THE UNIVERSITY OF WARWICK

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Toroidal Alfvén Eigenmodes

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One of the greatest technological challenges facing modern science is the world's energy needs and one possible solution is nuclear fusion. A test reactor, ITER, which uses strong magnetic fields to confine a Deuterium-Tritium plasma in a torus, is being constructed in southern France. The plasma needs to be heated to a temperature of the order of 100,000,000 K to achieve sustained fusion and so a major challenge is to reach this incredible temperature. A central aspect of the working and stability of a tokamak is the presence of magnetohydrodynamic waves, of which Toroidal Alfvén Eigenmodes (TAEs) are an important ingredient. They may influence the sustained fusion process in that they may be excited by the byproduct of the fusion reaction, alpha particles, and hence syphon-off energy available for reheating the plasma.

Introduction

Alfvén waves are a form of magnetohydrodynamic wave which propagate parallel to the magnetic field and where the ions and field lines oscillate transversely much like a wave on a taut string, and are essentially incompressible. Fig. 1 compares the phase and group speeds of Alfvén waves with the fast and slow magnetoacoustic wave modes. In each case, the angle is relative to the magnetic field and generally the Alfvén speed is between the slow and fast wave speeds. The group velocity of the Alfvén wave is always parallel (or antiparallel) to the field, the magnitude of which is always the Alfvén speed:

$$v_A = \sqrt{\frac{B}{\mu_0 \rho}}$$

where B is the magnetic field strength, μ_0 is the permeability of free space and ρ is the plasma density.

Tokamak Equilibrium

A tokamak is much like a torus (doughnut) and therefore has rotational symmetry about a central axis (through the hole in the doughnut), a rotation about which would be a toroidal motion. The distance from this central axis to the centre of a the circular cross-section of the torus shall be referred to as the major radius. The minor radius shall describe the size of the cross-section. A rotation about the axis through the centre of the cross-section and looping toroidal through the torus (defining a circle of major radius) would be poloidal. In other words, the poloidal direction is along the circumference of the cross-section. A useful tokamak, however, would not have circular cross-section and is more likely to resemble the cross-section seen in Fig. 2. The magnetic field within a tokamak is helical and so, in discussing tokamaks, a coordinate system must be chosen which simplifies the problem. It is useful to introduce the flux surface: the surface of a torus where both the pressure and electric current density are constant.

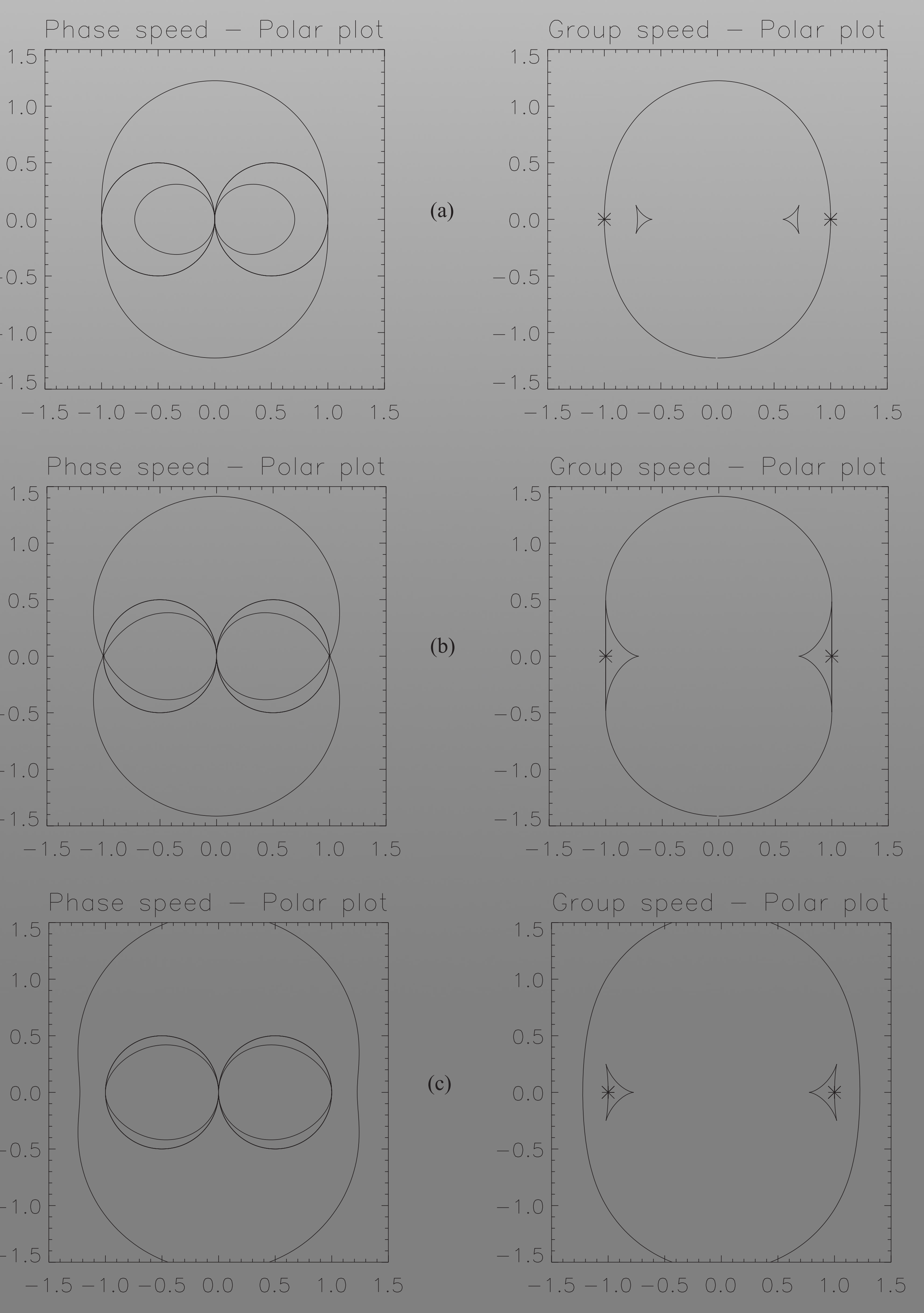

Figure 1: Comparison of Alfvén waves with fast and slow magnetosonic modes. All speeds are normalized to the Alvén speed, v_A. (a) β=0.5 (b) β=1.0 (c) β=1.5 where β is the ratio of gas pressure to magnetic pressure.

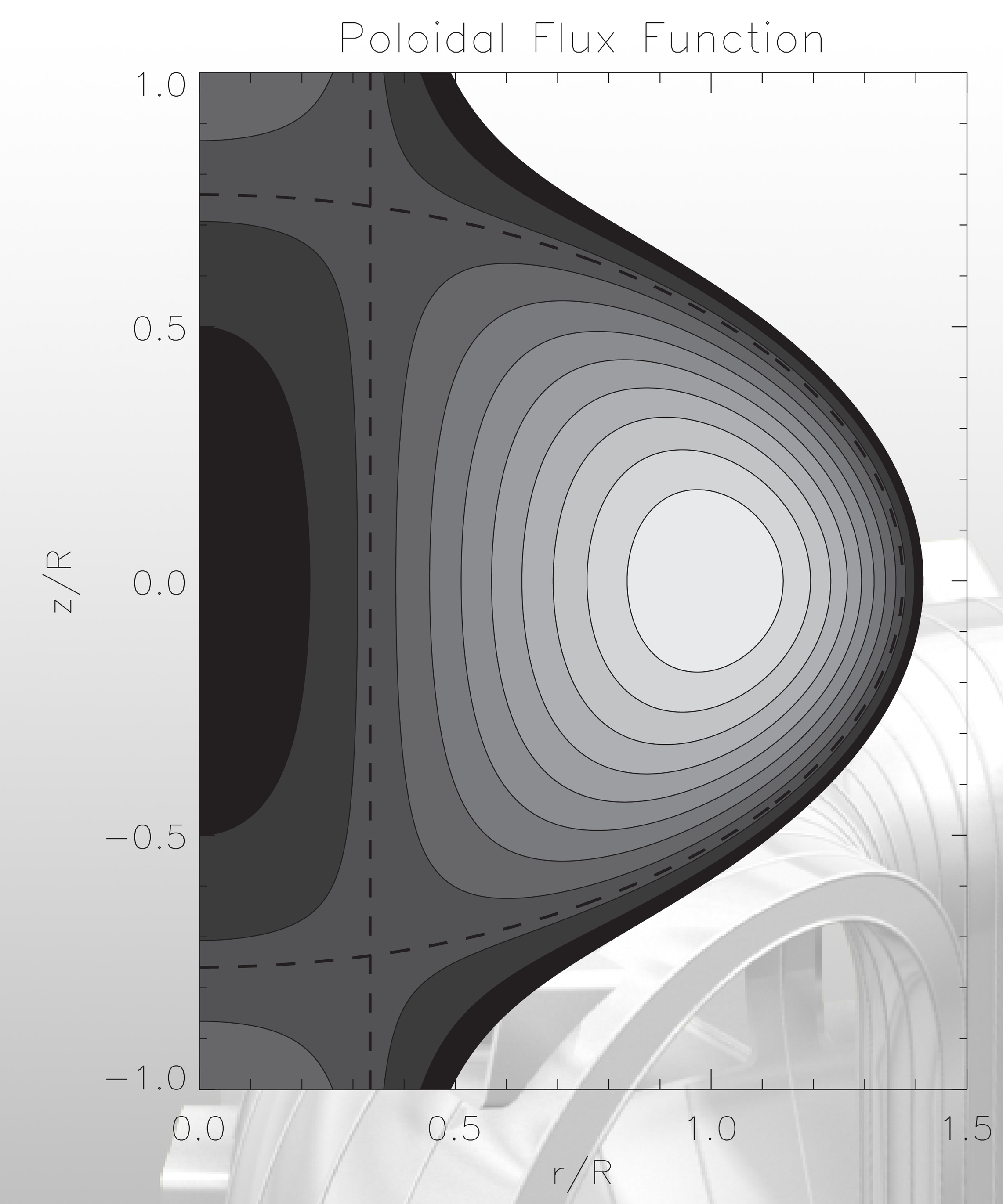

Figure 2: Flux surface plot for Solov'ev equilibrium normalised to the major radius of the torus. The dotted line marks the boundary of the plasma.and where they cross indicates a saddle point in the flux gradient.

The magnetic field is strongest in the centre of the torus and drops off approximately as $1/r$. The key equation for ideal magnetohydrodynamics in an axisymmetric plasma is the Grad-Shafranov equation:

$$L(\Psi) + \mu_0 r^2 \frac{dP}{d\Psi} + \frac{\mu_0^2}{8\pi^2}\frac{dI^2}{d\Psi} = 0$$

where L is the elliptic operator:

$$L(\Psi) = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial r}\right) + \frac{\partial^2 \Psi}{\partial z^2}$$

and where P is the plasma pressure, Ψ is the poloidal flux function, labeling the flux surfaces, and r and z are the usual cylindrical coordinates. I is called the poloidal current on each flux surface and is the current passing vertically through a circle whose radius is from the centre of the cross-section to the inner point (nearest the hole of the doughnut) of the flux surface.

A simple solution to the Grad-Shafranov equation, linear in Ψ, is the Solov'ev equilibrium given by:

$$P(\Psi) = P_b - \frac{\tilde{a}B_0}{\mu_0 R^2}(\Psi - \Psi_b)$$

$$I^2(\Psi) = I_b^2 - \frac{\tilde{b}8\pi^2 B_0}{\mu_0^2}(\Psi - \Psi_b)$$

where R is the major radius of the tokamak, a subscripted b indicates a value at the edge of the plasma, and $\tilde{a}$ and b are dimensionless control parameters determining the radial profile of the pressure and poloidal current. The flux surfaces defined by this equilibrium is given in Fig. 2.

Alfvén Waves in a Tokamak

To find the allowed Alfvén wave modes on a flux surface, the ideal MHD eigenmode equations are required:

$$\nabla\Psi \cdot \nabla \begin{pmatrix} P_1 \\ \xi_\Psi \end{pmatrix} = C \begin{pmatrix} P_1 \\ \xi_\Psi \end{pmatrix} + D \begin{pmatrix} \xi_s \\ \nabla\cdot\xi \end{pmatrix}$$

$$E \begin{pmatrix} \xi_s \\ \nabla\cdot\xi \end{pmatrix} = F \begin{pmatrix} P_1 \\ \xi_\Psi \end{pmatrix}$$

where ξ_s is a plasma displacement on a flux surface whilst also perpendicular to the magnetic field, ξ_Ψ is a radial plasma displacement and C, D, E and F are 2x2 matrix operators. Perturbations are assumed to be oscillatory in time as $exp(-i\omega t)$. The pressure perturbation, P_1, shall be ignored because the tokamak plasma is a low-beta plasma. radial displacement and plasma compression perturbation are ignored because we focus on the Alfvén waves. The only equation that remains after this is for matrix component E_{11}:

$$E_{11} = \frac{\omega^2 \mu_0 \rho |\nabla\Psi|^2}{B^2} + \mathbf{B}\cdot\nabla\left(\frac{|\nabla\Psi|^2 \mathbf{B}\cdot\nabla}{B^2}\right)$$

giving a much simpler eigenequation; $E_{11}\xi_s=0$. This is the wave equation for Alfvén waves in a tokamak equilibrium. Note that the wave equation does not involve derivatives with respect to Ψ although the equilibrium quantities depend on it. This means that for each flux surface separately the wave equation can have a solution uncoupled from the neighbouring flux surfaces. Hence, a continuum of modes is possible as a function of Ψ for each poloidal mode number. Due to the axisymmetric nature of the problem, perturbed quantities can be considered as periodic. The eigenequation can then be rewritten as:

$$\frac{\omega^2 \mu_0 \rho |\nabla\Psi|^2}{B^2} Y_1 + \frac{1}{J}\frac{\partial}{\partial\theta}\left(\frac{|\nabla\Psi|^2}{B^2 J}\frac{\partial}{\partial\theta}Y_1\right) = 0$$

where the Jacobian, J, has been introduced to change from cylindrical to magnetic flux coordinates (Ψ, θ, φ) and the surface pertubation has been replaced by the periodic Y_1:

$$Y_1(\theta) = \xi_s(\theta,\phi)\exp\left[in(\phi - q\theta)\right]$$

where n is the toroidal mode number, θ is the poloidal angle, φ is the toroidal angle and q (a function of Ψ) is the safety factor denoting the number of times a magnetic field line goes around the torus toroidally for each time poloidally; the higher it's value, the lower the risk of plasma instabilities, hence it's name. By adopting a variational principle and writing:

$$Y_1(\theta) = \sum_m y_m \exp\left[i(m - nq)\theta\right]$$

we find that the Alfvén continua are governed by the eigenvalue problem:

$$\sum_{m,m'} y_{m'}^* L_{m,m'} y_m = 0$$

where

$$L_{m'm} = \oint d\theta A_{11}\exp\left[i(m - m')\theta\right]$$

$$A_{11} = J\omega^2\mu_0\rho\frac{|\nabla\Psi|^2}{B^2} - \frac{|\nabla\Psi|^2}{JB^2}(m - nq)(m' - nq)$$

and m and m' are interfering poloidal mode numbers.

To find a particular solution, however, information about the tokamak geometry is still required. To obtain a spectrum as a result of toroidal coupling effects we use a high aspect ratio, low beta equilibrium for which:

$$J = \alpha(r)\left[1 + 2\epsilon\sigma(r)\cos\theta + O(\epsilon^2)\right]$$

$$\frac{|\nabla\Psi|^2}{B^2} = \frac{\epsilon^2 G(r)}{B_0^2}\left[1 + 2\epsilon\left(\frac{r}{R} + \Delta'\right)\cos\theta + O(\epsilon^2)\right]$$

where ε is the inverse aspect ratio and is therefore very small. This is very similar to the Solov'ev equilibrium but with an added realistic characteristic; Δ' which is the rate of change, with respect to r, of the shift of the centre of the surfaces from the magnetic axis.

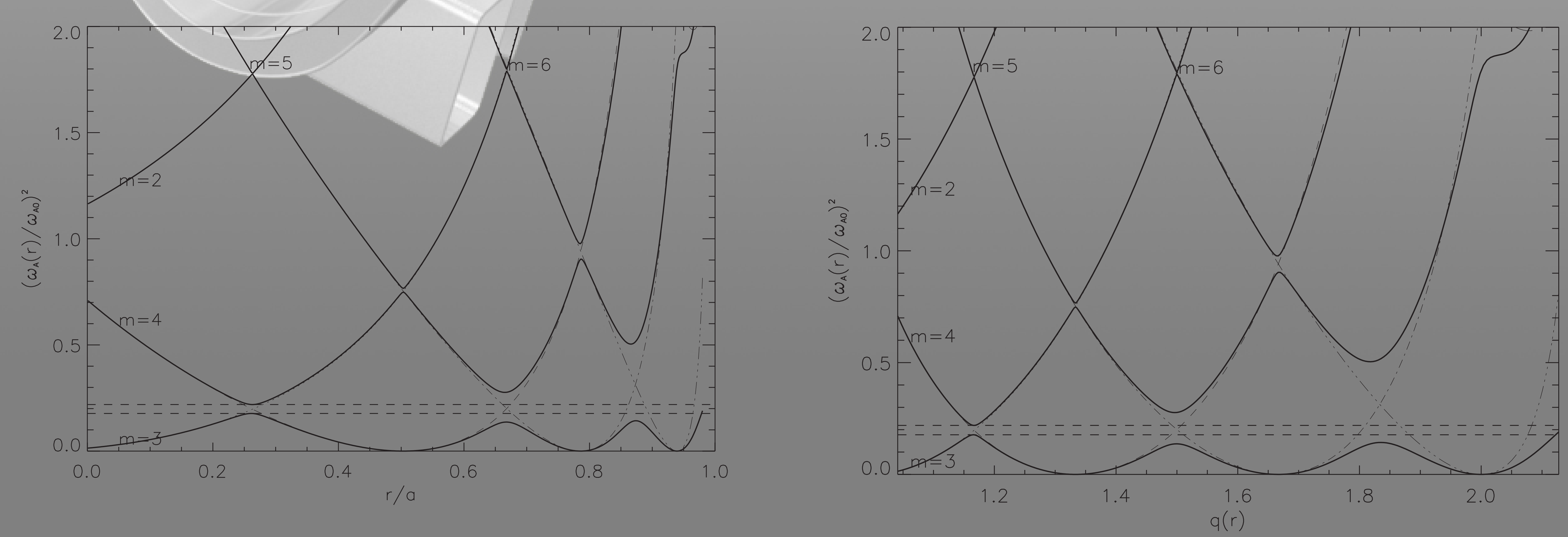

Figure 3: Allowed frequencies of an Alfvén wave on flux surfaces, labelled by the radius, r. Graphs shown both linear in r and in q, the safety factor. Frequencies normalised to the Alfvén frequency at B_0. ε=0.1, q_0=1.0408, q_a=2.2, n=3, $\beta=\epsilon^2$.

By inserting this geometry into the governing eigenequation, we find that coupling effects are only found for poloidal mode numbers differing by 1. This comes from integration of the terms in $L_{m'm}$ proportional to $cos(\theta)exp[i(m-m')\theta]$. Solutions to this particular case can then be found by forming the integrals into an infinite banded symmetric matrix, labeling rows and columns with m and m'; setting it's determinant to zero gives the dispersion relation. By varying r and searching for multiple solutions (branches) for each, the continuous spectrum shown in Fig. 3 is obtained. Also shown on the graphs (dotted) are the uncoupled mode frequencies hich are found by neglecting the $cos(\theta)$ in A_{11}, such that $L_{m'm}$ is only non-zero when m=m', i.e. no coupling between modes. The coupling is observed by the avoided crossing at uncoupled mode intersects. The key feature is the gap in the spectrum between the lowest branch and the next one up, which is of order ε in height, demonstrating that the coupling of neighbouring modes leads to a band of frequencies at which an Alfvén wave can not propagate.

The Coupled Pendulum

It is useful to investigate the properties of a coupled pendulum for an analogous result. The natural frequency of a pendulum is √(g/l). If two pendula of equal mass, m, with lengths l+½δl and l-½δl are joined at the ends by a spring, with spring constant k, the dispersion relation shown in Fig. 4 is found.

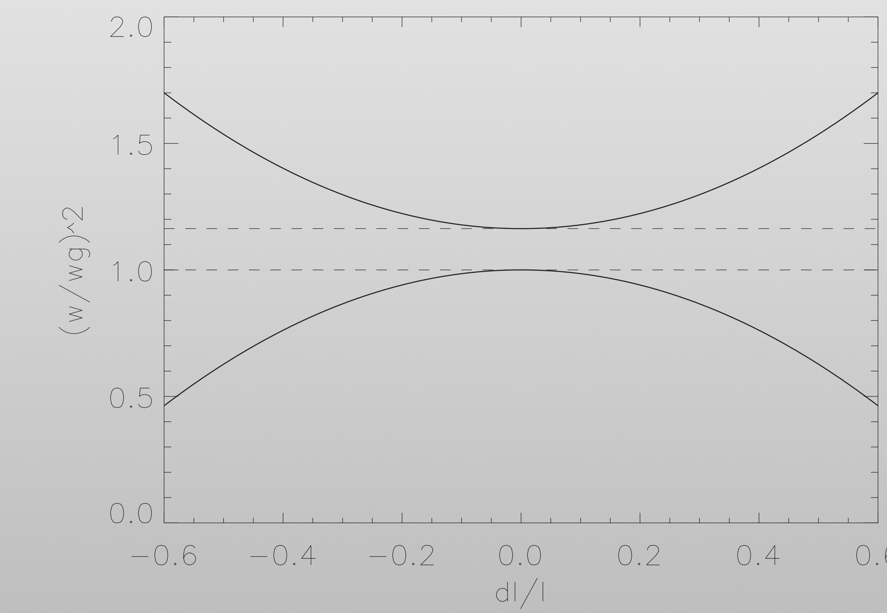

Figure 4: Graph showing allowed frequencies of two pendula of equal mass joined by a spring. δl is the difference in their lengths, l is their average and ω_g is the natural frequency of a pendulum of length l. A band of unallowed frequencies due to the coupling (dashed lines) from 1 to $[2(\omega_k/\omega_g)^2+1]$, where ω_k is the natural frequency of two masses on a spring.

Toroidal Alfvén Eigenmodes

Because of the gaps of the Alfvén continua created by the coupling between an m-continuum with $m+1$ or m-1 continua, a frequency gap is created free of continuum modes (between horizontal dashed lines in Fig. 3). Pairs of discrete long-lived Alfven modes, known as Toroidal Alfven Eigenmodes, can exist in those gaps. They have a certain radial spread around the location of the avoided crossing. Their name orginates from the fact that the gap in which they exist is created by the toroidal nature of the magnetic field topology, which gives rise to the $cos(\theta)$ terms and neighbouring m-coupling in $L_{m'm}$. Unfortunately, there was not sufficient time to explore the wave structure of TAEs in this summer project.

References:
- Cheng & Chance, 1986, Phys. Fluids 29, 1695
- Wesson, J., 2004: "Tokamaks", Oxford University Press