)^{\simeq} = \mathrm{Fun}(-,X)$ since X is a space. We are therefore reduced to show $\mathrm{Fun}(\tilde{D},X) \to \mathrm{Fun}(C,X)$ is an equivalence. But this factors as

$$\mathrm{Fun}(\tilde{D},X) \to \mathrm{Fun}(D,X) \to \mathrm{Fun}(C,X)$$

where the first map is a trivial fibration as $D \to \tilde{D}$ is inner anodyne. Using the definition of D as a pushout it is easy to reduce to showing that

$$\mathrm{Fun}(J,X) \to \mathrm{Fun}(\Delta^1,X) \tag{3.55}$$

is a trivial fibration. Now, it's certainly true that the 1-category J is obtained from $[1]$ by inverting the map $0 \to 1$ so if X was an ordinary groupoid then (3.55) would actually be an isomorphism. However, for an ∞-groupoid like our X, the inverse of a morphism is not unique so the statement is less obvious. Turning the idea that it is unique up to contractible choice into a proof requires some work that we omit.

Commentary 3.56. To sum up the previous two examples, the inclusion $\mathsf{Spc} \hookrightarrow \mathsf{Cat}_\infty$ admits both a left and a right adjoint, given by 'inverting all morphisms' and 'taking the core'.

Proposition 3.57. *For an ∞-category C the following are equivalent:*

1. *C admits I-shaped colimits.*
2. *The constant functor $\underline{(-)}\colon C \to C^I$ admits a left adjoint $\mathrm{colim}_I\colon C^I \to C$ such that $\mathrm{colim}_I(F)$ is a colimit of F, for every $F \in C^I$.*

Proof. We apply Remark 3.52. We take, given any $F\colon I \to C$ the object $\mathrm{colim}_I F \in C$ together with the colimit cone $F \to \underline{\mathrm{colim}_I F}$. Note how then (3.51) translates precisely into the condition of this being a colimit cone.

The reverse direction is easy and left as an exercise. □

Remark 3.58. Similarly, C admitting I-shaped limits is equivalent to the constant functor having a right adjoint $\lim_I$.

Example 3.59. Let C be an ∞-category and let G be a finite group, viewed as an ∞-category BG. (That is, it has a single object and morphisms given by the elements of G.) The ∞-category $\mathrm{Fun}(BG,C)$ is the ∞-category of G-objects in C: an object c in C together with an isomorphism $\overline{g}\colon c \xrightarrow{\sim} c$ for each $g \in G$, a homotopy $\overline{(g_1g_2)} \simeq \overline{g_1} \circ \overline{g_2}$ for each $g_1, g_2 \in G$, and higher coherences. Assume C admits BG-shaped limits and colimits. The left and right adjoints to the constant functor $C \to \mathrm{Fun}(BG,C)$ are called

(homotopy) orbits $(-)_{hG}\colon \mathrm{Fun}(BG,C) \to C$, *(homotopy) fixed points* $(-)^{hG}\colon \mathrm{Fun}(BG,C) \to C$,

respectively.

Proposition 3.60. *Let I, K be simplicial sets and C an ∞-category that admits I-shaped (co)limits. Then $\mathrm{Fun}(K, C)$ also admits I-shaped (co)limits and these are computed pointwise. That is, the family of functors, for $k \in K$,*

$$\mathrm{ev}_k \colon \mathrm{Fun}(K, C) \to C$$

preserves and reflects I-shaped (co)limits.

Proof. For the first statement (say, with colimits) we need to show, by Proposition 3.57, that the constant functor $\underline{(-)} \colon \mathrm{Fun}(K, C) \to \mathrm{Fun}(I, \mathrm{Fun}(K, C))$ has a left adjoint. We then take the functor

$$\mathrm{Fun}(I, \mathrm{Fun}(K, C)) \cong \mathrm{Fun}(K, \mathrm{Fun}(I, C)) \xrightarrow{\mathrm{Fun}(K, \mathrm{colim}_I)} \mathrm{Fun}(K, C)$$

and this is left adjoint to $\mathrm{Fun}(K, \underline{(-)})$ since the unit and counit transformations for $\mathrm{colim}_I \dashv \underline{(-)}$ induce unit and counit transformations after applying $\mathrm{Fun}(K, -)$.

The second statement follows from the description of this left adjoint. □

We now come to a (almost: the) fundamental property of adjoint functors.

Proposition 3.61. *Let $f \colon C \to D$ be a left (resp. right) adjoint. Then f preserves all colimits (resp. limits) that exist in C.*

Commentary 3.62. One would like to argue as follows (for colimits and left adjoints):

$$\begin{aligned}
\mathrm{Map}_D(f(\operatorname{colim}_i F(i)), d) &\simeq \mathrm{Map}_C(\operatorname{colim}_i F(i), g(d)) \\
&\simeq \lim_i \mathrm{Map}_C(F(i), g(d)) \\
&\simeq \lim_i \mathrm{Map}_D(f(F(i)), d) \\
&\simeq \mathrm{Map}_D(\operatorname{colim}_i f(F(i)), d)
\end{aligned}$$

As noted in Remark 2.67, this doesn't really make sense since the mapping spaces are not functors. There are ways to circumvent that (with functorial replacements) but we go down another route for the proof instead.

Proof. Assume given a diagram $F \colon I \to C$ which admits a colimit, represented by the diagram $\overline{F} \colon I^{\triangleright} \to C$. (The case of limits is dual.) By Proposition 3.41, we need to show that the restriction map induces a homotopy equivalence of spaces:

$$\mathrm{Map}_{D^{I^{\triangleright}}}(f_* \overline{F}, \underline{d}) \xrightarrow{\sim} \mathrm{Map}_{D^I}(f_* F, \underline{d}) \tag{3.63}$$

for each $d \in D$. As observed in the previous proof, the functor $f_* := f \circ - \colon C^K \to D^K$ is left adjoint to $g_* \colon D^K \to C^K$. From this you can check that (3.63) is homotopic to

$$\mathrm{Map}_{C^{I^{\triangleright}}}(\overline{F}, g_* \underline{d}) \xrightarrow{\sim} \mathrm{Map}_{C^I}(F, g_* \underline{d}). \tag{3.64}$$

But $g_* \underline{d} = \underline{g(d)}$ so the claim follows again from Proposition 3.41. □

Remark 3.65. Together with Commentary 3.56 this implies that the inclusion $\mathsf{Spc} \hookrightarrow \mathsf{Cat}_\infty$ preserves all limits and colimits. Something we already saw in Proposition 3.21.

Remark 3.66. We will not prove this here, but if $f \dashv g$ then:

1. the left adjoint f is fully faithful iff the unit $\mathrm{id} \to gf$ is an equivalence;
2. the right adjoint g is fully faithful iff the counit $fg \to \mathrm{id}$ is an equivalence.

4 Presentability

Commentary 4.1. We just saw in Proposition 3.61 that left adjoint functors preserve all colimits that exist in the domain, generalizing the same statement from ordinary category theory. There, 'the' adjoint functor theorem gives a partial converse: Under some assumptions on the categories involved, *every* colimit preserving functor is left adjoint.

In this section we are going to see an analogue of this statement for ∞-categories: For functors between *presentable* ∞-categories, being left adjoint and preserving colimits are equivalent conditions. This version of the adjoint functor theorem (there are more general ones, but we won't discuss these) is more powerful than in the context of ordinary categories. This is because constructing a functor 'by hand' is impossible. It also explains to some extent the importance of presentability in the theory of ∞-categories.

We will also discuss some related topics, starting with the promised universal property of the presheaf category.

4.1 Cocompletion

Commentary 4.2. In Commentary 2.1 we already noted that the presheaf category in ordinary category theory can be viewed as the *free cocompletion* of a small category. Now, we want to prove the same for ∞-categories. To make sense of it we use the combination of Propositions 3.21 and 3.60:

Corollary 4.3. *Let C be a small ∞-category. Then $\mathcal{P}(C)$ admits all (small) limits and colimits and these are computed pointwise.*

Convention 4.4. Let C, D be ∞-categories that are cocomplete. We denote by $\mathrm{Fun}^L(C, D)$ the full subcategory of $\mathrm{Fun}(C, D)$ spanned by the colimit preserving functors. (The superscript 'L' anticipates the adjoint functor theorem. If C and D are presentable, these are precisely the left adjoint functors.)

We can now state the main result of this subsection.

Theorem 4.5. *Let C be a small ∞-category, and let D be a cocomplete ∞-category. Then the restriction induces an equivalence:*

$$\mathrm{Fun}^L(\mathcal{P}(C), D) \xrightarrow{\sim} \mathrm{Fun}(C, D)$$

In other words, $\mathcal{P}(C)$ is the free cocompletion *of C.*

Commentary 4.6. Let C be an ordinary small category and $P(C)$ the category of presheaves of sets on C. Let us recall one way to prove the analogous statement in this context.

1. Every presheaf $H \cong \operatorname{colim}_{c\in C_{/H}} \mathrm{y}_c \in P(C)$ is a canonical colimit of representables over its *category of elements* $C_{/H}$. (See below.)
2. It follows that every colimit preserving functor $f\colon P(C) \to D$ is uniquely determined by its restriction to $f'\colon C \to D$. And conversely, given f', the formula $H \mapsto \operatorname{colim}_{c\in C_{/H}} f'(c)$ defines a colimit preserving functor $f\colon P(C) \to D$.

Both of these steps generalize to ∞-categories but they require quite a bit more work. We will do some of this work also in order to introduce an important technique in a situation where, hopefully, intuition is somewhat easier to come by. I invite you to have a look at Appendix A.3 for an introduction to the discrete straightening-unstraightening.

Commentary 4.7. To prove 1 we evaluate at some $c \in C$:

$$r\colon \operatorname*{colim}_{x'\in H(c')} \operatorname{Hom}_C(c, c') \to H(c) \tag{4.8}$$

which sends $f\colon c \to c'$ over x' to $H(f)(x') \in H(c)$. One then proceeds as follows:

- There is an obvious section s to (4.8) which sends $x \in H(c)$ to id_c over x.
- To show that the composite sr is also the identity start with $f\colon c \to c'$ over x'. Then $sr(f) = \mathrm{id}_c$ over $H(f)(x')$. But in the colimit these two elements are identified and we win.

I invite you to translate this proof into the language of discrete fibrations in order to view the analogy with the following argument in the ∞-categorical context.

Lemma 4.9. *Let $H \in \mathcal{P}(C)$. The canonical functor $(C_{/H})^{\triangleright} \to \mathcal{P}(C)$ exhibits H as a colimit of representables.*

Proof. By Corollary 4.3, it is enough to check that the induced map

$$(C_{/H})^{\triangleright} \to \mathcal{P}(C) \xrightarrow{\mathrm{ev}_c} \mathsf{Spc} \tag{4.10}$$

is a colimit diagram, for each $c \in C$. By the Yoneda lemma, evaluating at c is equivalent to mapping out of y_c, and as explained in Appendix A.3, the corresponding left fibration is given by $\mathcal{P}(C)_{c/} \to \mathcal{P}(C)$.[10] It follows that the functor (4.10) classifies the left fibration $E = (C_{/H})^{\triangleright} \times_{\mathcal{P}(C)} \mathcal{P}(C)_{c/}$ over $(C_{/H})^{\triangleright}$. In the language of left fibrations, being a colimit diagram translates to the inclusion

$$E^0 = C_{/H} \times_C C_{c/} \hookrightarrow E$$

[10]The ∞-category $\mathcal{P}(C)$ is not small but this isn't a problem. Either generalize the (un)straightening equivalence to fibrations with 'small fibers' or pass to a larger universe.

being a weak homotopy equivalence.[11] To see this we will show that both sides deformation retract onto $E^1 = C_{/H} \times_C \{\mathrm{id}_c\} \subseteq E^0$. (Note that $E^1 \simeq H(c)$ so this is analogous to the proof strategy in the 1-categorical situation before.) The idea is quite simple. An object $e \in E^0$, say, consists of a pair $e = (c' \to H, c \to c')$ and we may compose the two morphisms to get an object $(c \to c' \to H, \mathrm{id}_c) \in E^1$. This will give the retraction $r \colon E^0 \to E^1$. Noting that there is a canonical map $r(e) \to e$ in E^0,

$$(c \to c' \to H, \mathrm{id}_c) \to (c' \to H, c \to c'),$$

we expect to be able to construct a natural transformation $E^0 \times \Delta^1 \to E^0$ from $r \to \mathrm{id}_{E^0}$. (The situation for E is similar.) The actual construction of this natural transformation is, however, not as straightforward as one would hope. At least, I haven't found a slick argument. I suggest that, at least on first reading, you skip the following exercise that should produce the natural transformation. If you know a better argument, please let me know! □

Exercise 4.11.

1. Construct a functor $F \colon \mathcal{P}(C)_{/H} \times_{\mathcal{P}(C)} \mathcal{P}(C)_{c/} \to \mathcal{P}(C)^{\Delta^2}$ such that $\mathrm{ev}_0 \circ F = c$ and $\mathrm{ev}_2 \circ F = H$. It should send the object e in the proof above to a composite of $c \to c'$ and $c' \to H$. *Hint*: Replace the slices by their fat analogues (Lemma A.19) and use that for any ∞-category D, the canonical map $\mathrm{Fun}(\Delta^2, D) \to \mathrm{Fun}(\Lambda^2_1, D)$ admits a section (Lemma 2.14).
2. For any ∞-category D, construct a functor $D^{\Delta^2} \times \Delta^1 \to D^{\Delta^1}$ based on the idea that in a 2-simplex σ, the second face $d_2(\sigma)$ can be seen as a map $d_1(\sigma) \to d_0(\sigma)$. *Hint*: This functor is induced (by adjunction) from a map of simplicial sets $\Delta^1 \times \Delta^1 \to \Delta^2$.
3. Observe that there is a canonical functor

   $$\mathcal{P}(C)_{/H} \times_{\mathcal{P}(C)} \mathcal{P}(C)_{c/} \times \Delta^1 \to \mathcal{P}(C)_{c/} \times \Delta^1 \to \mathcal{P}(C)_{c/},$$

   first projection, and then the deformation retraction onto the initial object id_c.
4. Combine the previous points to get a functor

   $$\mathcal{P}(C)_{/H} \times_{\mathcal{P}(C)} \mathcal{P}(C)_{c/} \times \Delta^1 \to \mathcal{P}(C)_{/H} \times_{\mathcal{P}(C)} \mathcal{P}(C)_{c/}.$$

5. Restrict to suitable full subcategories to obtain the maps $E \times \Delta^1 \to E$ and $E^0 \times \Delta^1 \to E^0$ as claimed in the proof above.

Sketch of proof of Theorem 4.5. Let $f' \colon C \to D$ be an arbitrary functor. We wish to extend it to a colimit preserving functor $f \colon \mathcal{P}(C) \to D$. By Lemma 4.9 we are forced to set, for $H \in \mathcal{P}(C)$, $f(H) = \mathrm{colim}_{c \in C_{/H}} f'(c)$. In ordinary categories you would now argue that for $H \to H'$ there is a canonical map $f(H) \to f(H')$ by the definition of colimits. And using this canonicity, it can be shown to define a functor.

[11] A map $f \colon D \to D'$ between ∞-categories is a weak homotopy equivalence if the map between the associated Kan complexes (Example 3.54) is a (weak) homotopy equivalence. (Equivalently, it induces a (weak) homotopy equivalence on geometric realizations.) Such a map is not necessarily an equivalence of ∞-categories. (For example, consider the inclusion $\Delta^1 \hookrightarrow J$ of Example 3.54.) Here we use the easily verified fact that the inclusion of a deformation retract is a weak homotopy equivalence.

It turns out[12] that such an inductive construction of f also works for ∞-categories. With two caveats: You need a refined version of colimits and the canonicity of the simplices chosen in this process. This is provided by the theory of Kan extensions. And you need to choose the order of the simplices on which you successively define f rather carefully.

Once we have constructed such extensions, we may apply Remark 3.52 to show that $f \mapsto f'$ underlies a functor

$$\mathrm{y}_!\colon \operatorname{Fun}(C, D) \to \operatorname{Fun}^L(\mathcal{P}(C), D)$$

which is left adjoint to y^*, restriction along y. (This also uses some properties of Kan extensions.) By construction, the unit $\mathrm{id} \to \mathrm{y}^* \mathrm{y}_!$ is an equivalence, and the last thing to check is that the counit $\mathrm{y}_! \mathrm{y}^*(f) \to f$ is an equivalence whenever f is colimit preserving. This follows from Lemma 4.9 again. □

Corollary 4.12. *For any cocomplete D, there is a canonical equivalence*

$$\operatorname{Fun}^L(\mathsf{Spc}, D) \simeq D$$

given by evaluating at Δ^0. In other words, Spc is the free cocompletion of a point.

4.2 Ind objects

Commentary 4.13. Recall that an ordinary category C is filtered if

- for any finite[13] collection of objects $c_1, \ldots, c_n \in C$ there is $d \in C$ and morphisms $c_i \to d$,
- for any finite collection of parallel arrows $f_1, \ldots, f_n\colon c \to c'$ there exists $h\colon c' \to d$ such that $hf_i = hf_j$ for all i, j.

Exercise 4.14. Show that C is filtered iff for each $F\colon I \to N(C)$ from a finite simplicial set I there exists an extension $\bar{F}\colon I^{\triangleright} \to C$.

We can then take this as our definition of filtered ∞-categories.

Definition 4.15. An ∞-category C is called *filtered* if for any finite simplicial set I and any $F\colon I \to C$ there exists an extension $\bar{F}\colon I^{\triangleright} \to C$. A *filtered colimit* is one indexed by a filtered ∞-category.

Commentary 4.16. Following up on Remark 3.46 we could define an arbitrary simplicial set I to be filtered if there exists a categorical equivalence $I \to C$ to a filtered ∞-category. (Here, categorical equivalence means that for any ∞-category D, $\operatorname{Fun}(C, D) \xrightarrow{\sim} \operatorname{Fun}(I, D)$.) We won't have the need for this generality. Note, incidentally, that if $C \to C'$ is an equivalence of ∞-categories then C is filtered iff C' is.

Definition 4.17. Let C be a small ∞-category. We define the full subcategory $\operatorname{Ind}(C) \subseteq \mathcal{P}(C)$ spanned by right fibrations $E \to C$ such that E is filtered. Equivalently, spanned by those

[12] maybe quite surprisingly, given that I've been emphasizing the impossibility of constructing ∞-categories and functors between them "by hand" or "one simplex at a time"…

[13] possibly empty

presheaves H such that $C_{/H}$ is filtered. It is called the *Ind-completion* of C, and its objects are called Ind-objects. ("Ind" stands for "inductive" which is an old term for "filtered".)

Example 4.18. Let $c \in C$. Then $C_{/c}$ has a final object id_c. It is clear that every ∞-category with a final object is filtered. We conclude that all representable presheaves on C belong to $\mathrm{Ind}(C)$. In other words, the Yoneda embedding factors through

$$C \hookrightarrow \mathrm{Ind}(C).$$

Remark 4.19. By Lemma 4.9, we see that every Ind-object is a filtered colimit of representables. In fact, the converse is true as well: Any filtered colimit of representables in $\mathcal{P}(C)$ belongs to $\mathrm{Ind}(C)$. This follows from the fact that $\mathrm{Ind}(C) \subseteq \mathcal{P}(C)$ is closed under filtered colimits. (That, in turn, can be reduced to homotopy colimits in certain model categories of simplicial sets, and further to 1-categorical colimits. Eventually, this boils down to the observation that a filtered colimit of filtered 1-categories is filtered.)

We can now state the 'filtered' analogue of Theorem 4.5.

Corollary 4.20. *Let C be a small ∞-category and let D be an ∞-category that admits filtered colimits. Then the canonical restriction functor is an equivalence:*

$$\mathrm{Fun}^{\omega}(\mathrm{Ind}(C), D) \xrightarrow{\sim} \mathrm{Fun}(C, D)$$

(Here, the superscript refers to the fact that we only consider functors that preserve filtered colimits.) In other words, $\mathrm{Ind}(C)$ *is obtained from C by freely adjoining filtered colimits.*

Proof. Let $f' : C \to D$ be an arbitrary functor. There exists a fully faithful embedding $D \hookrightarrow D'$ which preserves and reflects all small colimits, and such that D' admits all small colimits. (For example, let SPC be the ∞-category of small spaces with respect to a larger universe, so that D becomes small. Set $D' = \mathrm{Fun}(D, \mathsf{SPC})^{\mathrm{op}}$.) By Theorem 4.5, we obtain a colimit preserving extension $f : \mathcal{P}(C) \to D'$ that we may restrict to a functor $\mathrm{Ind}(C) \to D'$. By our assumption, this factors through a functor $\mathrm{Ind}(C) \to D$ that preserves filtered colimits.

The rest of the proof is similar to the one of Theorem 4.5. □

4.3 Compactness

Commentary 4.21. The representable objects in $\mathrm{Ind}(C)$ share a covenient property. Namely, for each $c \in C$, the functor $\mathrm{Ind}(C) \to \mathsf{Spc}$ corepresented by y_c preserves filtered colimits. Indeed, this functor is the composite $\mathrm{Ind}(C) \hookrightarrow \mathcal{P}(C) \to \mathsf{Spc}$ of the canonical inclusion followed by evaluation at c. The inclusion preserves filtered colimits by Remark 4.19, and the second preserves all colimits, by Corollary 4.3. We take this as motivation to introduce the following notion.

Definition 4.22. Let D be an ∞-category which admits filtered colimits. An object $d \in D$ is called *compact* if the functor $D \to \mathsf{SPC}$ corepresented by d preserves filtered colimits.

Commentary 4.23. Here we passed to a larger universe so that D becomes small. SPC is the ∞-category of small spaces in that larger universe. The definition is independent of the choice of such an universe.

Exercise 4.24. (In contrast to our usual terminology, in this exercise we consider the 1-category of topological spaces.) Let X be a fixed topological space and consider the category (poset) C of open subsets of X. Then an object $U \in C$ is compact in the sense of Definition 4.22 iff it is (quasi-)compact in the usual sense: every open cover admits a finite subcover.

Example 4.25. As mentioned above, all representables are compact in $\mathrm{Ind}(C)$.

Before giving more examples we need one result. For this recall the following notion.

Definition 4.26. Let C be an ∞-category. A *retraction diagram* in C is a 2-simplex of the form:

$$\begin{tikzcd} & d \arrow[dr, "r"] & \\ c \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_c"'] \arrow[urr, phantom, "\sigma"] & & c \end{tikzcd}$$

If such a retraction diagram exists we say that c is a *retract* of d.

Proposition 4.27. *Let C be an ∞-category that admits filtered colimits. The full subcategory $C^\omega \subseteq C$ of compact objects is closed under finite colimits and retracts that exist in C.*

Commentary 4.28. We will not give a proof of this proposition but try to make it plausible. For this note that, in Spc, filtered colimits and finite limits commute. This can be reduced, via Commentary 3.19, to a question about homotopy colimits and homotopy limits in the Quillen model category on simplicial sets. And then to 1-categorical colimits and limits in sSet. Eventually, this boils down to the fact that filtered colimits and finite limits commute in Set.

So, informally, given a finite diagram $i \mapsto b_i \in C$ of compact objects and a filtered diagram $j \mapsto c_j \in C$, we have

$$\begin{aligned} \mathrm{Map}_C(\operatorname*{colim}_i b_i, \operatorname*{colim}_j c_j) &\simeq \lim_i \mathrm{Map}_C(b_i, \operatorname*{colim}_j c_j) && \text{Remark 3.17} \\ &\simeq \lim_i \operatorname*{colim}_j \mathrm{Map}_C(b_i, c_j) && \text{compactness of } b_i \\ &\simeq \operatorname*{colim}_j \lim_i \mathrm{Map}_C(b_i, c_j) && \text{observation in first paragraph} \\ &\simeq \operatorname*{colim}_j \mathrm{Map}_C(\operatorname*{colim}_i b_i, c_j) && \text{Remark 3.17,} \end{aligned}$$

"showing" that the colimit of F is compact too.

If c is a retract of d then $\mathrm{Map}_C(c, -)$ is a retract of $\mathrm{Map}_C(d, -)$. If the latter preserves I-shaped colimits then so does the former.

Example 4.29. An object $X \in \mathsf{Spc}$ is compact iff it is a retract of a finite space. Here, a finite space is an object of the smallest full subcategory containing the final object $*$ and closed under finite colimits.

Lemma 4.30. *Let D be an ∞-category that admits filtered colimits and let $f\colon C \to D$ be a fully faithful functor. If f factors through D^ω then the extension $F\colon \mathrm{Ind}(C) \to D$ (provided by Corollary 4.20) remains fully faithful.*

Commentary 4.31. Again, we only try to make this plausible. Let $X, X' \in \mathrm{Ind}(C)$. By Remark 4.19, we may write $X = \mathrm{colim}_i\, c_i$, $X' = \mathrm{colim}_j\, c'_j$, as filtered colimits of representables. Informally, we then have

$$\begin{aligned}
\mathrm{Map}_D(F(X), F(X')) &\simeq \mathrm{Map}_D(\operatorname*{colim}_i f(c_i), \operatorname*{colim}_j f(c'_j)) && \\
&\simeq \lim_i \mathrm{Map}_D(f(c_i), \operatorname*{colim}_j f(c'_j)) && \text{Remark 3.17} \\
&\simeq \lim_i \operatorname*{colim}_j \mathrm{Map}_D(f(c_i), f(c'_j)) && f(c_i) \text{ compact} \\
&\simeq \lim_i \operatorname*{colim}_j \mathrm{Map}_C(c_i, c'_j) && f \text{ fully faithful} \\
&\simeq \mathrm{Map}_C(X, X') && \text{as above}
\end{aligned}$$

Corollary 4.32. *Let C be a small ∞-category. Then $\mathcal{P}(C)^\omega$ is also a small ∞-category. Moreover, the canonical functor*

$$\mathrm{Ind}(\mathcal{P}(C)^\omega) \to \mathcal{P}(C) \tag{4.33}$$

is an equivalence.

Sketch of proof. The ∞-category $\mathcal{P}(C)$ is locally small. (This shouldn't be surprising: for a representable $c \in C$ and $H \in \mathcal{P}(C)$ we have $\mathrm{Map}_{\mathcal{P}(C)}(c, H) \simeq H(c)$ which is small. And a general presheaf is a (small) colimit of representables (Lemma 4.9) which translates, on mapping spaces, into a (small) limit of small spaces hence remains small.) For the first statement it therefore suffices to show the set of isomorphism classes of objects in $\mathcal{P}(C)^\omega$ is small. Similarly to Example 4.29, one can show that every compact object in $\mathcal{P}(C)$ is a retract of a finite colimit of representables. And there are indeed only a small set of these, up to isomorphism.

By Lemma 4.30, the functor (4.33) is fully faithful. Let $H \in \mathcal{P}(C)$ be an arbitrary presheaf. By Lemma 4.9, H is the colimit of a diagram $K \to C \xrightarrow{\mathrm{y}} \mathcal{P}(C)$. Writing $K = \cup_{i \in I} K_i$ as the union of its finite sub-simplicial sets one shows that H is the filtered colimit of finite colimits of representables. The latter belong to $\mathcal{P}(C)^\omega$ and we conclude that H belongs to the essential image of (4.33), finishing the proof. □

Commentary 4.34. In the terminology to be introduced below this says that $\mathcal{P}(C)$ is *compactly generated*. You see here already an important property of compactly generated ∞-categories (and more generally of presentable ∞-categories): While the Ind-category apriori only admits filtered colimits, in this case it actually has all colimits and limits. (Since it is equivalent to $\mathcal{P}(C)$.) In fact, this is a general phenomenon: If C is a small ∞-category that admits finite colimits then $\mathrm{Ind}(C)$ is complete and cocomplete.

Warning 4.35. Let C be a small ∞-category that admits finite colimits. We just mentioned that $\mathrm{Ind}(C)$ is cocomplete. Nevertheless $\mathrm{Ind}(C) \neq \mathcal{P}(C)$. The latter is the *free* cocompletion of C. That is, it does not 'remember' the finite colimits in C.

Rather, the fully faithful embedding $\mathrm{Ind}(C) \hookrightarrow \mathcal{P}(C)$ has a left adjoint $\mathcal{P}(C) \to \mathrm{Ind}(C)$ (which is the unique colimit preserving extension of the canonical inclusion $C \hookrightarrow \mathrm{Ind}(C)$) that we may interpret as 'enforcing' these finite colimits in C. This is an example of a localization that we discuss below.

4.4 Compact generation

Commentary 4.36. All that has be said in Sections 4.2 and 4.3 admits size-theoretic variants. The ideas remain exactly the same but the greater generality allows for many more examples of interest to be covered. Moreover, the theory becomes in some sense better if one considers 'all size variants' together. So, we indicate this generalization.

Remark 4.37. Let κ be a regular cardinal.[14] A set is called *κ-small* if its cardinality is less than κ. We may then speak of a *κ-small simplicial set*: one whose set of non-degenerate simplices is κ-small. Note that for $\kappa = \omega$ we recover the notion of a finite simplicial set.

You will have no trouble defining the following generalizations accordingly:

ω	κ
finite	κ-small
filtered	κ-filtered
$\mathrm{Ind}(C)$	$\mathrm{Ind}_\kappa(C)$
compact	κ-compact
D^ω	D^κ

Whenever it makes sense, the statements in Sections 4.2 and 4.3 remain true, with ω replaced by κ. More precisely, the 'κ-analogues' of Remark 4.19, Corollary 4.20, Proposition 4.27, Lemma 4.30, and Corollary 4.32 are also true.

We now come to the main definition of this section.

Definition 4.38. Let κ be a regular cardinal. An ∞-category C is called *κ-compactly generated* if there exists a small ∞-category C' that admits κ-small colimits, and an equivalence $C \simeq \mathrm{Ind}_\kappa(C')$. (For $\kappa = \omega$ we also just say *compactly generated*.) An ∞-category is *presentable* if it is κ-compactly generated for some κ.

Example 4.39. The ∞-category Spc is compactly generated hence presentable. Indeed, $\mathsf{Spc} = \mathcal{P}(\Delta^0)$ and we now apply Corollary 4.32.

Example 4.40. It is also true that Cat_∞ is compactly generated although we haven't proved that.

Exercise 4.41. If $C \simeq \mathrm{Ind}_\kappa(C')$ is κ-compactly generated then there is a canonical choice for C', namely C^κ. Indeed, show that there is a canonical equivalence $\mathrm{Ind}_\kappa(C') \simeq \mathrm{Ind}_\kappa(C^\kappa)$.

[14] By convention, regular cardinals here are always assumed infinite. Recall that a cardinal κ is called *regular* if it is not the union of fewer than κ-many subsets of cardinality less than κ. The smallest regular cardinal is ω, the only countable cardinal.

Commentary 4.42. The reason these ∞-categories are called (κ-)compactly generated is hopefully clear: They are generated (under (κ-)filtered colimits) by their (κ-)compact objects. This means that while they are not necessarily small, they are 'controlled' by a small amount of data, namely their subcategory of (κ-)compact objects.

To explain the term presentability we need some preparation.

Definition 4.43. A functor $f\colon C \to D$ between ∞-categories is a *localization* if it admits a fully faithful right adjoint.

Example 4.44. We saw in Example 3.54 that 'inverting all morphisms' provides a localization $\mathsf{Cat}_\infty \to \mathsf{Spc}$.

Definition 4.45. Let $f\colon C \to D$ be a functor between ∞-categories. We say that f is κ-*accessible* if C, D have κ-filtered colimits and f preserves these.

We say that f is *accessible* if it is κ-accessible for some regular cardinal κ.

Commentary 4.46. Why is that an interesting notion? Assume that $f\colon C \to D$ is an accessible functor between presentable ∞-categories. One can always find a large cardinal κ such that C, D are κ-compactly generated and $f\colon C \to D$ restricts to $f'\colon C^\kappa \to D^\kappa$. So, just as C and D are determined by a small amount of data, so f is determined by f' (see Corollary 4.20).

Commentary 4.47. If $C \simeq \mathrm{Ind}_\kappa(C')$ for some small ∞-category C' which admits κ-small colimits then the inclusion $\mathrm{Ind}_\kappa(C') \hookrightarrow \mathcal{P}(C')$ preserves κ-filtered colimits (Remark 4.19). As remarked (without proof) in Warning 4.35, this inclusion admits a left adjoint. This shows the forward implication in the following characterization of presentable ∞-categories. (We omit the proof of the backward implication.)

Proposition 4.48. *Let C be an ∞-category. Tfae:*

1. *C is presentable.*
2. *C is an accessible localization of a presheaf ∞-category. That is, there exists a small ∞-category C' and a localization $\mathcal{P}(C') \to C$ whose (fully faithful) right adjoint is accessible.*

Commentary 4.49. The presheaf category $\mathcal{P}(C')$ is freely *generated* by C', and the localization can be thought of as imposing *relations*. This is analogous to how one presents a group by generators and relations. In other words, Proposition 4.48 says that C is presentable iff it admits a *presentation*. This hopefully explains the terminology.

Corollary 4.50. *Every presentable ∞-category admits all (small) limits and colimits.*

Sketch of proof. We already know that $\mathcal{P}(C')$ is complete and cocomplete (Corollary 4.3). A colimit in C is computed in $\mathcal{P}(C')$ followed by applying the localization functor. A limit in C is computed in $\mathcal{P}(C')$. $\square$

The accessibility condition in the characterization of Proposition 4.48 is forced upon us, as you can see from the following adjoint functor theorem. We omit the proof.

Theorem 4.51. *Let $f: C \to D$ be a functor between presentable ∞-categories.*

1. *f is a left adjoint iff it preserves colimits.*
2. *f is a right adjoint iff it preserves limits and is accessible.*

Example 4.52. As the inclusion $\mathsf{Spc} \hookrightarrow \mathsf{Cat}_\infty$ preserves all limits and colimits (Proposition 3.21) we deduce that it admits both left and right adjoints. This is something we proved 'by hand' before, see Commentary 3.56.

Exercise 4.53. Let C be a presentable ∞-category, and let C' be a small ∞-category. Show that $\mathrm{Fun}(C', C)$ is also presentable.

4.5 The ∞-category of presentable ∞-categories

Commentary 4.54. We will now study the collection of all presentable ∞-categories. A morphism between two presentable ∞-categories C and D should be given by an adjunction:

$$\begin{tikzcd} C \arrow[r, bend left] \arrow[r, phantom, "\perp"] & D \arrow[l, bend left] \end{tikzcd}$$

The left adjoint determines the right adjoint and vice-versa. This leads to two equivalent ways of encoding the collection of presentable ∞-categories, see Proposition 4.56 below.

Definition 4.55. The ∞-category of *presentable ∞-categories and left adjoint functors* Pr^{L} has as objects presentable ∞-categories, and as morphisms the colimit preserving functors. The ∞-category of *presentable ∞-categories and right adjoint functors* Pr^{R} has as objects presentable ∞-categories, and as morphisms the limit preserving functors that are accessible.

Both of these are defined as subcategories of the ∞-category CAT_∞ of not necessarily small ∞-categories.

We now state a number of facts about these ∞-categories. For time reasons most proofs will be omitted.

Proposition 4.56. *There is an equivalence $(\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}} \simeq \mathrm{Pr}^{\mathrm{R}}$ that can be thought of as exchanging left and right adjoints.*

In other words, it is the identity on objects, and sends a left adjoint $f: C \to D \in \mathrm{Pr}^{\mathrm{L}}$ to its right adjoint $g: D \to C \in \mathrm{Pr}^{\mathrm{R}}$.

Proposition 4.57. *The ∞-categories Pr^{L} and Pr^{R} admit all small limits. Moreover, the forgetful functor to CAT_∞ preserves these.*

Corollary 4.58. *The ∞-categories Pr^{L} and Pr^{R} admit all small colimits. These are computed as follows: Pass to the adjoint diagram via Proposition 4.56 and compute the limit in CAT_∞.*

Proof. This is immediate from Propositions 4.56 and 4.57. □

Warning 4.59. The forgetful functors $\mathrm{Pr}^{\mathrm{L}}, \mathrm{Pr}^{\mathrm{R}} \to \mathrm{CAT}_\infty$ do *not* preserve colimits. Therefore, it is important to specify in each case where such a colimit is computed.

We know from Proposition 4.57 that Pr^{R} is closed under all small limits in CAT_∞. In fact, the collection of presentable ∞-categories is stable under many other constructions. It is therefore a very convenient framework to work in. We give the following examples.

Proposition 4.60. *Assume that C and D are presentable ∞-categories, and K a small simplicial set.*

1. $\mathrm{Fun}(K, C)$ *and* $\mathrm{Fun}^L(C, D)$ *are presentable.*
2. $C_{/p}$ *and* $C_{p/}$ *are presentable for any diagram* $p\colon K \to C$.

Remark 4.61. Let C, D, E be presentable ∞-categories and consider the full subcat $\mathrm{Fun}'(C \times D, E) \subseteq \mathrm{Fun}(C \times D, E)$ spanned by functors that preserve colimits in each variable separately. One can show that

$$\mathrm{Fun}'(C \times D, E) \simeq \mathrm{Fun}^L(C \otimes D, E)$$

for some presentable ∞-category $C \otimes D$. This is the underlying tensor product for a symmetric monoidal structure on Pr^{L}. (I won't be able to discuss symmetric monoidal structures on ∞-categories but for now you can think of a symmetric monoidal structure on the homotopy category that comes with additional coherence data.) We can then interpret $\mathrm{Fun}^L(D, E)$ of Proposition 4.60 as an internal mapping object, in the sense that

$$\mathrm{Map}_{\mathrm{Pr}^{\mathrm{L}}}(C, \mathrm{Fun}^L(D, E)) \simeq \mathrm{Map}_{\mathrm{Pr}^{\mathrm{L}}}(C \otimes D, E).$$

In fact, the same equivalence holds with the mapping spaces replaced by the ∞-categories $\mathrm{Fun}^L(-,-)$.

Example 4.62. The unit for this tensor product is given by the presentable ∞-category Spc. Indeed,

$$\mathrm{Map}_{\mathrm{Pr}^{\mathrm{L}}}(C \otimes \mathsf{Spc}, E) \simeq \mathrm{Map}_{\mathrm{Pr}^{\mathrm{L}}}(C, \mathrm{Fun}^L(\mathsf{Spc}, E)) \simeq \mathrm{Map}_{\mathrm{Pr}^{\mathrm{L}}}(C, E),$$

where I used Corollary 4.12 for the last equivalence. This easily implies that $C \otimes \mathsf{Spc} \simeq C$ for arbitrary presentable ∞-categories C.

Exercise 4.63. Let C, C' be two small ∞-categories. Show that $\mathcal{P}(C) \otimes \mathcal{P}(C') \simeq \mathcal{P}(C \times C')$.

Commentary 4.64. Starting with this information and given that every presentable ∞-category is an accessible localization of a presheaf category (Proposition 4.48) you will have no difficulty guessing what the tensor product of any two presentable ∞-categories is.

5 Further topics

In the remainder of the course I want to discuss two successes of the theory of ∞-categories: spectra and K-theory. For lack of time the discussion will be even sketchier than before. In particular, if you don't know a little bit about these two topics already, you might not be too impressed.

5.1 The ∞-category of spectra

Commentary 5.1. Recall that a spectrum E is a collection of pointed topological spaces $(E_n)_{n\geq 0}$ together with bonding maps $\Sigma E_n \to E_{n+1}$ from the reduced suspension of the preceding one to the next. (With an obvious notion of maps between such spectra.) Such a spectrum attempts to be a (generalized) cohomology theory on, say, pointed CW-complexes:

$$E^n(X) := [X, E_n]$$

the set of homotopy classes of pointed maps. It is not difficult to clarify under what assumptions on E this succeeds. Namely, one needs that $E^{n+1}(\Sigma X) \cong E^n(X)$ which translates into the condition that the map adjoint to the bonding map is a homotopy equivalence

$$\Omega E^{n+1} \simeq E^n. \tag{5.2}$$

(Sometimes, this condition goes by the term Ω-*spectrum*.) Conversely, every cohomology theory can be obtained through this process (Brown representability). If we formally invert the maps of (Ω-)spectra $E \to F$ that induce isomorphisms on corresponding cohomology theories we obtain the *stable homotopy category* SH. By the preceding discussion, we have a canonical bijection:

$$\{\text{objects in } \mathsf{SH}\}/\cong \quad \longleftrightarrow \quad \{\text{cohomology theories}\}/\cong$$

Example 5.3.

1. Let X be a pointed topological space. Its *suspension spectrum* $\Sigma^\infty X$ is the spectrum $(\Sigma^n X)_n$ with identity bonding maps.
2. The *sphere spectrum* $\mathbb{S}$ is the suspension spectrum $\Sigma^\infty(S^0) = (S^n)_n$ of the 0-sphere. We also denote $\mathbb{S}^k = \Sigma^\infty(S^k) = (S^{n+k})_n$. The groups $\hom_{\mathsf{SH}}(\mathbb{S}^k, \mathbb{S}) = \pi_k(\mathbb{S}) = \operatorname{colim}_n \pi_{n+k}(S^n)$ are the *stable homotopy groups of spheres*.
3. Let G be an abelian group. The spectrum $(K(G,n))_n$ of Eilenberg-MacLane spaces with bonding maps adjoint to the canonical homotopy equivalence $K(G,n) \simeq \Omega K(G, n+1)$ defines the *Eilenberg-MacLane spectrum* associated with G. (In particular, this is an Ω-spectrum.) It represents singular cohomology with coefficients in G.

Commentary 5.4. The category SH is triangulated and comes with a compatible symmetric monoidal structure. It took a lot of effort to make this structure available at the level of model categories, and even then it is arguably not an entirely satisfactory solution. In this section we are going to see how an ∞-categorical model can be constructed that has, in addition, many other nice properties. This is due to Lurie. You can find it in his *Higher Algebra*.

Definition 5.5. An ∞-category is *pointed* if it has a *zero object*, that is, an object that is both initial and final. We typically denote it by 0.

Exercise 5.6. Show that this is equivalent to the existence of an initial object $\emptyset$, a final object $*$, and a morphism $* \to \emptyset$.

Remark 5.7. If C has a final object we can define the ∞-category of *pointed objects* C_* as the full subcategory of C^{Δ^1} spanned by morphisms $c \to d$ where c is a final object. By the following exercise together with Proposition 4.60, this is a presentable ∞-category whenever C is. For example, this is true of Spc_*, the ∞-category of *pointed spaces.*

Exercise 5.8. Pick a final object $1 \in C$. Show that $C_* \simeq C_{1/}$.

Construction 5.9. Let C be a pointed ∞-category and $c \in C$. Assuming the relevant limits or colimits exist, we define the *suspension* and *loop* functors $\Sigma, \Omega \colon C \to C$ by[15] the following pushout and pullback diagram, respectively:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd} \qquad \begin{tikzcd} \Omega X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

Example 5.10. Let $C = \mathsf{Spc}_*$ be the ∞-category of pointed spaces. Then these functors recover the usual (reduced) suspension and loop functors. Note that there is an adjunction $\Sigma \dashv \Omega$.

Definition 5.11. A pointed ∞-category C is *stable* if it satisfies any of the following equivalent conditions:

1. C admits pushouts and the suspension functor $\Sigma \colon C \to C$ is an equivalence.
2. C admits pullbacks and the loop functor $\Omega \colon C \to C$ is an equivalence.
3. C admits finite limits and colimits and a square is a pushout square iff it is a pullback square.

Commentary 5.12. There is an obvious idea to turn an ∞-category C with finite limits into a stable ∞-category, namely by *inverting* the loop functor. That is, by taking the sequential limit,

$$C_* \xleftarrow{\Omega} C_* \xleftarrow{\Omega} C_* \xleftarrow{\Omega} \cdots \tag{5.13}$$

Definition 5.14. Let C be an ∞-category with finite limits. We define its *stabilization* $\mathsf{Sp}(C)$ to be the limit in CAT_∞ of the sequence (5.13). If $C = \mathsf{Spc}$ we denote the result simply by $\mathsf{Sp} = \mathsf{Sp}(\mathsf{Spc})$ and call it the *∞-category of spectra.*

Commentary 5.15. An object of Sp can thus be identified with a collection $(E_n)_n$ of pointed spaces together with equivalences $\Omega E_{n+1} \simeq E_n$ as in (5.2). In other words, the objects of Sp can be identified with Ω-spectra. In fact, it is true that

$$\mathrm{h}(\mathsf{Sp}) \simeq \mathsf{SH}\,.$$

Definition 5.16. Let C and D be two ∞-categories with finite limits (resp. colimits). We say that a functor $f \colon C \to D$ is *left-exact* (resp. right-exact) if it preserves these finite limits (resp. colimits). It is *exact* if it is both left- and right-exact.

[15] One needs to explain how these are *functors*. It isn't terribly hard but requires technology we haven't discussed.

Exercise 5.17. Assume $f: C \to D$ is a functor between stable ∞-categories. Show that the following are equivalent:

1. f is exact.
2. f is left-exact.
3. f is right-exact.

Proposition 5.18. *Let C be an ∞-category with finite limits. The stabilization $\mathsf{Sp}(C)$ is stable. Moreover, the canonical functor $\Omega^\infty: \mathsf{Sp}(C) \to C$ induces an equivalence*

$$\mathrm{Fun}^{\mathrm{lex}}(D, \mathsf{Sp}(C)) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}(D, C)$$

on ∞-categories of left-exact functors (defined as usual) for any stable ∞-category D.

Remark 5.19. The ∞-category of spectra Sp is presentable. This follows immediately from Proposition 4.57 and the fact that $\Omega: \mathsf{Spc}_* \to \mathsf{Spc}_*$ is right adjoint to the suspension functor. It also follows from Corollary 4.58 that it is the colimit of the sequence

$$\mathsf{Spc}_* \xrightarrow{\Sigma} \mathsf{Spc}_* \xrightarrow{\Sigma} \mathsf{Spc}_* \xrightarrow{\Sigma} \cdots$$

in Pr^{L}. We conclude that $\Omega^\infty: \mathsf{Sp} \to \mathsf{Spc}_*$ admits a left adjoint, the infinite suspension functor $\Sigma^\infty: \mathsf{Spc}_* \to \mathsf{Sp}$.

There is another way to describe Sp which is also reminiscent of classical constructions of SH. Namely, consider the sequence

$$\mathsf{Spc}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathsf{Spc}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathsf{Spc}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \cdots$$

at the level of finite pointed spaces (Example 4.29). Take the colimit of this sequence in Cat_∞. (This is (maybe up to idempotent completion) the category of finite spectra.) Finally, take the Ind-completion:

$$\mathsf{Sp} \simeq \mathrm{Ind}(\mathrm{colim}(\mathsf{Spc}_*^{\mathrm{fin}}, \Sigma))$$

(This is a general fact about filtered colimits in Pr^{L} of diagrams of compactly generated ∞-categories with compact-preserving left adjoints.)

Corollary 5.20. *Let D be a stable presentable ∞-category. Evaluation at the sphere spectrum $\mathbb{S}$ induces an equivalence*

$$\mathrm{Fun}^L(\mathsf{Sp}, D) \xrightarrow{\sim} D.$$

In other words, Sp is the stable ∞-category freely generated under colimits by a point.

Sketch of proof. Let $\mathrm{Fun}^R(-,-)$ denote the full subcategory on right adjoints. We then have

$$\mathrm{Fun}^L(\mathsf{Sp}, D) \simeq \mathrm{Fun}^R(D, \mathsf{Sp})^{\mathrm{op}},$$

which, by Proposition 5.18, we may identify with a full subcategory of $\mathrm{Fun}^{\mathrm{lex}}(D, \mathsf{Spc}_*)^{\mathrm{op}}$. It turns out that it is exactly the subcategory

$$\mathrm{Fun}^R(D, \mathsf{Spc}_*)^{\mathrm{op}} \simeq \mathrm{Fun}^L(\mathsf{Spc}_*, D) \simeq \mathrm{Fun}^L(\mathsf{Spc}, D) \simeq D,$$

where we used that passing to pointed objects is left adjoint to the canonical forgetful functor, as well as Corollary 4.12. Tracing through the proof one observes that the composite equivalence is evaluation at $\Sigma^\infty(* \amalg *) = \mathbb{S}$. $\square$

Remark 5.21. Let C be a stable ∞-category. For any two objects $c, d \in C$ consider 'the' composition map

$$\operatorname{Map}_C(0,d) \times \operatorname{Map}_C(c,0) \to \operatorname{Map}_C(c,d).$$

Since the domain of this map is contractible we obtain a well-defined element $0 \in \pi_0 \operatorname{Map}_C(c,d) \cong \operatorname{Hom}_{h(C)}(c,d)$ which we call the *zero map* from c to d. In fact, note that we have equivalences (cf. Example 3.24)

$$\operatorname{Map}_C(\Sigma c, d) \simeq \Omega \operatorname{Map}_C(c,d)$$

so that $\pi_0 \operatorname{Map}_C(\Sigma c, d) \cong \pi_1 \operatorname{Map}_C(c,d)$ has a group structure. Similarly, $\pi_0 \operatorname{Map}_C(\Sigma^2 c, d) \cong \pi_2 \operatorname{Map}_C(c,d)$ has the structure of an abelian group. But since $\Sigma\colon C \to C$ is an equivalence, we can choose c' so that $\Sigma^2 c' = c$ and deduce that the preceding discussion works for any choice of two objects. It is not difficult to prove that in fact, $h(C)$ is an additive category.

Remark 5.22. Recall that a triangulated category T is an additive category which comes with an equivalence $[1]\colon T \xrightarrow{\sim} T$ and a collection of *distinguished triangles* $X \to Y \to Z \to X[1]$ satisfying a bunch of axioms. Now, let C be a stable ∞-category. We will indicate how its homotopy category $h(C)$ inherits a triangulated structure. For $C = \mathsf{Sp}$ this recovers the classical triangulated structure on the stable homotopy category.

We already saw in the previous remark that $h(C)$ is additive. The translation functor is induced by the suspension functor $\Sigma\colon C \to C$ which is an equivalence by assumption. And, roughly speaking, the distinguished triangles are those that fit into a pushout (or, equivalently, a pullback) square of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd} \tag{5.23}$$

together with the canonical map $Z \to X[1]$ in $h(C)$ induced by the functoriality of pushouts. You might find it instructive to try to prove that this defines indeed a triangulated structure. Be warned, however, that some of the axioms are rather subtle to check.

Commentary 5.24. We now turn to the symmetric monoidal structure on Sp. For this recall that Pr^{L} has a symmetric monoidal structure, see Remark 4.61. Let us denote by $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ the full subcategory spanned by stable presentable ∞-categories. (Note incidentally that, by Exercise 5.17, every morphism in this ∞-category is exact.) Our goal is to exhibit this as a localization of Pr^{L} and show that the symmetric monoidal structure on Pr^{L} can be localized to one on $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ so that the localization $\mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ becomes symmetric monoidal.

Lemma 5.25. *Let C be any presentable ∞-category. We have, canonically,*

$$C \otimes \mathsf{Sp} \simeq \mathsf{Sp}(C)$$

Sketch of proof. As mentioned in Remark 4.61, the symmetric monoidal structure on Pr^{L} is closed (admits internal mapping spaces) so, in particular, the tensor product preserves colimits in each variable. It follows (Remark 5.19) that $C \otimes \mathsf{Sp}$ is the sequential colimit in Pr^{L} of

$$C \otimes \mathsf{Spc}_* \xrightarrow{\Sigma} C \otimes \mathsf{Spc}_* \xrightarrow{\Sigma} \cdots$$

One now proves similarly that $C \otimes \mathsf{Spc}_* \simeq C_*$ canonically, and the claim follows. □

Proposition 5.26. *The inclusion* $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} \hookrightarrow \mathrm{Pr}^{\mathrm{L}}$ *admits a left adjoint given by* $\mathsf{Sp}(-) = \mathsf{Sp} \otimes -$.

Proof. Given a presentable ∞-category C, consider the canonical functor $C \to \mathsf{Sp}(C)$ in Pr^{L}. We need to show (Remark 3.52) that precomposition with this functor induces an equivalence

$$\mathrm{Map}_{\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}}(\mathsf{Sp}(C), D) \to \mathrm{Map}_{\mathrm{Pr}^{\mathrm{L}}}(C, D)$$

for any stable presentable ∞-category D. This follows from Proposition 5.18, as in in the proof of Corollary 5.20. □

Construction 5.27. It follows from the last two results (and their proofs) that $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ inherits a symmetric monoidal structure such that the stabilization functor $\mathsf{Sp}\colon \mathrm{Pr}^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ is symmetric monoidal. The tensor product of stable presentable ∞-categories C, D is given by $\mathsf{Sp}(C \otimes D)$ with unit the ∞-category of spectra Sp. The right adjoint to a symmetric monoidal functor is always lax symmetric monoidal and therefore preserves commutative algebra objects. In other words, the ∞-category $\mathsf{Sp} \in \mathrm{Pr}^{\mathrm{L}}$ underlies a commutative algebra object, whose tensor product we can identify with a bifunctor

$$\wedge\colon \mathsf{Sp} \times \mathsf{Sp} \to \mathsf{Sp}$$

that can be seen as the *smash product* of spectra. Tracing through the construction we see that it 'extends', colimit-preserving in each variable, the smash product on finite pointed spaces, induced from the Cartesian product on spaces.

Commentary 5.28. What we have really shown is that the functor $\Sigma^{\infty}_{+}\colon \mathsf{Spc} \to \mathsf{Sp}$ exhibits the latter as an idempotent algebra in Pr^{L}. One also deduces that modules over this algebra canonically identify with $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$:

$$\mathrm{Mod}_{\mathrm{Pr}^{\mathrm{L}}}(\mathsf{Sp}) \simeq \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$$

5.2 The universal property of K-theory

Commentary 5.29. We now turn to algebraic K-theory. Let C be a category with a notion of 'exact sequence' (for example, an abelian or triangulated category). An *Euler characteristic* (on C) is a function $\chi\colon C_0/\cong \to A$ from isomorphism classes of objects to an abelian group A such that

$$\chi(c) - \chi(d) + \chi(e) = 0 \qquad (5.30)$$

whenever there is an exact sequence $c \to d \to e$. The Grothendieck group $K_0(C)$ is the target of the universal Euler characteristic in that

$$\mathrm{Hom}_{\mathsf{Ab}}(K_0(C), A) \xrightarrow{\sim} \{\chi\colon C_0/\cong \to A \text{ Euler characteristic}\}$$

via the Euler characteristic $C_0/\cong \to K_0(C)$. (Of course, you can describe $K_0(C)$ as the free abelian group on $C_0/\cong$ modulo the obvious relation suggested by (5.30).)

Example 5.31. If X is a (paracompact) topological space we may consider the category of complex vector bundles $\mathsf{VB}(X)$ on X in which exact sequences are split. Then $K^0(X) := K_0(\mathsf{VB}(X))$ is better known as the Grothendieck group of vector bundles on X. If (X, x) is pointed, one may consider the kernel

$$\tilde{K}^0(X) := \ker\left(K^0(X) \to K^0(x) = \mathbb{Z}\right),$$

and set $\tilde{K}^{-n}(X) = \tilde{K}^0(\Sigma^n X)$. This satisfies *Bott periodicity*, $KU^{2-n}(X) \cong KU^{-n}(X)$, which one may take to extend this to positive degrees. In fact, this extends to a generalized cohomology theory as in Commentary 5.1, called *topological K-theory*. Therefore it is represented by a spectrum, typically denoted KU, with

$$KU_n = \begin{cases} \mathbb{Z} \times BU & : n \text{ even} \\ U & : n \text{ odd} \end{cases}$$

Topological K-theory was one of the first *generalized* cohomology theories to be discovered and put to use. For example, it can be used to give an elementary solution to the *Hopf invariant one* problem.

Commentary 5.32. Later on, higher K-theory groups $K_i(C)$ (even for negative i) were realized as the homotopy groups of the *K-theory spectrum* $K(C)$. (We will not discuss the non-connective K-theory spectrum.) This turns out to be an unreasonably powerful invariant in many areas of algebraic topology, algebraic geometry and number theory.

Example 5.33. Let R be a ring of integers in a number field. The K-groups $K_i(R)$ are defined as the K-groups $K_i(\mathrm{Proj}^{\mathrm{fg}}(R))$ of the category of finitely generated projective R-modules (that is, vector bundles on $\mathrm{Spec}(R)$; every exact sequence splits).

1. $K_0(R) = \mathrm{Pic}(R) \times \mathbb{Z}$ is essentially the ideal class group.
2. $K_1(R) = R^\times$ is the group of units.
3. $K_2(R)$ is the group of certain symbols in class field theory.
4. All $K_i(R)$ are finitely generated. Their ranks are known and related to the number of real and complex embeddings of the number field.
5. The torsion part of $K_i(R)$ for $i > 2$ is not known in general. A lot of progress has come from the relation (*Bloch-Kato conjecture*, now a theorem) with étale cohomology. For the ring $R = \mathbb{Z}$, the order of the known groups exhibit beautiful number-theoretic properties (they are related to Bernoulli numbers). The remaining effort centers around the *Kummer-Vandiver conjecture*.

Example 5.34. The K-theory groups in the previous example could have been defined in a similar way for any ring. And globalizing we may also define an invariant of schemes. Namely, if X is a scheme, let $\mathsf{VB}(X)$ be the category of vector bundles on X, or equivalently, locally free $\mathcal{O}_X$-module of finite rank. There is an obvious notion of exact sequence (coming from exactness in $\mathcal{O}_X$-modules; these don't necessarily split) thus we may define the algebraic K-theory of X as $K(X) := K(\mathsf{VB}(X))$.

If X is a smooth complex projective variety there is a natural map $K_0(X) \to K^0(X(\mathbb{C}))$ to the topological K-theory of the underlying manifold $X(\mathbb{C})$. The *Hodge conjecture* describes the image of this map after tensoring with $\mathbb{Q}$.

Commentary 5.35. Not only are the groups $K_i(C)$ hard to compute but, in contrast to $K_0(C)$, the spectrum $K(C)$ isn't defined through a universal property anymore, at least not in an obvious way. While there was work in that direction, these attempts lacked the right formalism. The definite treatment was given using the language of ∞-categories, by Blumberg-Gepner-Tabuada in the article *A universal characterization of higher algebraic K-theory*. My goal is to sketch this characterization and what goes into the proof.

Commentary 5.36. The notion of an exact sequence makes sense in a stable ∞-category, namely those fitting into a biCartesian square as in (5.23). So, it is natural to feed stable ∞-categories into the K-theory machine. It is a general phenomenon (known as 'Eilenberg swindle') that K-theory of large categories tends to be ill-behaved (often trivial). We will therefore be interested in small stable ∞-categories. These assemble into $\mathsf{Cat}_\infty^{\mathrm{ex}}$, the subcategory of Cat_∞ on stable ∞-categories and exact functors.

Construction 5.37. Let C be a stable ∞-category. We are going to construct the spectrum $K(C)$. For exact (ordinary) categories (or more generally for Waldhausen categories), this is known as the (iterated) *$S_\bullet$-construction*, due to Waldhausen. The exact details are not as important. The point is that this is a concrete construction and will look very ad-hoc to you if you haven't worked with K-theory before.

Let $\mathrm{Gap}_n(C)$ be the full subcategory of $\mathrm{Fun}(N([n]^{[1]}), C)$ spanned by functors F such that $F(i \to i) \simeq 0$ for each $0 \leq i \leq n$ and the square

$$\begin{tikzcd} F(i \to j) \arrow[r] \arrow[d] & F(i \to k) \arrow[d] \\ F(j \to j) \arrow[r] & F(j \to k) \end{tikzcd}$$

is pushout/pullback. This is still a stable ∞-category.

One then defines a simplicial ∞-category $S_n(C) = \mathrm{Gap}_n(C)$. Finally, $\Omega|(S_\bullet C)^{\simeq}|$ where $|-|$ is the geometric realization (colimit), is the K-theory space. One can reiterate this construction, and $K(C) := (|(S_\bullet^n(C))^{\simeq}|)_n$ is a spectrum, the *K-theory spectrum*. Moreover, it isn't hard to exhibit a functor $K\colon \mathsf{Cat}_\infty^{\mathrm{ex}} \to \mathsf{Sp}$ which sends $C \mapsto K(C)$.

Definition 5.38. In an *ad-hoc* fashion, we define the *idempotent completion* of $C \in \mathsf{Cat}_\infty^{\mathrm{ex}}$ to be $C^\natural := \mathrm{Ind}(C)^\omega$. This is also a small stable ∞-category. And we will say that $C \in \mathsf{Cat}_\infty^{\mathrm{ex}}$

is *idempotent complete* if the canonical fully faithful functor $C \to C^\natural$ is an equivalence. Small idempotent complete stable ∞-categories will also be called *perfect* in the following.

Commentary 5.39. Let $C \in \mathsf{Cat}_\infty^{\mathrm{ex}}$. The map $K(C) \to K(C^\natural)$ induced by the idempotent completion is an isomorphism in all positive degrees and an injection $K_0(C) \hookrightarrow K_0(C^\natural)$. In fact, as a full *dense* stable subcategory of $C^\natural$ (that is, whose idempotent completion is $C^\natural$), C is characterized by the subgroup $K_0(C) \subseteq K_0(C^\natural)$. Hence, the computation of $K(C)$ splits into two steps: first compute $K(C^\natural)$, and then identify this subgroup of $K_0(C^\natural)$. The first one is the more interesting and harder one, of course, so we focus on that one.

This suggests restricting to perfect ∞-categories which form a full subcategory $\mathsf{Cat}_\infty^{\mathrm{perf}} \subseteq \mathsf{Cat}_\infty^{\mathrm{ex}}$. Idempotent completion defines a left adjoint $(-)^\natural\colon \mathsf{Cat}_\infty^{\mathrm{ex}} \to \mathsf{Cat}_\infty^{\mathrm{perf}}$ which is thus a localization. In fact, both ∞-categories are compactly generated and this is an accessible localization. (In particular, both are complete and cocomplete.)

Commentary 5.40. Algebraic K-theory then gives a functor $K\colon \mathsf{Cat}_\infty^{\mathrm{perf}} \to \mathsf{Sp}$ and our goal is to characterize this functor. One of the most important and useful computational tools is Waldhausen's *additivity theorem* which we want to formulate as the K-theory functor taking split exact sequences to cofiber sequences. Together with the basic fact that it preserves filtered colimits this will characterize the functor already!

Definition 5.41. Let $\iota\colon A \hookrightarrow B$ be a fully faithful in $\mathsf{Cat}_\infty^{\mathrm{perf}}$. We define the *Verdier quotient B/A* as the *cofiber* of ι, that is, as the pushout in $\mathsf{Cat}_\infty^{\mathrm{perf}}$:

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & B/A \end{tikzcd}$$

(B/A is the localization of B at the morphisms whose cofibers lie in A. The homotopy category of B/A is the idempotent completion of $\mathrm{h}(B)/\mathrm{h}(A)$, the classical Verdier quotient of triangulated categories.) We then say more generally that a sequence $A \xrightarrow{\iota} B \xrightarrow{\pi} C$ is an *exact sequence* if the composite is the zero functor, ι is fully faithful, and the induced $B/A \to C$ is an equivalence. We say that the sequence *splits* if both ι and π admit right adjoints.

Remark 5.42. It is enough to ask for one of the right adjoints. The second is then automatic. Moreover, it is then also automatic that π is a localization in the sense of Definition 4.43.

Definition 5.43. An *additive invariant* is a functor $E\colon \mathsf{Cat}_\infty^{\mathrm{perf}} \to D$ into a stable presentable ∞-category D with the following properties:

- E preserves filtered colimits.
- E takes split-exact sequences to cofiber sequences in D.

We denote by $\mathrm{Fun}^{\mathrm{add}}(\mathsf{Cat}_\infty^{\mathrm{perf}}, D)$ the ∞-category of additive invariants.

Remark 5.44. As announced above, K-theory is an additive invariant. Indeed, $K\colon \mathsf{Cat}_\infty^{\mathrm{perf}} \to \mathsf{Sp}$ preserves filtered colimits since the ∞-categories that enter in the construction ($N([n]^{[1]})$) are compact in Cat_∞ and filtered colimits in $\mathsf{Cat}_\infty^{\mathrm{perf}}$ are computed in Cat_∞ (and $(-)^{\simeq}$ also preserves filtered colimits). The second property is implied by Waldhausen's additivity theorem for the K-theory of Waldhausen categories.

Exercise 5.45. The second condition for an additive invariant can be phrased differently. Let $A \xrightarrow{\iota} B \xrightarrow{\pi} C$ be a split-exact sequence, and ρ a right adjoint to π. Show that E takes this sequence to a cofiber sequence if and only if

$$E(A) \amalg E(C) \xrightarrow{\iota \amalg \rho} E(B)$$

is an equivalence.

Proposition 5.46. *There is a universal additive invariant* $[-]\colon \mathsf{Cat}_\infty^{\mathrm{perf}} \to \mathcal{M}_{add}$ *in that*

$$[-]^*\colon \mathrm{Fun}^{\mathrm{add}}(\mathsf{Cat}_\infty^{\mathrm{perf}}, D) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{L}}(\mathcal{M}_{add}, D)$$

for any stable presentable D.

Sketch of proof. How does one construct this universal additive invariant? Suppose we dropped the second condition in the definition of additive invariants. (That is, we consider functors that preserve filtered colimits.) Then we have

$$\mathrm{Fun}^{\omega}(\mathsf{Cat}_\infty^{\mathrm{perf}}, D) \simeq \mathrm{Fun}((\mathsf{Cat}_\infty^{\mathrm{perf}})^{\omega}, D) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}((\mathsf{Cat}_\infty^{\mathrm{perf}})^{\omega}), D) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathsf{Sp}(\mathcal{P}((\mathsf{Cat}_\infty^{\mathrm{perf}})^{\omega})), D)$$

so that the universal invariant would be given by

$$\psi\colon \mathsf{Cat}_\infty^{\mathrm{perf}} \to \mathsf{Sp}(\mathcal{P}((\mathsf{Cat}_\infty^{\mathrm{perf}})^{\omega})) \simeq \mathrm{Fun}((\mathsf{Cat}_\infty^{\mathrm{perf}})^{\omega,\mathrm{op}}, \mathsf{Sp}).$$ [16]

Now we add in the second condition to the mix. We should invert the canonical maps $\gamma_e\colon \psi(B)/\psi(A) \to \psi(C)$ for any split-exact sequence e of the form $A \to B \to C$ in $\mathsf{Cat}_\infty^{\mathrm{perf}}$. It turns out that every split-exact sequence can be written as a filtered colimit of split-exact sequences belonging to a small set $\mathcal{E}$ of representatives. It follows that the localization with respect to $\{\gamma_e \mid e \in \mathcal{E}\}$ is enough.

In conclusion,

$$\mathcal{M}_{add} = \mathrm{Fun}((\mathsf{Cat}_\infty^{\mathrm{perf}})^{\omega,\mathrm{op}}, \mathsf{Sp})[\gamma_e^{-1} \mid e \in \mathcal{E}]$$

with the canonical functor $\mathsf{Cat}_\infty^{\mathrm{perf}} \to \mathcal{M}_{add}$. □

Exercise 5.47. Let I be a small simplicial set. Show that $\mathsf{Sp}(\mathrm{Fun}(I, \mathsf{Spc})) \simeq \mathrm{Fun}(I, \mathsf{Sp})$ canonically.

Remark 5.48. It follows from Propositions 5.46 and 4.60 that $\mathrm{Fun}^{\mathrm{add}}(\mathsf{Cat}_\infty^{\mathrm{perf}}, D)$ is a stable presentable ∞-category.

[16] See Exercise 5.47 below.

Remark 5.49. The ∞-category $\mathsf{Cat}_\infty^{\mathrm{perf}}$ is equivalent to compactly generated stable ∞-categories and compact-preserving left adjoints. In one direction you take $\mathrm{Ind}(-)$, in the other $(-)^\omega$. It follows that the ∞-category of compact spectra Sp^ω has a convenient universal property in $\mathsf{Cat}_\infty^{\mathrm{perf}}$. Namely, given any perfect ∞-category C, we have

$$\mathrm{Fun}^{\mathrm{ex}}(\mathsf{Sp}^\omega, C) \simeq \mathrm{Fun}^{\mathrm{L,cpt}}(\mathsf{Sp}, \mathrm{Ind}(C)) \simeq \mathrm{Ind}(C)^\omega \simeq C$$

given by evaluating at the sphere spectrum, see Corollary 5.20. This explains why compact spectra will appear frequently below.

Before stating the main theorem we need to observe that mapping spaces in stable ∞-categories naturally underlie *spectra.*

Construction 5.50. Let C be a small stable ∞-category. Essentially by definition, $\Omega^\infty \colon \mathsf{Sp}(C) \xrightarrow{\sim} C$. Note also that the stabilization is functorial in left exact functors. We may therefore consider the following composite:

$$\tilde{\mathrm{y}} \colon C \simeq \mathsf{Sp}(C) \xrightarrow{\mathrm{y}} \mathsf{Sp}(\mathrm{Fun}(C^{\mathrm{op}}, \mathsf{Spc})) \simeq \mathrm{Fun}(C^{\mathrm{op}}, \mathsf{Sp})$$

which is the *stable Yoneda* functor. (The last equivalence holds by the previous exercise.) The adjoint functor

$$\mathrm{map}_C(-,-) \colon C^{\mathrm{op}} \times C \to \mathsf{Sp}$$

then defines mapping spectra. Note that by construction we have $\Omega^\infty \mathrm{map}_C(c,d) \simeq \mathrm{Map}_C(c,d)$ for any two $c, d \in C$

Theorem 5.51. *For any perfect ∞-category C there is a canonical equivalence of spectra*

$$\mathrm{map}_{\mathcal{M}_{add}}([\mathsf{Sp}^\omega], [C]) \simeq K(C).$$

Commentary 5.52. We may interpret this statement as a corepresentability result for algebraic K-theory.

Sketch of proof. Consider the functor $K_C \colon (\mathsf{Cat}_\infty^{\mathrm{perf}})^{\omega,\mathrm{op}} \to \mathsf{Sp}$ defined by

$$K_C(B) = K(\mathrm{Fun}^{\mathrm{ex}}(B, C)).$$

This functor sends split-exact sequences to cofiber sequences and thus defines an object of $\mathcal{M}_{add}$. The main computation is that this object is nothing but $[C]$. The slogan is that Waldhausen's $S_\bullet$-construction acts as the suspension on $\mathcal{M}_{add}$. It now follows that

$$K(C) = K_C(\mathsf{Sp}^\omega) \simeq \mathrm{map}_{\mathrm{Fun}((\mathsf{Cat}_\infty^{\mathrm{perf}})^{\omega,\mathrm{op}}, \mathsf{Sp})}(\mathsf{Sp}^\omega, K_C)$$

by the spectral Yoneda lemma. Note we used here Remark 5.49 which also implies that Sp^ω is indeed compact in $\mathsf{Cat}_\infty^{\mathrm{perf}}$. Because K_C sends split-exact sequences to cofiber sequences, this last mapping spectrum is also

$$\mathrm{map}_{\mathcal{M}_{add}}([\mathsf{Sp}^\omega], K_C) \simeq \mathrm{map}_{\mathcal{M}_{add}}([\mathsf{Sp}^\omega], [C]),$$

concluding the proof. ◻

Convention 5.53. Let $E, F\colon \mathsf{Cat}_\infty^{\mathrm{perf}} \to D$ be two additive invariants with same target. We denote by

$$\mathrm{Nat}(E,F) = \mathrm{map}_{\mathrm{Fun}^{\mathrm{add}}(\mathsf{Cat}_\infty^{\mathrm{perf}},D)}(E,F)$$

the spectrum of natural transformations.

Corollary 5.54. *Let $E\colon \mathsf{Cat}_\infty^{\mathrm{perf}} \to \mathsf{Sp}$ be an additive invariant valued in spectra. Then there is a natural equivalence of spectra*

$$\mathrm{Nat}(K,E) \simeq E(\mathsf{Sp}^\omega).$$

Proof. By Proposition 5.46, the mapping spectrum $\mathrm{Nat}(K,E)$ can be computed in $\mathrm{Fun}^{\mathrm{L}}(\mathcal{M}_{add}, \mathsf{Sp})$. More precisely, K, E factor essentially uniquely through colimit-preserving functors $\bar{K}, \bar{E}\colon \mathcal{M}_{add} \to \mathsf{Sp}$, and

$$\mathrm{Nat}(K,E) \simeq \mathrm{map}_{\mathrm{Fun}^{\mathrm{L}}(\mathcal{M}_{add},\mathsf{Sp})}(\bar{K},\bar{E}). \tag{5.55}$$

By Theorem 5.51, $\bar{K}$ is corepresented by $[\mathsf{Sp}^\omega]$ so that, by the spectral version of the Yoneda lemma, the spectrum (5.55) identifies with

$$\bar{E}([\mathsf{Sp}^\omega]) \simeq E(\mathsf{Sp}^\omega).$$

This concludes the proof. □

Definition 5.56. An (associative) algebra object in Sp (with the symmetric monoidal structure of Section 5.1) is called a *ring spectrum*. For such an R one can consider its module category $\mathrm{Mod}(R)$ whose compact objects are called perfect R-modules, that is, $\mathrm{Perf}(R) := \mathrm{Mod}(R)^\omega$. This is a perfect ∞-category (small, stable, idempotent-complete). For any additive invariant $E\colon \mathsf{Cat}_\infty^{\mathrm{perf}} \to D$ we then define $E(R) := E(\mathrm{Perf}(R)) \in D$.

Example 5.57.

1. If $R = \mathbb{S}$ is the sphere spectrum (the tensor unit) then $\mathrm{Perf}(\mathbb{S}) = \mathsf{Sp}^\omega$ so we can reformulate Corollary 5.54 as

$$\mathrm{Nat}(K,E) \simeq E(\mathbb{S}).$$

2. If R is a (discrete) ring in the usual sense there is an associated ring spectrum HR which represents the (multiplicative) cohomology theory $\mathrm{H}^\bullet(-,R)$. The module category $\mathrm{Mod}(R)$ is an ∞-categorical enhancement of the unbounded derived category $D(R)$. Therefore, $\mathrm{Perf}(R)$ is an ∞-categorical enhancement of the derived category of perfect complexes on R. These are exactly the bounded complexes of finitely generated projective R-modules. From this, one deduces that $K(R) \simeq K(HR)$.

Commentary 5.58. As mentioned, computing K-theory is hard. A lot of progress has come from *trace maps*, natural transformations $K \to E$ to invariants that are easier to compute. An example of such an invariant is topological Hochschild homology.

Construction 5.59. Let R be a commutative ring spectrum. One defines the *topological Hochschild homology* of R as

$$\mathrm{THH}(R) := R \otimes_{R \wedge R} R \in \mathsf{Sp}$$

where R is viewed as an (R, R)-bimodule in an obvious way. (Actually, this can be done for arbitrary ring spectra. Also, $\mathrm{THH}(R)$ comes with more structure that we won't discuss.)

Commentary 5.60. In fact, there is an additive invariant $\mathrm{THH}\colon \mathsf{Cat}_{\infty}^{\mathrm{perf}} \to \mathsf{Sp}$ which spits out $\mathrm{THH}(R)$ upon feeding it $\mathrm{Perf}(R)$. In that case, the relevant trace map

$$K \to \mathrm{THH}$$

is known as the *Dennis* trace map. Several constructions of this map are known but Corollary 5.54 allows one to explain its existence and to characterize it:

Proposition 5.61. *The unit in $\mathbb{Z} = \pi_0(\mathbb{S})$ corresponds, via*

$$\mathrm{Nat}(K, \mathrm{THH}) \simeq \mathrm{THH}(\mathbb{S}) \simeq \mathbb{S},$$

to the Dennis trace map.

A Appendix

A.1 Joins

Commentary A.1. We will start with the *join* construction. Recall that for two ordinary categories C, D, their join $C \star D$ is the category essentially uniquely determined by the following properties:

1. There are fully faithful embeddings $C, D \hookrightarrow C \star D$ and together they are surjective on objects.
2. There are no maps from objects in D to objects in C.
3. There is a single map from any object in C to any object in D.

Example A.2. If $D = [0]$ is the final category then $C \star [0] =: C^{\triangleright}$ adds a new final object to C. Similarly, $[0] \star C =: C^{\triangleleft}$ adds a new initial object.

Exercise A.3. Show that there are canonical isomorphisms of categories $[m] \star [n] \cong [m+n+1]$.

Commentary A.4. We now want to generalize the join construction to simplicial sets. In particular, we want the nerve functor to preserve this construction. So we define it on the simplex category as the bifunctor

$$\star \colon \Delta \times \Delta \to \Delta, \qquad ([m], [n]) \mapsto N([m] \star [n]). \tag{A.5}$$

Recall now that $\mathsf{sSet} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set})$ so that every simplicial set is a (canonical) colimit of representables Δ^n.

Definition A.6. The *join* operation $\star \colon \mathsf{sSet} \times \mathsf{sSet} \to \mathsf{sSet}$ is defined as the unique such functor extending (A.5) and such that for every simplicial set I, the functors $I \star (-), (-) \star I \colon \mathsf{sSet} \to \mathsf{sSet}_{I/}$ preserve arbitrary colimits.

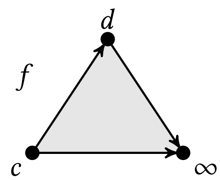
Remark A.7. One can easily deduce an explicit formula for the join construction. Namely,

$$(I \star I')_n = I_n \cup \left(\bigcup_{i+j=n-1} I_i \times I_j \right) \cup I_n.$$

We notice the following easy consequence: If C and D are ∞-categories then so is $C \star D$. Moreover, in that case the canonical functors $C, D \to C \star D$ are fully faithful. (Exercises!)

Example A.8. Let I be a simplicial set and consider the *right cone on* I, $I^{\triangleright} := I \star \Delta^0$. We describe its low-dimensional *non-degenerate* simplices:

0. All objects of C and one additional object that we denote by ∞. It is called the cone point.
1. All non-degenerate morphisms in C, and a unique morphism from any object in C to ∞.
2. The non-degenerate 2-simplices of C and a unique 2-simplex



for each morphism $f: c \to d$ in C.

Exercise A.9. Show that $(\Lambda^2_0)^{\triangleright} \cong \Delta^1 \times \Delta^1$ (which is what you might have expected anyway!).

Exercise A.10. Assume that I is an ∞-category. Show that the cone point $\infty \in I^{\triangleright}$ is a final object.

A.2 Slices

Commentary A.11. We now define an ∞-categorical version of an overcategory. Recall that if $d \in C$ is an object in an ordinary category then the overcategory $C_{/d}$ has as objects morphisms $c \to d$ and as morphisms morphisms in the domain that make the obvious triangle commute. This can be expressed by the following universal property, for each ordinary category K:

$$\mathrm{Hom}_{\mathsf{Cat}}(K, C_{/d}) = \mathrm{Hom}_d(K^{\triangleright}, C)$$

where the right-hand side consists of those functors that take the cone point to $d \in C$.

Definition A.12. Let $F: I \to C$ be a map of simplicial sets. By construction, the functor $(-) \star I: \mathsf{sSet} \to \mathsf{sSet}_{I/}$ commutes with colimits, hence by the adjoint functor theorem admits a right adjoint. We define the *slice over* F, denoted $C_{/F}$, as the value of this right adjoint at F. That is, it is the simplicial set $C_{/F}$ with the following universal property, for each $K \in \mathsf{sSet}$:

$$\mathrm{Hom}_{\mathsf{sSet}}(K, C_{/F}) = \mathrm{Hom}_F(K \star I, C)$$

where on the right-hand side we only consider maps of simplicial sets that restrict to F on I. Note that restricting along $K \hookrightarrow K \star I$ produces a map $\mathrm{Hom}_F(K \star I, C) \to \mathrm{Hom}(K, C)$ which corresponds to a map of simplicial sets $C_{/F} \to C$.

Remark A.13. If C is an ∞-category then so is $C_{/F}$. The same holds for the dual construction of *slice under* F, denoted $C_{F/}$. In fact, the projection $C_{/F} \to C$ is always a right fibration while $C_{F/} \to C$ is a left fibration.

Example A.14. Let $d \in C$ be an object in an ∞-category, classified by the functor $d: \Delta^0 \to C$. The corresponding slice category $C_{/d}$ is called the *overcategory*. Note that its simplices are as follows:

0. The objects of $C_{/d}$ are morphisms $c \to d$ in C.
1. The morphisms of $C_{/d}$ are 2-simplices in C with final vertex d.
2. More generally, the n-simplices of $C_{/d}$ are the $(n+1)$-simplices in C with final vertex d.

Note that in particular $N(C_{/d}) \cong N(C)_{/d}$ for any ordinary category C.

We now turn to a 'fatter' variant of slices.

Construction A.15. Let $d \in C$ where C is an ∞-category. We define the *fat overcategory* $C^{/d}$ by the following pullback diagram in sSet:

$$\begin{tikzcd} C^{/d} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, C) \arrow[d, "\mathrm{ev}_0 \times \mathrm{ev}_1"] \\ C \arrow[r, "\mathrm{id}_C \times d"] & C \times C \end{tikzcd}$$

Remark A.16. Recall our definition of mapping spaces in Definition 2.18. Note that for any $c \in C$, we have a pullback diagram

$$\begin{tikzcd} \mathrm{Map}_C(c,d) \arrow[r] \arrow[d] & C^{/d} \arrow[d] \\ \Delta^0 \arrow[r, "c"] & C \end{tikzcd}$$

In other words, we may think of $C^{/d}$ as a parametrized version of the mapping spaces $\mathrm{Map}_C(c,d)$ for varying c. We will make this more precise in Appendix A.3.

Definition A.17. More generally, let $F \colon I \to C$ be a map of simplicial sets. We define the *fat slice over F* in a similar way, through the following pullback diagram in sSet:

$$\begin{tikzcd} C^{/F} \arrow[r] \arrow[d] & \mathrm{Fun}(I \times \Delta^1, C) \arrow[d, "\mathrm{ev}_0 \times \mathrm{ev}_1"] \\ C = C \times \Delta^0 \arrow[r, "\underline{(-)} \times F"] & \mathrm{Fun}(I, C) \times \mathrm{Fun}(I, C) \end{tikzcd}$$

Remark A.18. Just as with the ordinary slices, the canonical map $C^{/F} \to C$ is a right fibration and the dual $C^{F/} \to C$ is a left fibration. In fact, the ordinary and fat slices are connected in another intimate way:

Lemma A.19. *Let C be an ∞-category and let $F \colon I \to C$ be a diagram in C. There are canonical equivalences of ∞-categories*

$$C_{/F} \xrightarrow{\sim} C^{/F}, \qquad C_{F/} \xrightarrow{\sim} C^{F/}.$$

Sketch of proof. By definition, the ordinary slice is defined as a right adjoint to $(-) \star I \colon \mathsf{sSet} \to \mathsf{sSet}_{I/}$. Turning this around we define a new operator $(-) \diamond (-) \colon \mathsf{sSet} \times \mathsf{sSet} \to \mathsf{sSet}$ (the 'fat join') by a similar adjointness property from the fat slice. With model category some work, the statement then translates to one about the left adjoints, namely the canonical map $X \diamond Y \to X \star Y$. A longer dévissage reduces further to a single map $\Delta^1 \diamond \Delta^1 \to \Delta^1 \star \Delta^1$, which is a very explicit calculation. $\square$

Remark A.20. Lemma A.19 together with Remark A.16 suggests another definition of mapping spaces, namely as fibers of the ordinary slice. In the literature, these are known as *left and right mapping spaces* (for $c, d \in C$ in an ∞-category):

$$\begin{tikzcd} \mathrm{Map}^l_C(c,d) \arrow[r] \arrow[d] & C_{c/} \arrow[d] \\ \Delta^0 \arrow[r, "d"] & C \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Map}^r_C(c,d) \arrow[r] \arrow[d] & C_{/d} \arrow[d] \\ \Delta^0 \arrow[r, "c"] & C \end{tikzcd}$$

Since the right vertical maps are left (resp. right) fibrations so are the left vertical ones. But this implies that these left/right mapping spaces are indeed Kan complexes.

Passing to fibers in the equivalences of Lemma A.19 we get the following result:

Corollary A.21. *There are canonical homotopy equivalences of spaces*

$$\mathrm{Map}^l_C(c,d) \xrightarrow{\sim} \mathrm{Map}_C(c,d) \xleftarrow{\sim} \mathrm{Map}^r_C(c,d).$$

A.3 (Un)straightening, discrete

Commentary A.22. We start in the 1-categorical context. Let C be a small ordinary category and $H \in P(C)$ a presheaf of sets on C. The *category of elements* $C_{/H}$ can be thought of in various (equivalent) ways:

(i) Explicitly, it has objects elements $x \in H(c)$ and morphisms from $x \in H(c)$ to $x' \in H(c')$ morphisms $f\colon c \to c'$ such that $H(f)(x') = x$.

(ii) As the full subcategory $C_{/H} \subset P(C)_{/H}$ spanned by representables.

(iii) As the *Grothendieck construction* applied to $H\colon C^{\mathrm{op}} \to \mathsf{Set}$. Note that the category of elements comes with an obvious projection $\pi\colon C_{/H} \to C$ and this has two prominent features:

- Each fiber $\pi^{-1}(c) := (C_{/H}) \times_C \{c\} \cong H(c)$ is a discrete category (that is, a set).
- For each map $f\colon c \to c'$ 'downstairs' and any object $x' \in H(c') \cong \pi^{-1}(c')$ 'over' c', there is a unique 'lift' $x \to x'$ of f. (The latter means that $\pi(x \to x') = f$.)

A functor $\pi\colon E \to C$ satisfying these two properties is known as a *discrete fibration*. It is clear that you can reconstruct the functor H from its category of elements and in fact, this induces an equivalence of categories

$$P(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathsf{Set}) \simeq \mathrm{DFib}(C) \tag{A.23}$$

where the right-hand side denotes the full subcategory $\mathrm{DFib}(C) \subset \mathsf{Cat}_{/C}$ spanned by discrete fibrations.

Exercise A.24. If you haven't seen (A.23) before show this.

There is a direct analogue of this equivalence for ∞-categories. For this recall that a right fibration of simplicial sets has the RLP with respect to right horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$, $0 < i \leq n$.

Proposition A.25 (Straightening-unstraightening, discrete version). *Let C be a small ∞-category. There is an equivalence of ∞-categories*

$$\mathcal{P}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathsf{Spc}) \simeq \mathrm{RFib}(C)$$

where $\mathrm{RFib}(C) \subset (\mathsf{Cat}_\infty)_{/C}$ *denotes the full subcategory spanned by right fibrations.*

Similarly, there is an equivalence for left *fibrations:*

$$\mathrm{Fun}(C, \mathsf{Spc}) \simeq \mathrm{LFib}(C)$$

Commentary A.26. This is not an abstract existence statement. The equivalence can be given quite explicitly (although we won't do that here). One says that a fibration *straightens to* or *is classified by* a functor. And the functor, in turn, *unstraightens to* or *classifies* the fibration.

From the explicit description one deduces that pullbacks on the left fibration side correspond to pre-composition on the functor side. When $C = \Delta^0$ is a point then the equivalence is trivial:

$$\mathcal{P}(\Delta^0) \cong \mathsf{Spc} \xleftarrow{\sim} \mathsf{Spc}_{/\Delta^0} = \mathrm{RFib}(\Delta^0)$$

Together the properties alluded to above imply that $\pi^{-1}(c) \simeq f(c)$.

which, as a first approximation, can be thought of as sending a functor $G\colon C \to \mathsf{Spc}$ to the similarly defined $C_{G/} \to C$.

Commentary A.27. In fact, on objects, an explicit incarnation of this equivalence takes a presheaf $H \in \mathcal{P}(C)$ and sends it to the right fibration $C_{/H} \to C$, the full subcategory of the slice $\mathcal{P}(C)_{/H}$ on representables.[17] Note how the fiber over $c \in C$ is indeed given by

$$\begin{aligned} C_{/H} \times_C \{c\} &\simeq \mathrm{Map}^r_{\mathcal{P}(C)}(\mathrm{y}_c, H) && \text{Remark A.20} \\ &\simeq \mathrm{Map}_{\mathcal{P}(C)}(\mathrm{y}_c, H) && \text{Corollary A.21} \\ &\simeq H(c) && (2.61). \end{aligned}$$

Now, suppose we are given a morphism $f\colon c \to c' \in C$. We should construct a map of spaces $H(f)\colon H(c') \to H(c)$ from the right fibration. For example, given a point $x' \in H(c')$ viewed as a map $\Delta^0 = \Lambda^1_1 \to C_{/H}$ we may consider

$$\begin{tikzcd} \Lambda^1_1 \arrow[r, "x'"] \arrow[d, hook] & C_{/H} \arrow[d] \\ \Delta^1 \arrow[r, "f"] \arrow[ur, dotted] & C \end{tikzcd}$$

[17] It is the pullback of the right fibration from Remark A.13 hence itself a right fibration:

$$\begin{tikzcd} C_{/H} \arrow[r] \arrow[d] & \mathcal{P}(C)_{/H} \arrow[d] \\ C \arrow[r, "\mathrm{y}"] & \mathcal{P}(C) \end{tikzcd}$$

and since the right vertical map is a right fibration, a dotted lift exists. (In contrast to the 1-categorical case we lost uniqueness of a lift but that was to be expected.) The lower commutative triangle says that it lifts f and the upper commutative triangle says that it ends in x'. In particular, it is of the form $x \to x'$ for some $x \in H(c)$ as sought for. Of course, constructing such a map of spaces simplex-by-simplex $H(c') \to H(c)$ would be difficult which is why Proposition A.25 is a powerful statement.

Example A.28. Consider the special case where H is representable, say by $d \in C$. Consider the right fat slice fibration $C^{/d} \to C$ (Remark A.18). By straightening, this is classified by a functor $C^{\mathrm{op}} \to \mathsf{Spc}$ whose value at $c \in C$ is equivalent to

$$C_{/d} \times_C \{c\} \simeq \mathrm{Map}_C(c, d).$$

This explains Remark A.16 where we described $C^{/d}$ as a *parametrized* mapping space.

Commentary A.29. The previous example suggests that we might be able to define a Yoneda functor $C^{\mathrm{op}} \times C \to \mathsf{Spc}$ by constructing a suitable left/right fibration over $C^{\mathrm{op}} \times C$ and then apply straightening. This is indeed possible, and we sketch this now.

Construction A.30. Let C be an ∞-category. The *twisted arrow category* $\mathrm{Tw}(C)$ of C is the simplicial set

$$\mathrm{Tw}(C)_n := \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n \star (\Delta^n)^{\mathrm{op}}, C)$$

Note that $\Delta^n \star (\Delta^n)^{\mathrm{op}} \cong \Delta^n \star \Delta^n \cong \Delta^{2n+1}$ so that the n-simplices of the twisted arrow category are in bijection with the $(2n+1)$-simplices of C. But writing it in this way makes plain the face and degeneracy maps.

Exercise A.31. Figure out what's going on here. Let C be an ordinary category and describe $\mathrm{Tw}(N(C))$. Explain why it is called the twisted arrow category.

Remark A.32. The inclusions $\Delta^n, (\Delta^n)^{\mathrm{op}} \hookrightarrow \Delta^n \star (\Delta^n)^{\mathrm{op}}$ induce a map of simplicial sets

$$\mathrm{Tw}(C) \to C \times C^{\mathrm{op}}$$

which turns out to be a right fibration. Hence, by straightening, it is classified by a functor $C^{\mathrm{op}} \times C \to \mathsf{Spc}$ which one can take as an alternative to the construction of the Yoneda functor in (2.65).