The Navier-Stokes equations: symmetries, transport and turbulence modeling

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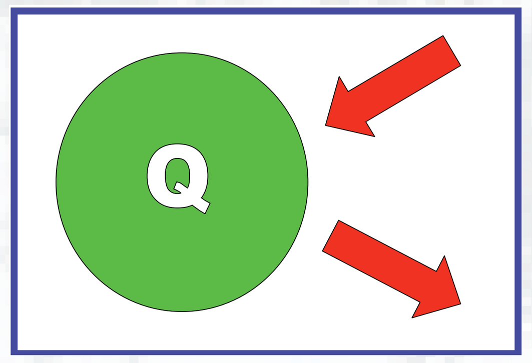
Introduction: Symmetries of a partial differential equation can be enough to understand them completely. The Navier-Stokes eqns are PDEs for which even existence of a general solution is unknown. Before we try to find solutions, a study of conservation laws deepens understanding of fluids and the NSE. However, particular solutions or no, the physics of turbulence (erratic swirling in the fluid) requires a statistical representation that drives us towards turbulence modeling.

Transport equations

We use a fundamental premise, the Reynolds Transport Theorem:

$$\boxed{\frac{\partial L}{\partial t}} + \boxed{\nabla.(Lv)} + \boxed{Q} = 0$$

Change over time
Amount leaving/entering
Creation/destruction inside



for a quantity ***L*** inside some volume. Setting ***L***=*ρ* (density) then ***L=U*** (velocity) and assuming *ρ=const*, the simplest *incompressibility condition*, we derive the **incompressible Navier-Stokes equations (NSE)** (1) & (2). It is not immediately obvious that the pressure term, *P*, depends on ***U*** but if we take the divergence of (1) we get a Poisson equation:

$$\rho(\partial_t + \mathbf{U}.\nabla)\mathbf{U} = \nabla P + \mu\nabla^2\mathbf{U} \quad (1) \qquad \nabla\mathbf{U} = 0 \quad (2) \qquad \nabla^2 P = \nabla.[(\mathbf{U}.\nabla)\mathbf{U}] \ \textit{and initially}\ \nabla\mathbf{U} = 0$$

Solving this tells us a lot about the pressure term; it is a *non-local operator* on ***U***; it transforms as the non-linear term; it propagates incompressibility from an initial condition to all time, **the fluid stays incompressible.**

We are interested in swirling motions of turbulence so the vorticity ***ω***=curl(***U***) is useful. The transport equation follows from (1),

$$(\partial_t + \mathbf{U}.\nabla)\mathbf{w} = \boxed{(\mathbf{w}.\nabla)\mathbf{v}} + \nu\nabla^2\mathbf{w}$$

It also has no pressure term. We choose a form similar to the RTT so that we can deduce the form of Q. Noteworthy is the non diffusive term in Q, it represents vortex twisting and stretching; **the vortices deform smoothly**.

Symmetries of the Navier-Stokes equations

The know symmetries of the NSE are:

Translations: $(\mathbf{x}, t, \mathbf{U}) = (\tilde{\mathbf{x}} + \mathbf{X}, \tilde{t} + T, \tilde{\mathbf{U}})$

Parity: $(\mathbf{x}, t, \mathbf{U}) = (\tilde{\mathbf{x}}, \tilde{t}, -\tilde{\mathbf{U}})$

Rotations and Reflections: $(\mathbf{x}, t, \mathbf{U}) = (A\tilde{\mathbf{x}}, \tilde{t}, A\tilde{\mathbf{U}})$

Galilean transformation: $(\mathbf{x}, t, \mathbf{U}) = (\tilde{\mathbf{x}} + \mathbf{c}t, \tilde{t}, \tilde{\mathbf{U}} + \mathbf{c})$

Scaling: (if $\mu = 0$) $(\mathbf{x}, t, \mathbf{U}) = (\lambda^{-1}\tilde{\mathbf{x}}, \lambda^{h-1}\tilde{t}, \mathbf{u} = \lambda^{-h}\tilde{\mathbf{U}})$.

Symmetries:
Transformations
$$(\mathbf{x}, t, \mathbf{U}) \to (\tilde{\mathbf{x}}, \tilde{t}, \tilde{\mathbf{U}})$$
under which the NSE are unchanged. On the left ***X, U*** & ***c*** are vectors, *λ* is a number and *A* is a *matrix*. Note we are considering an **infinite fluid**

There is a variation of Galilean transformations that preserve isotropy; random Galilean transformations take U to be randomly isotropically distributed. This transformation helped improve models which originally broke the symmetry.

The more complicated ***anomalous scaling*** is a symmetry of Navier Stokes (irrespective of *μ*) it is anomalous since it is inferred by observing simulations or real flows. The symmetry is thought to come from the non-linear mixing dynamics of the non-linear term, i.e. the second term, and the pressure term.

Given symmetries we can calculate the *infinitesimal symmetry generators* - operators that make the infinitesimal changes of the transformation. For example, the generators corresponding to time and Galilean translations are

$$\partial_t \ \& \quad t\partial_i + \partial_{u^i} \quad \text{where} \quad \partial_i = \frac{\partial}{\partial x_i}$$

Altogether they generate the complete symmetry algebra of the NSE. This algebra has various uses & the fact that there is no specific method contributes to why we do not know more about general solutions of the NSE.

A general approach is to try and *linearise* the pde (this is often influenced by the algebra) reduce this to an ode (ordinary d.e.) and solve using a *solvable sub-algebra* of the ODE's symmetry algebra. For the NSE the computational demands make this a difficult process and often certain ansatz are used.

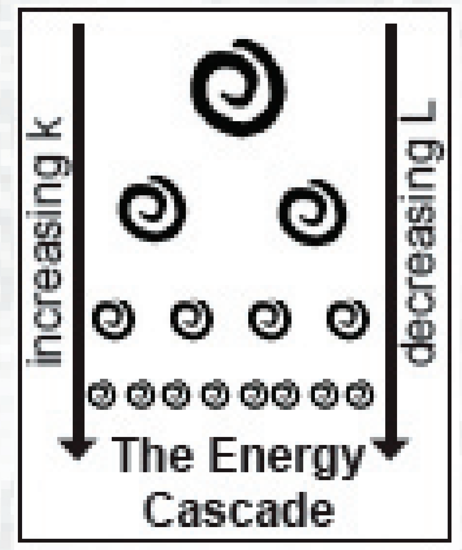
Turbulent kinetic energy

The nature of turbulence favors averaging (angular braces) and thus the **Reynolds decomposition.** ***u*** is called the *fluctuating velocity* & <***U***> the *bulk*. <***U***> and ***u*** have transport eqns derived from the NSE. In these appears the **Reynoldsstress tensor** *ρ*<***uu***>; ***u*** transports mean-momentum across the boundary of a fluid volume giving an apparent stress (and hence force). With this we can write down the turbulent (fluctuating) K.E. eqn

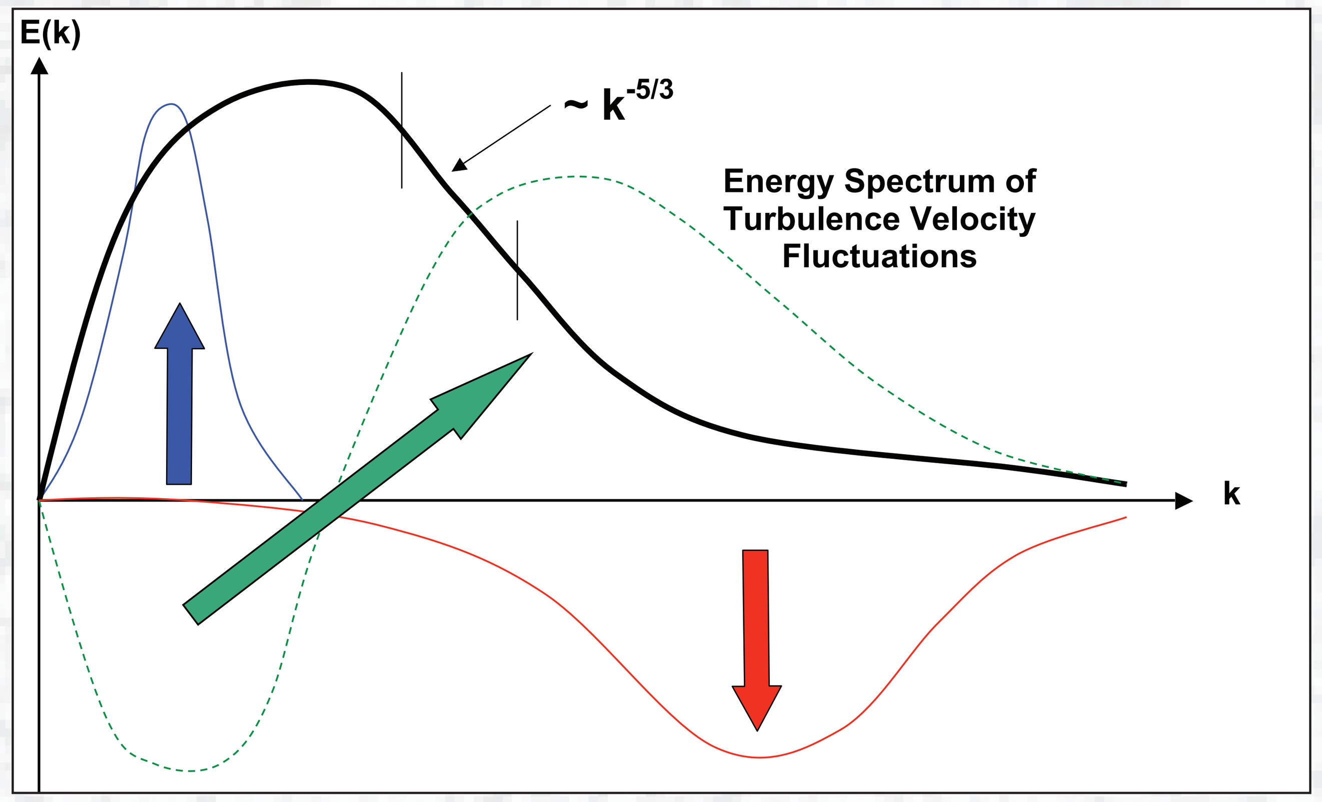
$$\mathbf{U} = \langle\mathbf{U}\rangle + \mathbf{u} \qquad P = \langle P\rangle + p$$

$$T := \frac{1}{2}\langle\mathbf{u}.\mathbf{u}\rangle, \quad (\partial_t + \boxed{\mathbf{U}.\nabla})T + \nabla.\Psi = \boxed{\pi} - \boxed{\epsilon}$$

Inertail transfer (mixing)
K.E. production
dissipation.



The production term comes from the Reynolds stress i.e. the bulk K.E. goes into the turbulent K.E. The dissipation is the loss of K.E. to viscosity; this occurs at the *small scales* or eddies. A reasonable approximation of the mixing is that only eddies of comparable size interact (otherwise one transports the other). All this leads to the concept of the **energy cascade**.



k, the ***wavenumber***, is the variable of the Fourier transformed eqn, k is inversely proportional to length **small k = large scales, large k = small scales.**

This graph is an empirically informed account of how the energy transfers through turbulence via the energy cascade.

The 5/3 relation is an interesting result from dimensional analysis in the regimes where the energy spectrum (i.e. in terms of wavenumber) only depends on dissipation

The problem of closure and turbulence modeling

First off we define <***UU...U***>, an averaged product of n realizations of the velocity, as an **n-th order moment.**

To outline the problem we first simplify (1) and then average. This eqn involves a 2nd order moment and we have no other equations to hand so the only way to find an expression for this is to multiply through by ***U*** and average, this process goes on.

$$(1) \to \mathcal{L}U = \mathcal{N}UU \to \mathcal{L}\langle U\rangle = \mathcal{N}\langle UU\rangle \to \mathcal{L}\langle UU\rangle = \mathcal{N}\langle UUU\rangle \to \ldots$$

Simplifying the NSE:
L is the operator for *linear* terms and ***N*** for *non-linear terms.* Since we can rewrite *P* using (2) gives our form

This leads to the **moment hierarchy**, a series of equations that needs **closure**. Solving this problem is the basis of turbulence modeling. Many different approximations are used but one of the simplest and most widely used is the **turbulent viscosity hypothesis**. This hypothesis says we can treat the Reynolds stress as a real stress, proportional to strain. This leads to an eqn for <***UU***> and we are done. That said the approximation will not always be valid.

Applying the hypothesis to the K.E. equations & using empirical observation gives the ***K-ε* model**: closed coupled PDEs that can be solved numerically or using the same sort of methods as for the NSE to give predictions for the **statistical development of turbulence.**

References

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