Train track geometry

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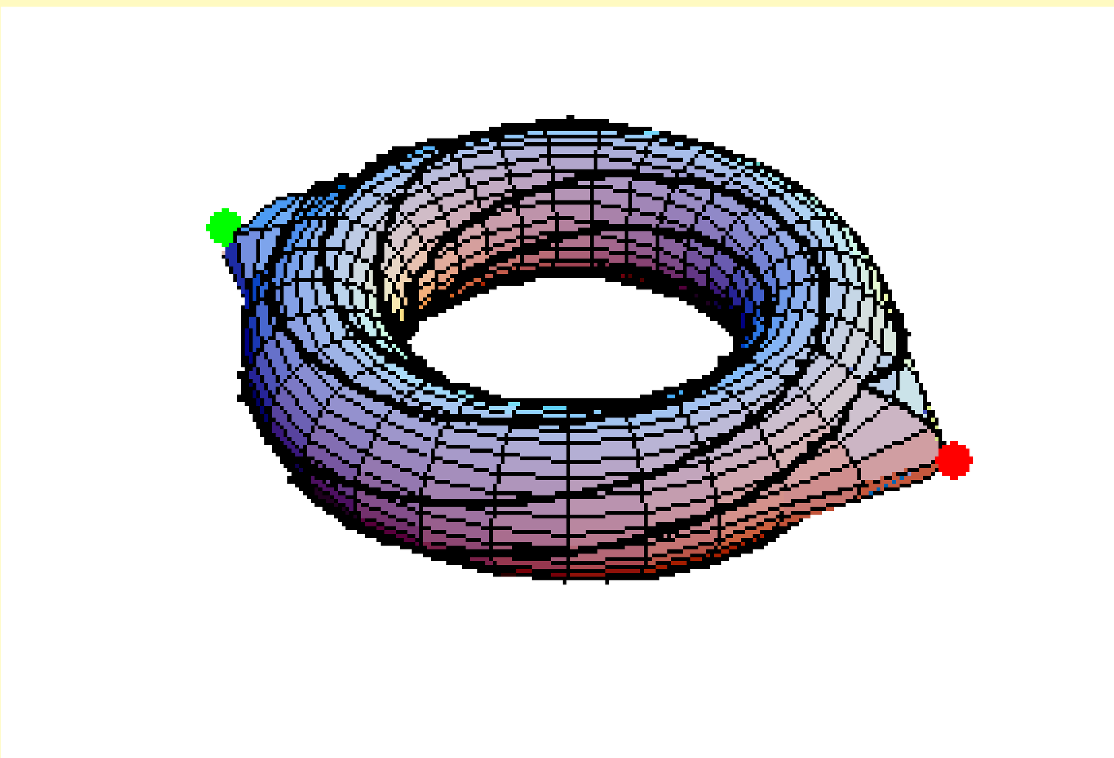
Introduction

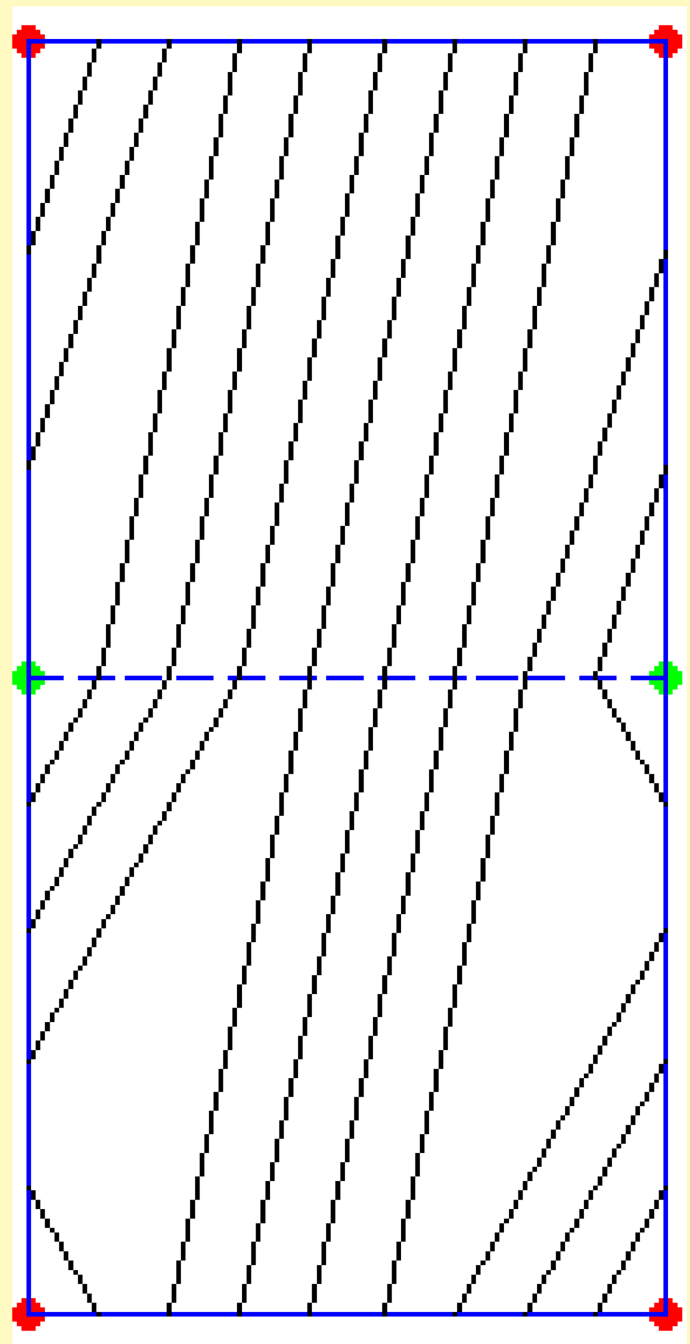
This project looked at the simple, closed curves on the twice punctured torus. A curve is simple if it does not intersect itself and closed if it forms a loop. A torus is most often described as like a doughnut, however, due to the presence here of punctures, perhaps an inner tube is more apt. A crucial tool in our study was the train track, first introduced by Thurston. Using a certain type of train track, Keen, Parker and Series found that curves could be represented by four coordinates, something which uncovered a rich seam of results. To develop those results further, it became necessary to study the wheels of curves and develop a catalog of such wheels, something this project has accomplished.

Surface curves



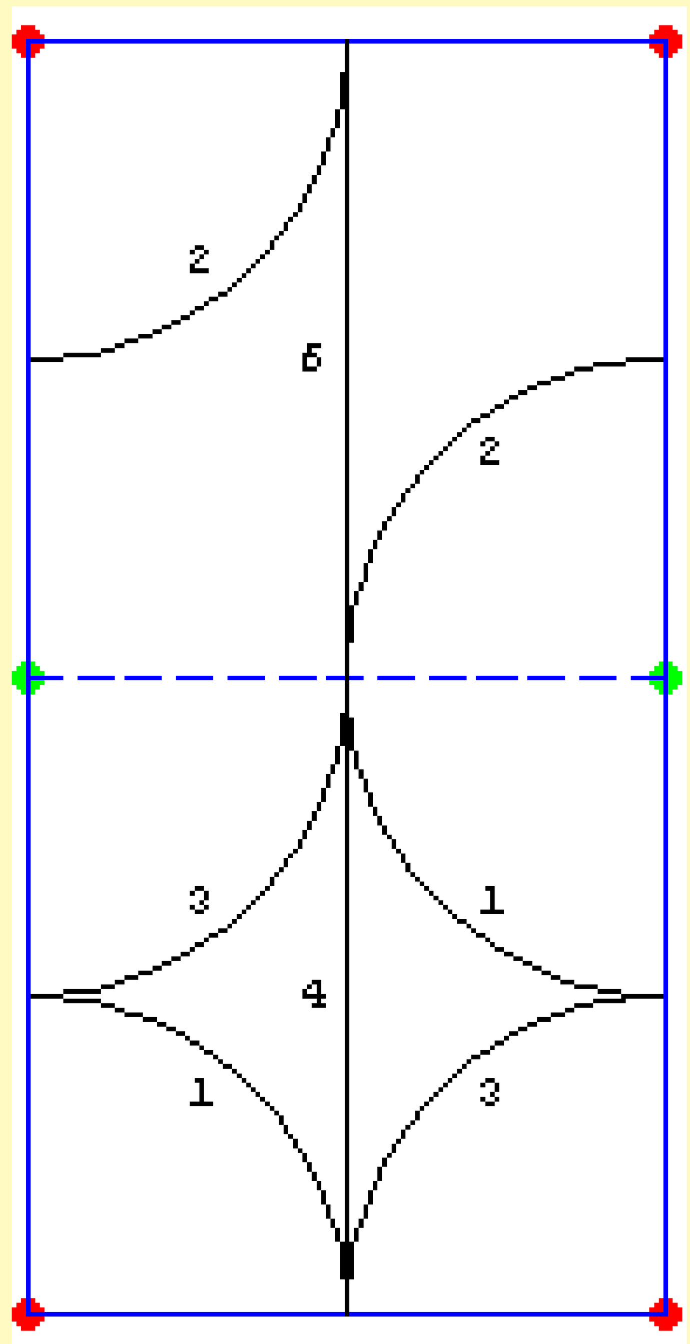
A twice punctured torus, one puncture red, the other green, with a surface curve winding around it. This curve is called separating, were you to cut along it, you would be left with two distinct pieces, one a disc containing both punctures, the other a torus with a hole. If, when you cut along a curve, the surface remains in one piece we shall say the curve is non-separating.

Train tracks

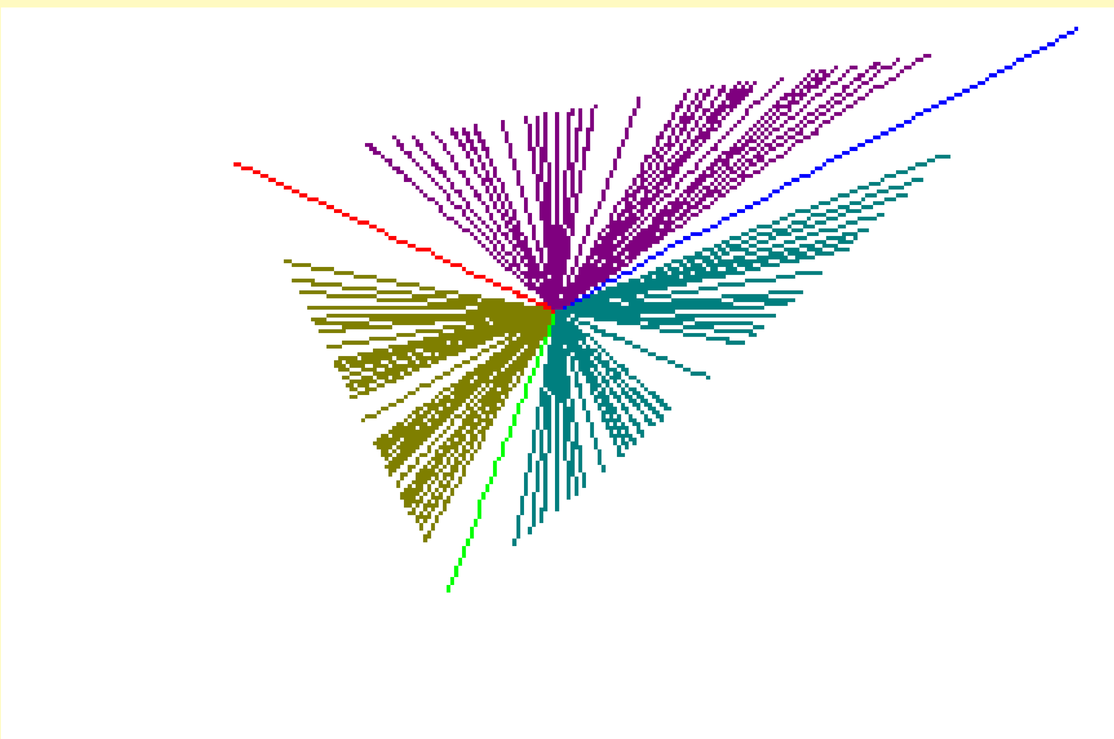


Imagine cutting the surface through the red puncture so it opens into a cylinder, the green puncture at its midpoint. Now cut again, along the length of the cylinder, through the green puncture. The surface unfolds into the diagram on the left; the same curve runs across it. Doing so has made it far easier to identify this curve as separating.

If we collect together strands that join the same sides and label the result with that number of strands, we get the diagram on the right. This is a train track; the numbers are called weights. A set of simple formulae relating the weights gives the coordinates of the curve, in this case (8,2,6,2).

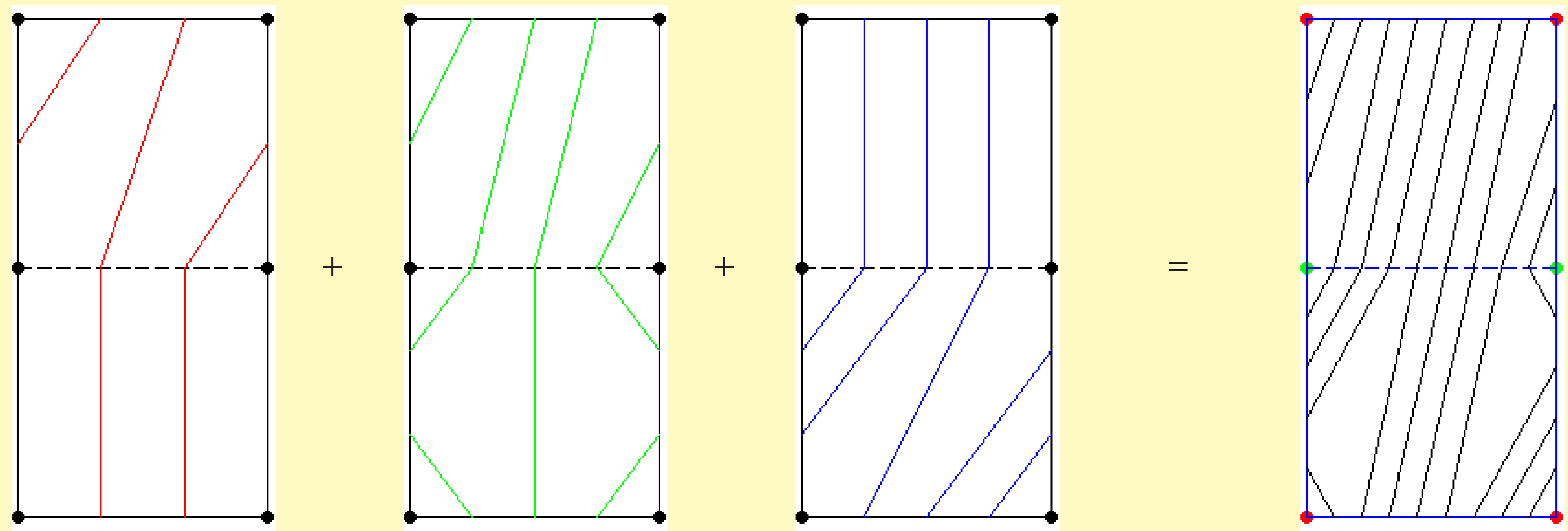


Wheels



A wheel is a given curve and all curves disjoint from it. The given curve is the centre, those disjoint from it make up the rim. A line joining the centre to a point on the rim is called a spoke. The picture is an abstraction, the coordinates have allowed us to project to 3-space and plot curves as points.

Corners



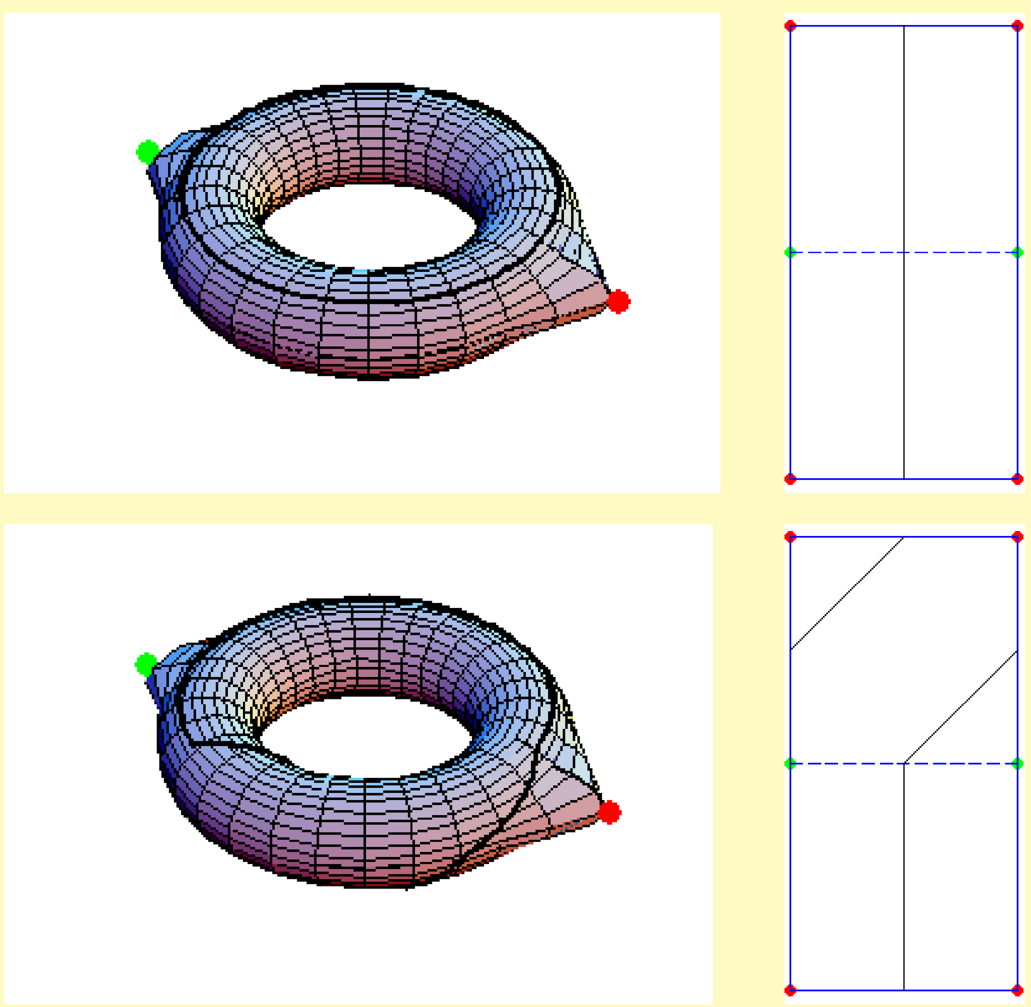
The first three curve diagrams above are the corners of the wheel shown on the left, they are coloured accordingly. These are end points of rim segments. Curves can be combined; every point on a given rim segment is a combination of its endpoints. Finding the corners allows us to calculate all curves in the wheel. It is a key result of this project that an algorithm was developed that, given a separating curve, identifies the corners and draws the wheel.

As is suggested by the pictorial equation, this centre is the sum of its corners. Many similar fascinating relationships exist.

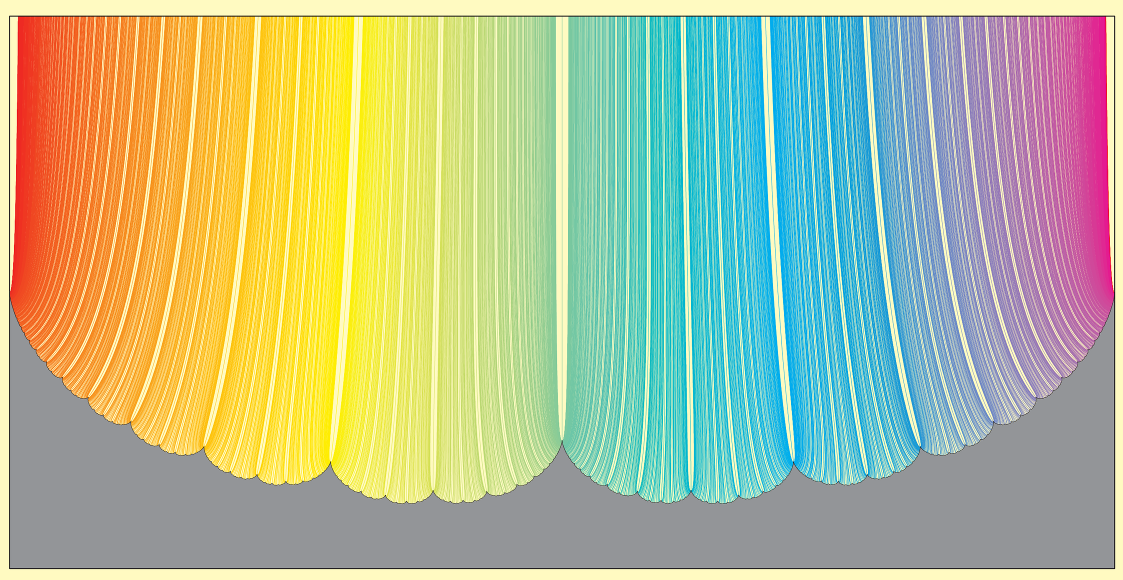
Dehn twists

If the centre is non-separating, there is less structure, and the corners are more difficult to identify. We can, however, take a very simple wheel and transform it into a more complicated one. This is achieved using Dehn twists, functions which take a curve and wrap it an additional number of times round the surface in a particular direction.

The diagrams on the right show the effect of one particular Dehn twist, the top is before, the bottom after. It was a major achievement of the project that a computer program has been written that performs any combination of these intricate transformations. Remarkably, any topology preserving transformation of the surface and its curves can be represented as a sequence of Dehn twists.



Maskit embedding



The Maskit slice of the once punctured torus (Keen, Series and Wright). An explanation would not fit here, the interested are directed to the excellent Indra's Pearls by Mumford, Series and Wright. This project laid foundations for a similar visualisation, but for the twice punctured torus.