The Origin of Particle Properties from an Action Principle

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1. Introduction

Modern Particle Physics ,the study of the fundamental elements of matter, including electrons and protons, is based upon the Standard Model, a theory which has been rigorously proven by experiment to be an extremely accurate model of nature. There is a problem however: many of the properties of the particles, specifically their masses and mixing angles (to be discusses later), are not predicted by the Standard Model and must be added in. This stands in stark contrast to the fundamental aim of physics: to reduce the laws of physics to more general ones with, very importantly, a sense of inevitability about them i.e. that once the law has been discovered it should seem as if there could have been no other way of describing nature. The purpose of this project was to solve this problem, to find a way to predict these masses and mixing angles, and the relations between them, which would otherwise have been added in, using something known as an Action principle. This is essentially a mathematical device, which when minimized (i.e. made as small as possible) gives the correct laws of nature. An analogue can be made with light, where the action is dependent upon the time the light takes to travel between two points. This means that when the action is minimzed the light takes the path for which it gets between two points the quickest, which is a straight line in free space.

2. Theory

The weak force, one of the four fundamental forces, is described by the Weak Lagrangian, as shown in the figure below.

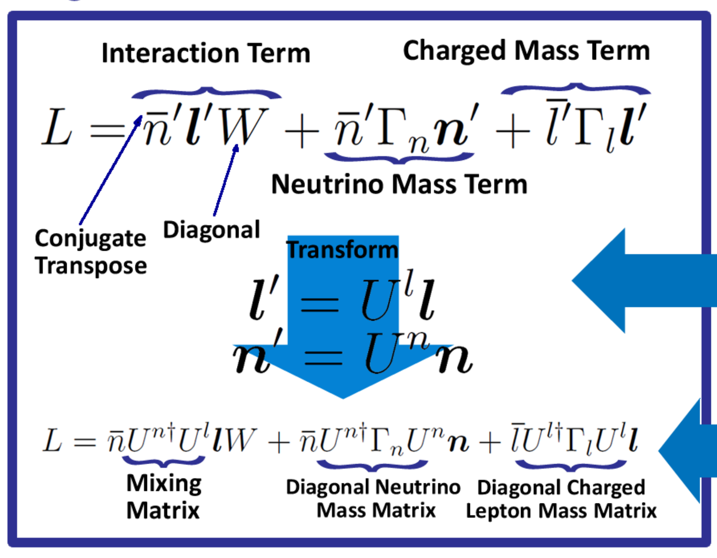


Figure 1: Transforming the basis.

This is the Lagrangian in the basis in which the weak interaction matrix is diagonal, but the mass matrices, given by gamma, are not. This is the weak basis; note that if the mass matrices were also diagonal in this basis there would be no mixing.

Transform the basis to that in which the mass matrices are diagonal, using the unitary transforms given by the U matrices.

This is the basis in which flavour states are eigenstates. Note now W is not diagonal and is proportional to the mixing matrix.

The presence of this mixing matrix means that interaction and mass states are not simultaneously both eigenstates of the Weak Lagrangian. This causes the phenomenon known as mixing, whereby a particle produced in a certain flavour state is in a linear combination of the three mass states. Mixing is most obviously manifest in neutrino oscillations, whereby a neutrino produced in a certain flavour state can be measured as any one of the three mass states: electron, mu and tau, with varying probability. The mixing matrix depends on the three mixing angles mentioned previously, and one phase, which determines the amount of CP violation.

3. The Method

This action to be extremised took on a specific form, due to the constraint that it must be basis invariant (see "Why Basis Invariance?"). A method of doing this was to write the action in terms of traces of the mass matrices, given by gammas above and here by L and N for charged leptons and neutrinos respectively.

As an example the action A = Det C was extremised, where C = -i[L, N]. This action is proportional to the Jarlskog invariant, a quantity which is proportional to the amount of CP violation in the weak interaction. Using the fact that Det C = Tr C^3 for a traceless matrix like C, this action was extremised with respect to both the L and N matrices. Two relations were then obtained:

$$\partial_L TrC^3/3 = -i[N, C^2]^T = 0$$

$$\partial_N TrC^3/3 = i[L, C^2]^T = 0$$

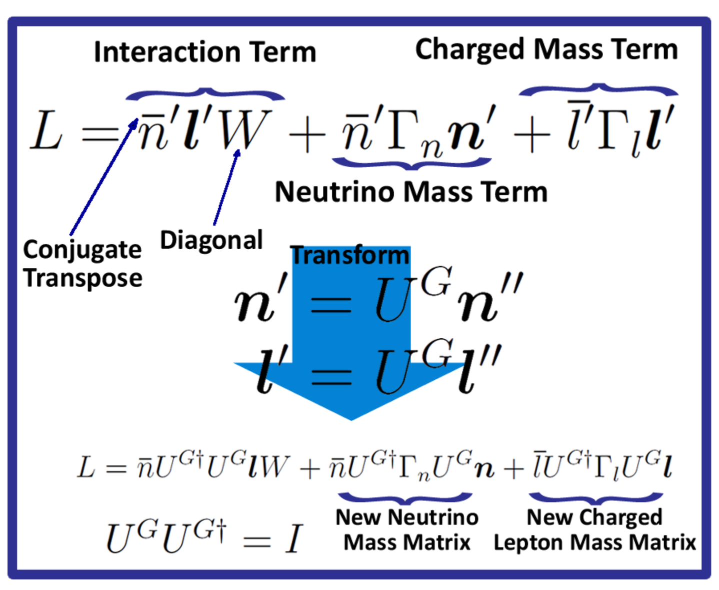
These conditions gave a mixing matrix which maximised CP violation, as expected. This is not consistent with the data, however, so a more complicated action was used. This was formed from elements of the Q matrix (see later), which are also related to CP violation.

This action was extremised in Prof. Harrison's paper (hep-ph/0508012) using Lagrange multipliers, which forced the masses to stay fixed and reduced the number of constraints considerably. This allowed a solution to the above conditions to be obtained which predicted the correct lepton mixing matrix and neutrino masses. This form of the mixing matrix is known as the Tribimaximal matrix (shown below) and is consistent with all current neutrino oscillation data. It is so named as one column contains three elements of maximum size and another contains two, considering also that the columns must sum to one. Obtaining this matrix, along with all six lepton masses was the ultimate goal of this project.

$$\begin{array}{c} \\ e \\ \mu \\ \tau \end{array}\begin{array}{c} \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \\ \left(\begin{array}{ccc} 2/3 & 1/3 & 0 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/3 & 1/2 \end{array}\right) \end{array}$$

Figure 2: The Tribimaximal form of the lepton mixing matrix (known as the PMNS matrix). Obtaining this matrix from extremisation of an action is highly indicative that the method was on the right track.

4. Why Basis Invariance?



Start with the Weak Lagrangian in the basis in which the interaction matrix W is diagonal, and the mass matrices are not.

Now perform the same general unitary transform on both neutrino and charged lepton fields at once, thereby changing the basis.

Now the lagrangian is in a new basis, but note the identity in the bottom left hand corner. This means that W is still diagonal and so for all intents and purposes we are still in the same basis. This means that whatever conclusions are made for the mass matrices in one basis must work for all, hence basis invariance.

5. Extending the Action

The initial stage of the project entailed extending the action formed from elements of the Q matrix, mentioned in section 3, to predict the correct masses and mixing angles without the Lagrange multipliers. Removing the Lagrange multipliers resulted in the extremisation method effectively gaining six new constraint equations: three from the diagonal elements of the derivative with respect to the charged lepton mass matrix (L) and three from the on-diagonal elements of the neutrino mass matrix (N) derivative. It was found that the solutions with the Lagrange multipliers used in Prof. Harrison's paper did not satisfy all of the new constraints and so could not be used. In fact, there was no way of satisfying all the constraints, and so this action could not be the one we were looking for. This action was A = Q_{11} + qQ_{21}, where q was a free parameter; it seemed like a good idea to try the action A = Q_{11} + qQ_{31} next, to see if that would work. These Q's are elements of the Q matrix, defined below. Note also that L and N were taken to be in the basis in which L was diagonal and N was not, being in the epsilon basis instead. This does not violate basis invariance; as long as the extremisation is done basis invariantly we are free to chose a basis afterward.

$$Q = \frac{1}{2}\left(\begin{array}{ccc} \mathrm{Tr}\,[L,N]^2 & \mathrm{Tr}\,[L,N][L,N^2] & \mathrm{Tr}\,[L,N^2]^2 \\ \mathrm{Tr}\,[L,N][L^2,N] & \mathrm{Tr}\,[L,N][L^2,N^2] & \mathrm{Tr}\,[L,N^2][L^2,N^2] \\ \mathrm{Tr}\,[L^2,N]^2 & \mathrm{Tr}\,[L^2,N][L^2,N^2] & \mathrm{Tr}\,[L^2,N^2]^2 \end{array}\right)$$

Figure 4: The Q matrix. The elements of this matrix are related by a simple transform to the K matrix, which is the CP conserving analogue of the Jarlskog invariant J. In this way extremising Q elements also involves extremisation of CP violation.

Extremisation of A = Q_{11} + qQ_{31} was done in the same fashion as in section 3, still without Lagrange multipliers. It was found again that the solutions failed to satisfy the conditions from the N derivatives. Clearly a new approach was needed. At this point it should be noted that, in order to make finding solutions easier and to match observational evidence, solutions were assumed to obey S3 symmetry. This meant assuming that the rows and columns of N summed to the same amount. Could this have been wrong?

6. Expanding from S3

It is known that the N matrix obeys S3 symmetry to a high degree, but it is possible that it isn't quite S3 symmetric. This hinted at the idea of Taylor expanding the solutions to the action A = Q_{11} + qQ_{21} in small deviations from S3 symmetry. First the solution before applying S3 was taken (x, a, b and c are elements of N below):

$$x = \pm\sqrt{(a-b)(c-a)E}$$

E is a function of charged lepton masses and q.

S3 implies that (a-b) = (x-y) and (c-a)=(z-x). To extend the solution from S3 these relations became: (a-b) = (x-y) + α and (c-a) = (z-x) + β, where α and β are small parameters. The above expression, and those for y and z, was Taylor expanded about α = 0 and β = 0 i.e. the S3 solution.

$$N = \begin{array}{c} \\ e \\ \mu \\ \tau \end{array}\begin{array}{c} \begin{array}{ccc} e & \mu & \tau \end{array} \\ \left(\begin{array}{ccc} a & z+id & y-id \\ z-id & b & x+id \\ y+id & x-id & c \end{array}\right) \end{array}$$

This expansion gave x as a power series:

$$x = x_{S3} + a_1\alpha + b_1\beta + \mathcal{O}(\alpha\beta, \alpha^2, \beta^2)$$

Since α and β were small the higher order terms were ignored, giving a simple expression for small S3 violation. This form for x, and those for y and z, were substituted into the N derivative constraints which gave the troulbe before. Unfortunately the solutions were still not viable and so this method could not work.

7. New Actions: The D Matrix

The failure of the S3 extension prompted the search for new actions to spread further afield. Previous actions had focused on elements of the Q matrix, which is formed from elements of the K matrix. These elements are the real parts of the matrix of mixing plaquettes, Π, whose imaginary parts are the Jarlskog invariant J, which measures CP violation. It seemed possible, therefore, to perform a similar transform which gave Q from K on the matrix formed entirely of J's to give a new matrix D (where D_{11} = 9 Det[L, N]).

Theoretically this D would conserve or violate CP oppositely to Q; an action could then be formed from both Q and D elements which, when extremised, would neither totally violate or conserve CP. The action A = Q_{11} + qQ_{21} + r(D_{11} + εD_{21}) was extremised, where the free parameter r determined the relative contribution from the CP conserving and violating parts. This meant that r should be either very large or very small depending on the nature of the extremum, as CP is violated to only a small degree in nature. Another motivation for using this action was that Q_{11} + qQ_{21} worked very well until it had to satisfy the extra N derivative constraints; the implication was that this action was incomplete and formed part of a larger action, such as the one above. Cyclic solutions for x, y and z were used:

$$x = \frac{\sqrt{XYZ}}{X} \quad y = \frac{\sqrt{XYZ}}{Y} \quad z = \frac{\sqrt{XYZ}}{Z}$$

X, Y and Z are functions of the charged lepton masses and r, ε and q.

This action was more successful than any previously at solving all the constraints, but was still not ultimately successful. The action must therefore have been too simple, or perhaps it did not take account of enough parameters other than CP violation. New actions had to be analysed.

8. New Actions: FSMOs and the Koide Relation

Previously all actions focused on the extremisation of CP violation, but the mixing matrix exhibits other symmetries of interest. These are described by flavour symmetric mixing observables (FSMOs), which are quantities similar to the Jarlskog invariant, J, but for different symmetries. For example the FSMO F determines the degree to which the mixing matrix has a trimaximally mixed column i.e. when F = 0 one of the columns of the PMNS matrix has all elements equal to 1/3. Note also that F = 0 when S3 symmetry is obeyed by N. It is possible that a potential action could be formed from a function which, when extremised, caused all of the significant FSMOs to become small i.e. nearly zero. This could be found by expanding the relevant FSMOs and seeing if any have a leading term in common.

Another possible hint at the true action is the Koide relation, shown below. This is a very siginficant and mysterious result, showing that a seemingly random combination of lepton masses gives zero, an exact quantity, only for their experimental values. This seems to be closely related to the precise nature of the Tribimaximal matrix. Is it possible that the correct action should predict this relation?

$$512L_3L_1 - 64L_2^2 - 656L_2L_1^2 + 207L_1^4 = 0$$

L_1 = Tr L
L_2 = Tr L^2
L_3 = Tr L^3

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