Circle Packing

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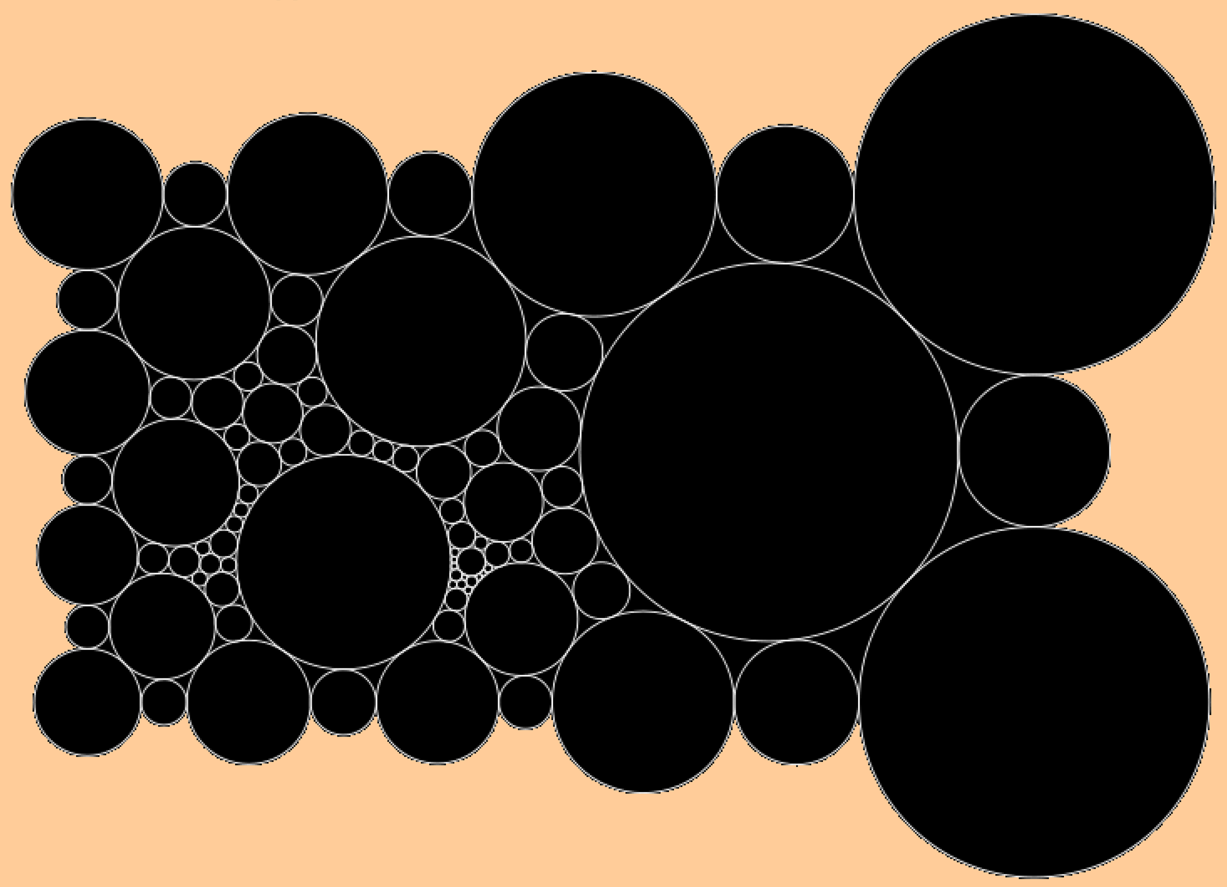
Circle packing is a subject in mathematics with both a visual and a purely mathematical appeal, having originally been introduced by William Thurston whilst studying 3-manifolds, and more recently being treated as its own subject in Kenneth Stephenson's *Introduction to Circle Packing*.[1]

What is Circle Packing?

A circle packing may be described as a configuration of circles with a specified tangency pattern. The tangency pattern is a list of all circle centres and for each centre an ordered list of the neighbouring circles. This may be encoded as a *triangulation*. A triangulation is a set of triangles glued together, sharing common edges. The vertices represent the circle centres, and the edges between them signify they are neighbouring circles. This information is purely combinatorial; there are no geometric details such as lengths and coordinates.



An example of a circle packing

For a given triangulation one may give a *label*, or a set of real numbers associated to each vertex. This label serves as tentative radii for the circles. The label is called a *packing label* for the triangulation if the angle sum arising at each vertex from the given label is 360˚, in which case we have a circle packing.

Aims

Given a circle packing we can obtain a triangulation by means of encoding the tangency pattern. In the topic of circle packing there is an important theorem that tells us that this function has an inverse, i.e. given any triangulation we can find a packing label.

My task consisted an implementation of the algorithm, due to Thurston, for finding the packing label of a triangulation, and thus obtain a circle packing. The objective was to design a computer program using the C++ language that would draw an image arising out of a set of combinatorial data representing planar graphs, the latter being part of my supervisor's research.[2]

Initial Data

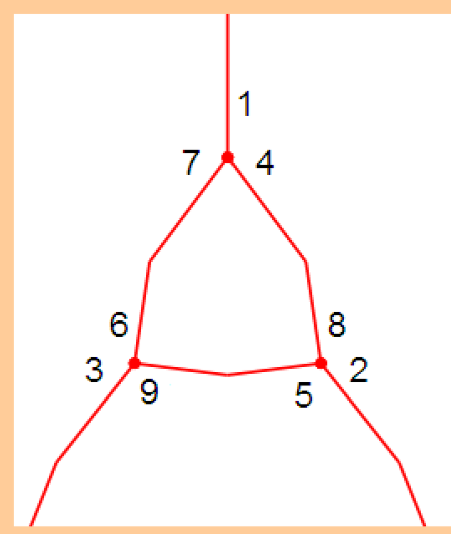
The combinatorial data that was the starting point consisted of four parts:

•two functions: a rotation *c* and an involution *e*

•two integer values *B*, *E*

	1	2	3	4	5	6	7	8	9
c	4	5	6	7	8	9	1	2	3
e	-	-	-	8	9	7	6	4	5

B = 3 E = 6



A planar graph defined by the above combinatorial data

Out of these data an abstract planar graph arises. By performing the process of barycentric subdivision the planar graph is triangulated, and from the theorem we know there is a circle packing to be had from this.

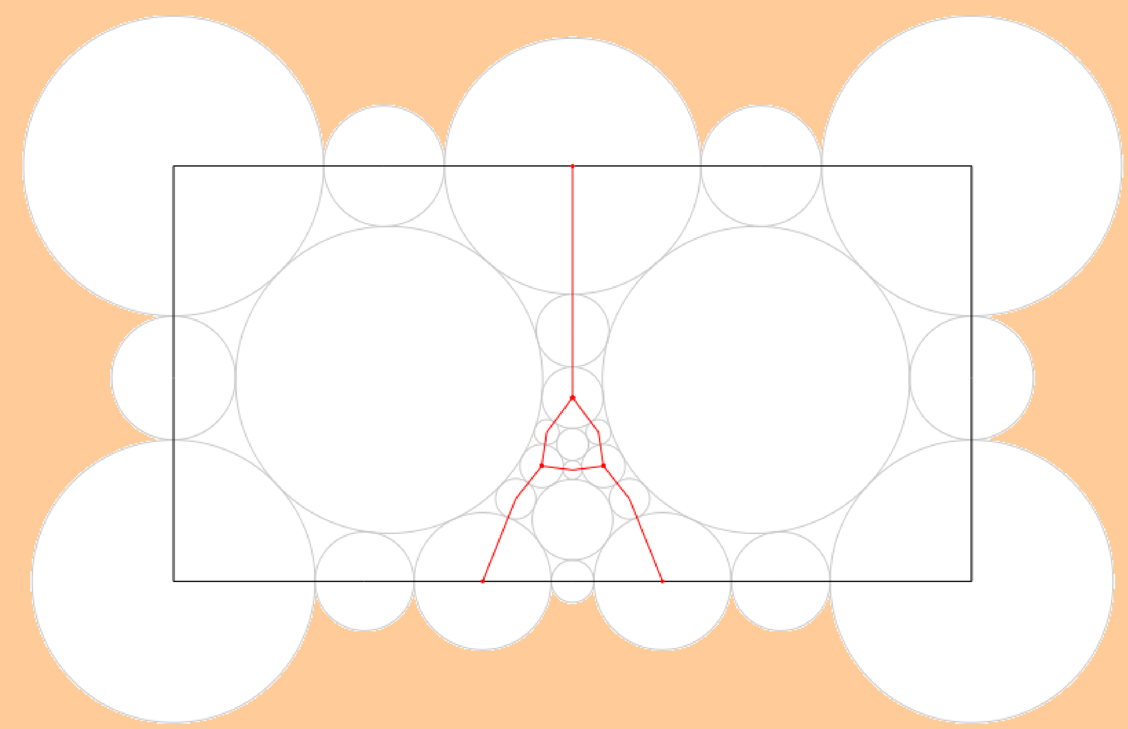
Applying the algorithm

The algorithm goes through two stages.

In the first stage the label of the triangulation is made into a packing label, thus giving the radii of all the circles. At this stage there is still no information on how the circles can be placed. The coordinates of the circle centres are calculated in a process called the *layout*; this is the second stage of the algorithm.

Final Image

Each circle represents a vertex in the triangulation. Having stored additional information concerning the individual vertices, the program is in a position to distinguish the vertices of the planar graph and the barycentres coming from the barycentric subdivision, and thus proceeds to draw the edges and vertices of the planar graph.



The produced image, with the underlying circles in the background

Conclusion

The computer program is now operational, providing the user with a tool that quickly produces the image of the planar graph arising from the given combinatorial data.

On a final note, I would like to thank the URSS for their financial support, and I would also like to thank Dr. Bruce Westbury for his guidance, as well as Kenneth Stephenson for his excellent book on the subject of Circle Packing.

References

1. Stephenson, Kenneth. *Introduction to Circle Packing*. Cambridge University Press, 2005
2. Sikora, Adam S., Westbury, Bruce W. *Confluence Theory for Graphs*. Algebraic & Geometric Topology, 7:439-478, 2007

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