new $(n-2)$-set has an expected value no greater than the original, and since the partial sums are equal, this method must be at least as good as the former combination.

(2b) If 1 and r are combined, where r is 3, then one of two situations can arise. If $(p_1+p_3)>p_4$ the proof is analogous to (2a). If however, $(p_1+p_3)<p_4$, then, after we have combined 1 and 3 we would combine 2 and the combination of 1 and 3. In the second method, after we had combined 1 and 2, we would combine 3 with the combination of 1 and 2. The resulting $(n-2)$-sets would be the same, and the partial sum of the 1 and 2 combination would be lower than that of the 1 and 3 combination. Therefore the latter would be at least as good as the former method.

In the proof of (2) it is possible to interchange the values for p_1 and p_2 so as to prove the theorem for the combination of 2 and r. All possible first combinations in the n-set have been treated and hence the theorem is proved.

A rather interesting situation is that of nonuniqueness. If, at any stage of the method, there exist two or more probabilities which are equal, we can arrange them in any order we like, as long as they are adjacent. Further, if these probabilities represent not individual objects but groups, then the order will affect the final combination procedure. The result will be two or more arrangements which are equally acceptable.

As final remarks, let me indicate the outcome of the example presented at the outset, that of the determination of a letter of the alphabet. We are given a set of probabilities corresponding to each letter and the blank space. They range in size from .001 for the letter Q to .200 for the blank. Using the rule of combination, the expected number of divisions or questions is 4.154.

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SOLUTION OF A SET OF GAMES

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1. Introduction. This paper deals with a family of zero-sum two-person games in the Von Neumann sense [1]. The rules of the games are simple, but their implications are sufficiently obscure that the methods of game-theoretic analysis are necessary to determine the solutions. In this respect, the games fall into a category which, at present, has too few representatives known; namely, those games which (unlike tic-tac-toe) are not so simple that a theoretical analysis is superfluous, and (unlike chess) not so complicated that the analysis is inadequate to provide the solution.

2. Description of the games. The games considered are known by the designation of the "n-coin game," where n is a positive integer. The particular case of the three-coin game will be used to illustrate the rules of play; the general case is completely analogous.

In the three-coin game, each player A and B is supplied initially with three identical coins or other counters.

First Move. Each player chooses, unknown to the other, an integer between 0 to 3 (inclusive) and places that number of his coins to his right hand, which is thereafter kept closed. The number of coins selected by A and B, respectively, in this move will be denoted A_1 and B_1.

Second Move. Player A attempts to guess the total number of coins now held in both player's right hands. He announces his guess, A_2, an integer between 0 and 6.

Last Move. Player B, having heard A's second move, now attempts himself to guess the total. His guess, B_2, must differ from A_2.

The hands are now opened and the correct total ascertained. If either player has guessed correctly, he wins one arbitrary unit from the other. Otherwise, the play is a draw. In practice, when a draw occurs, it is customary to repeat the game with A and B exchanging roles, until one or the other wins. However, this convention need not affect our theoretical studies, which are concerned simply with optimal strategies for A and B in a single play of the game.

3. Heuristic discussion. Each player has some benefits under the rules, and it is not immediately obvious which has the greater. Player A has the choice of seven numbers (0, 1, 2, 3, 4, 5, or 6) for his second move, while B has only six. Moreover, if A guesses the sum correctly, then B's last move is completely futile, since he is defeated already. However, B has the advantage of hearing A_2 before naming B_2, and A_2 may give B some clue as to A_1, A's choice at the first move. For example, if $A_2=6$, then B can be reasonably sure that A's right hand holds three coins. This information will be useless in the case when B is also holding 3, but in any other case, B can be assured of a win.

Note that A can cancel this advantage to B by making a guess of $A_2=3$ at his second move. This reveals nothing to B about A's first move. On the other hand, in order to utilize this method of annulling his opponent's advantage, A must sacrifice his own advantage of complete freedom of choice at Move 2.

The relative importance of each of these features of the game will be revealed in the theoretical analysis below.

4. A simple special case. Before undertaking the analysis of the general n-coin game, we shall consider some special cases. Our first example is the one-coin game. Observe first, that among the possible strategies for either player, there are some which offer no possibility of winning. For example, for A to play $A_1=1$, $A_2=0$ is manifestly absurd. It is possible to exclude such strategies by employment of the dominance principle [1], but it is hardly necessary to use so sophisticated a technique to effect so elementary a result. In the remainder of this paper, only strategies which give the player a possibility of success ("feasible" strategies), will be considered. This does not destroy the generality of the results, since, as remarked above, all others may be excluded by dominance. Henceforth, we shall use the term strategy to denote feasible strategy, unless otherwise specified. With this convention, A has exactly four strategies:

$$A\text{–I}\ : A_1 = 0,\ A_2 = 0, \qquad A\text{–III}: A_1 = 1,\ A_2 = 1,$$

$$A\text{–II}\ : A_1 = 0,\ A_2 = 1, \qquad A\text{–IV}: A_1 = 1,\ A_2 = 2;$$

and B has only two strategies

$$B\text{–I}:\quad B_1 = 0,\ B_2 = \begin{cases} x \\ 0 \\ 1 \end{cases} \text{as}\quad A_2 = \begin{cases} 0 \\ 1, \\ 2 \end{cases}$$

$$B\text{–II}:\ B_1 = 1,\ B_2 = \begin{cases} 1 \\ 2 \\ x \end{cases} \text{as}\quad A_2 = \begin{cases} 0 \\ 1. \\ 2 \end{cases}$$

In case $B_1=0$ and $A_2=0$, B presumably has a lost game, and his second move is immaterial; a similar comment holds for the case $B_1=1$, $A_2=2$. These cases are denoted above by the (purely arbitrary) notation $B_1=x$. As a matter of practical expediency, B may be best advised to take $B_2=1$ in these two cases since this will give him an opportunity to win in the unlikely event that A is playing one of the nonfeasible strategies.

The payoff matrix for the game is given in Table 1.

TABLE 1. PAYOFF FOR THE ONE-COIN GAME (PAYOFF TO A)

	B-I	B-II
A-I	1	−1
A-II	−1	1
A-III	1	−1
A-IV	−1	1

From this table, it can be seen that basic optimal strategy mixes [3] for A include (and are limited to)

I and II in equal proportions,
I and IV in equal proportions,
II and III in equal proportions,
III and IV in equal proportions;

and B has only one optimal strategy, which is to mix B-I and B-II in equal proportions. The value of the game is zero; no play will end in a draw, and neither player can protect himself against loss if he attempts to take advantage of an opponent's error. In several of these respects, we shall see that the one-coin game is atypical.

5. The two-coin game. When $n=2$, a game is obtained which is typical in all important respects of all higher values of n, and still has few enough possibilities that detailed examination is practical.

It will be convenient to extend our notation as follows: Let $\overline{B}_1$ be player A's guess of B's first move, B_1, *i.e.*, $\overline{B}_1=A_2-A_1$. Then A's strategies in the two-coin game are nine in number and may be tabulated as in Table 2.

TABLE 2. STRATEGIES FOR PLAYER A IN THE TWO-COIN GAME

Strategy number	$\overline{B}_1$	A_1	A_2
A-I	0	0	0
A-II	0	1	1
A-III	0	2	2
A-IV	1	0	1
A-V	1	1	2
A-VI	1	2	3
A-VII	2	0	2
A-VIII	2	1	3
A-IX	2	2	4

The strategies for Player B may be described by the following convention. Any strategy for B determines a column in the payoff matrix, provided the A-strategies are listed down the side as in Table 2. Conversely, any column of nine entries of $+1$, -1, or 0 determines a B-strategy provided certain conditions are observed. The restrictions are stated below for the two-coin game. The form these take in the general case (with parameter n) is given in brackets.

Condition 1. Let the A-strategies be classed into the following three $[n+1]$ mutually exclusive sets:

All strategies for which either $\overline{B}_1=0$, or $\overline{B}_1=1, \cdots$, or $\overline{B}_1=n$. Then there must be $+1$ entries in the column defining the B-strategy against all A-strategies in exactly one of these sets, and no other $+1$ entries may appear. Since any B-strategy whatsoever must have a value of B_1, one of the three $[n+1]$ estimates $\overline{B}_1$ must be correct, *i.e.*, lead to a payoff to A of $+1$. We shall denote a B-strategy in which $B_1=i$ as a strategy of Type i.

Condition 2. The remaining 6 $[2(n+1)]$ entries in the column may be filled with 0 and -1 entries, but one and only one -1 entry* may be inserted against

* If nonfeasible B-strategies are considered, this condition should be modified to read "... not more than one -1 entry ...."

any single value of A_2. Thus, strategy A-VI and A-VIII both display $A_2=3$. In the column defining the B-strategy (say of Type 0), player B must select his call B_2 corresponding to $A_2=3$. He may select it so that he will win against A-VI, *i.e.*, $B_2=2$, or against A-VIII, *i.e.*, $B_2=1$, but he cannot choose a strategy which will win against both.

Brief consideration will establish that any column conforming to conditions (1) and (2) above will define a B-strategy, and conversely. It is not difficult to determine all B-strategies in the two-coin game, of which there are exactly ten. These strategies and the payoff matrix for the game appear in Table 3. (In this table $+1$ and -1 are abbreviated to $+$ and $-$ respectively, for brevity.) The reader is urged to construct Table 3 for himself. By so doing, he will greatly facilitate his understanding of the proofs given in subsequent sections.

TABLE 3. PAYOFF MATRIX, TWO-COIN GAME (PAYOFF TO A)

A-Strategies			*B*-Strategies									
			Type 0				Type 1		Type 2			
Strategy number	$\bar{B}_1$	A_2	*B*-I	*B*-II	*B*-III	*B*-IV	*B*-V	*B*-VI	*B*-VII	*B*-VIII	*B*-IX	*B*-X
A-I	0	0	+	+	+	+	−	−	−	−	−	−
A-II	0	1	+	+	+	+	−	−	0	−	−	0
A-III	0	2	+	+	+	+	−	0	−	−	0	0
A-IV	1	1	−	−	−	−	+	+	−	0	0	−
A-V	1	2	−	−	0	0	+	+	0	0	−	−
A-VI	1	3	−	0	0	−	+	+	−	−	−	−
A-VII	2	2	0	0	−	−	0	−	+	+	+	+
A-VIII	2	3	0	−	−	0	−	−	+	+	+	+
A-IX	2	4	−	−	−	−	−	−	+	+	+	+

A somewhat tedious but straightforward examination of this payoff matrix suffices to establish the following results.

PROPOSITION 1. *The game is fair.* [2]

PROPOSITION 2. *Player B has several optimal mixed strategies. Every one of the ten B-strategies is active in at least one of the optimal strategy mixes available to B.*

PROPOSITION 3. *Player A has only one optimal mixed strategy; namely, A-III, A-V, and A-VII in equal proportions.*

6. The general theorems. Statements analogous to Propositions 1, 2, and 3 hold in the general case. We shall present proofs of these general theorems. Because of the awkward nature of the operations, it will prove convenient to present the proofs in the language of a specific game, and we shall use $n=3$, the three-coin game, for this purpose. However, it will be observed that the methods we employ may immediately be applied to any value of n.

THEOREM 1. *If $n>1$, the n-coin game is fair.*

Proof for the three-coin game. Using notation like that introduced in the preceding section, we first show a mixed strategy for B in which the sum of the entries in any row is nonpositive. This implies that if the strategies so selected are played in equal proportions against any A-strategy, pure or mixed, the expected payoff to A will not exceed zero [2].

Select any nonnegative integers, $i \neq j$ $(i, j \leqq 3)$. We now choose B-strategies of Type i and j. In the column defining the Type i strategy, we insert -1 against every A-strategy which has $\overline{B}_1 = j$; similarly, the Type j strategy has -1's inserted in those positions where $\overline{B}_1 = i$. The other -1 entries are immaterial as long as Condition 2 is observed. It is obvious that such strategies satisfy the Conditions 1 and 2 of the preceding section and assure that the row sums are nonpositive. B can now play these two pure strategies with equal weight in a mixed strategy. This completes the proof that B can force an expected payoff of 0 or less.

Player A can force an expected payoff of 0 or more by playing an equal weight mix of the four stragegies:

$$\overline{B}_1 = 0,\ A_2 = 3; \qquad \overline{B}_1 = 1,\ A_2 = 3; \qquad \overline{B}_1 = 2,\ A_2 = 3; \qquad \overline{B}_1 = 3,\ A_2 = 3.$$

By Condition 1, every B-strategy must lose against exactly one of these four. Also, according to Condition 2, each B-strategy will win against one and only one of them. Therefore, any column sum is zero.* This completes the proof. The restriction $n>1$ is, of course, unnecessary since we have already proved the one-coin game to be fair. However, the method of proof used here requires $n>1$.

THEOREM 2. *In the n-coin game, any pure B-strategy is active in some mixed optimal strategy for B.*

Proof for the three-coin game. Let S be a given pure B-strategy. We shall produce a four-strategy mix of B-strategies including S which has all row sums nonpositive.

Strategy S is of some type, say i; rename $S = S_i$, and begin construction of the other three strategies in the mix by choosing them to be one of each of the three types among Types 0, 1, 2, and 3 obtained by omitting Type i. Name each strategy by a subscript defining its type, *i.e.*, S_0, S_1, S_2, and S_3, of which one is S and is completely defined, and the others still have -1 entries to be inserted in the columns of the payoff matrix defining them.

We shall insert these -1 entries such that each row has at least one such -1 entry. Since each row (defined by $\overline{B}_1$ and A_2) has exactly one $+1$ entry (by construction) the four strategies S_0, S_1, S_2, and S_3 may be played with equal weight, giving nonpositive payoff to A. This will prove the theorem. We will assign these -1 entries systematically. For each value of A_2, there are one or more A-

* If nonfeasible B-strategies are considered, any column sum is nonnegative.

strategies, *i.e.*, one or more rows in the matrix. Table 4 shows the number of occurrences of each A_2 value.

TABLE 4. OCCURRENCES OF VALUES OF A_2 IN FEASIBLE A-STRATEGIES IN THE THREE-COIN GAME

Values of A_2	0	1	2	3	4	5	6
Number of occurrences	1	2	3	4	3	2	1

We begin by assigning the four -1's which fall on rows where $A_2=3$. One of these already is entered, in the S_i column (since it is a feasible strategy). This lies on a row for which $\overline{B}_1=j$ (say), $j\neq i$. In column S_j insert a -1 against an A-strategy for which $A_2=3$, and $B_1\neq i, j$, say at $\overline{B}_1=k$. This is always possible since for each $\overline{B}_1$, an A-strategy exists with $A_2=3$. Next, in column S_k, insert -1 against an A-strategy for which $A_2=3$, $\overline{B}_1\neq i, j, k$, say at $\overline{B}_1=m$. Lastly, in column S_m, insert a -1 against strategy $A_2=3$, $\overline{B}_1=i$. It is obvious that this construction is always possible and complies with Conditions 1 and 2.

In similar manner, three -1 entries are made against A-strategies with $A_2=4$. One of these is already determined in column S_i. Suppose it occurs against the A-strategy with $\overline{B}_1=j'$. In column $S_{j'}$, insert a -1 against an A-strategy with $A_2=4$, $\overline{B}_1\neq i$, say at $B_1=k'$. This is always possible. In column $S_{k'}$, insert a -1 against an A-strategy with $A_2=4$. This may or may not occur where $\overline{B}_1=i$, but it, too, is always possible, since three distinct values of $\overline{B}_1$ ($\overline{B}_1=1$, 2, and 3) permit A-strategies with $A_2=4$.

Similarly, the three -1 entries for $A_2=2$, the two entries for $A_2=1$, and the two entries for $A_2=5$ can be assigned. The one entry for $A_2=0$ is inserted in any one (or more) of columns S_1, S_2, and S_3; the entry for $A_2=6$ is inserted in any one (or more) of columns S_0, S_1, and S_2. This completes the construction. It can readily be seen that the required conditions are satisfied and the theorem proved.

THEOREM 3. *If $n>1$, in the n-coin game, no optimal strategy for Player A has any active strategies except*:

$$\overline{B}_1 = 0,\ A_2 = n;\ \overline{B}_1 = 1,\ A_2 = n;\ \cdots;\ \overline{B}_1 = n,\ A_2 = n.$$

Proof for the three-coin game. These strategies have been shown to be active in one optimal strategy mix. We now prove that no other pure strategy can be active in an optimal mixed strategy for A in the three-coin game.

Let T be an A-strategy with $A_2\neq 3$. First consider the case $A_2<2$. We choose the following two-strategy mix for B: (1) S_2, a Type 2 strategy with -1 entries against T and every A-strategy for which $\overline{B}_1=3$, (2) S_3, a Type 3 strategy with -1 entries against every A-strategy for which $\overline{B}_1=2$. Other -1 entries are immaterial as long as condition 2 is satisfied. If these strategies are played with equal weights, a nonpositive expected payoff to A is assured for every pure A-strategy, and a negative payoff of $-1/2$ is obtained when strategy T is played in particular. Similarly, if $A_2>4$, a mix of a Type 0 and Type 1 strategy suffices to provide a negative expected payoff.

It remains to consider the case $A_2=2$ or 4 (in general, $A_2=n\pm 1$). Suppose $A_2=2$. For this case, we construct a four strategy optimal mix for B in a manner similar to that employed in the proof of Theorem 2. We begin by defining four strategies of Types 0, 1, 2, and 3, denoted S_0, S_1, S_2, and S_3, respectively. S_3 will be called the key strategy. Suppose strategy T is defined by $\overline{B}_1=i$, $A_2=2$. Choose an integer $j\leqq 3$, $j\neq i$. In column S_j, insert a -1 against strategy T. This is possible since $\overline{B}_1(T)=i$, and $j\neq i$. Now in column S_i, find an A-strategy with $A_2=2$, $\overline{B}_1\neq i$, j. Such a strategy exists, say $\overline{B}_1=k$, since three occurrences of $A_2=2$ are shown in Table 4. Furthermore, $k\neq 3$, since the A-strategy $\overline{B}_1=3$, $A_2=2$ is not feasible. In column S_k, assign a -1 against the A-strategy with $A_2=2$, $\overline{B}_1=j$. Also, in column S_i, assign a -1 against the A-strategy with $A_2=2$, $\overline{B}_1=k$.

Now enter a -1 in column S_3 (the key strategy) against T. This is the fourth -1 entry against A-strategies having $A_2=2$, and therefore, the second such entry against T. The remaining entries in column S_3 are entered in any manner to create a feasible strategy and satisfy Condition 2. The remaining entries in columns S_0, S_1, and S_2 are filled in the manner described in the proof of Theorem 2 (except for the -1's on lines where $A_2=2$ which have already been defined). The resulting strategy mix has one $+1$ and one or more -1 on every line, and in particular on line T, it has two -1's. Hence, when these strategies are played in equal proportion against any A pure strategy, the payoff to A is nonpositive, and against T in particular, it is negative and equal to $-1/4$.

This proves the theorem for the case $A_2=2$. For $A_2=4$, the proof is similar, except that S_0 is the key strategy. This completes the proof of the theorem.

Remark. It is clear from Theorem 3 and the proof of Theorem 1 that A must play these strategies with equal weights.

7. Concluding remarks. Despite the content of Theorem 1, it seems apparent that the advantage in the game lies with Player B wherever $n>1$. He has a wide selection of optimal strategy mixes, and in view of Theorems 2 and 3, he can select them in such a way that he can penalize A for any departure from the strategies in A's single optimal mix, while simultaneously protecting himself against any loss. Player A, on the other hand, is limited to his single mixed strategy, and may not deviate from it without risking loss. Unfortunately for A, while his strategy assures a nonnegative expected payoff, it also assures a nonpositive payoff, even if B blunders. While a quantitative measure of the value of such a property of a game is not generally agreed on [3], it is difficult to imagine any interpretation which would not make this circumstance a plus value for B.

It has been pointed out to the author that in view of this, a wiley A-player may choose a nonfeasible strategy such as $A_1=3$, $A_2=2$ in order to force a drawn play. In accordance with the convention mentioned in Section 2, this player would then attain on the next play the B-role, which is preferable. The interpretation of this strategem will be left to the reader.

It will be noted that in the three-coin game, one half of the plays end in draws (provided both players play perfectly). This may be seen by examination of the proof of Theorems 1 and 3. The single feasible strategy mix which A will employ will win one play out of four against any B-strategy, and lose one out of four against any feasible B-strategy, and draw in the remaining two plays. In general, the ratio of draws to total plays is readily seen to be $(n-1)/(n+1)$.

An interesting consequence of the theory is that B may play the game with only one coin for any n. In the proof of Theorem 1, a construction is given which provides an optimal mixed strategy for B comprising Type 0 and 1 strategies only. The content of Theorems 2 and 3 must be modified, of course, if B has only one coin. However, the methods employed in the proof of the theorems may readily be applied to determine the new relationships.

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MATHEMATICAL NOTES

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NOTE ON STIRLING'S FORMULA

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Stirling's asymptotic formula, namely,

$$n! \sim \sqrt{(2\pi n)}\cdot(n/e)^n, \qquad n \to \infty,$$

is usually proved by showing that

$$(1) \qquad n! = \sqrt{(2\pi n)}\cdot(n/e)^n\cdot e^{\gamma_n}, \qquad 0 < \gamma_n < 1/(12n), \qquad n = 1, 2, \cdots.$$

Herbert Robbins (this Monthly, vol. 62, 1955, pp. 26–29) has shown by an elementary method that the estimate for γ_n in (1) can be replaced by the improved estimate

$$(2) \qquad 1/(12n+1) < \gamma_n < 1/(12n), \qquad n = 1, 2, \cdots.$$

In this note, we shall prove, by a simple refinement of the argument of Robbins, the stronger and more precise result