

ES2A7 - Fluid Mechanics Example Classes

Example Questions (Set IV)

Question 1: Dimensional analysis

a) It is observed that the velocity 'V' of a liquid leaving a nozzle depends upon the pressure drop 'P' and its density ρ . Show that the relationship between them is of the form $V = C\sqrt{P/\rho}$

b) It is observed that the frequency of oscillation of a guitar string 'f' depends upon the mass 'm', the length 'l' and tension 'F'. Show that the relationship between them is $f = C\sqrt{F/ml}$

c) Find the dimension of the bulk modulus K, knowing its relationship with the speed of the sound 'a' in a liquid and the density ' ρ ': $a = C\sqrt{K/\rho}$

a) Dimension:

$$[V] = L/T, \quad [P] = M/LT^2, \quad [\rho] = M/L^3$$

The relationship should verify:

$$[V]^a [P]^b [\rho]^c = 1$$

$$L^{a-b-3c} M^{b+c} T^{-2b-a} = 1$$

This leads to the following system:

$$\begin{cases} a - b - 3c = 0 \\ b + c = 0 \\ -2b - a = 0 \end{cases} \quad \begin{cases} -2b - b + 3b = 0 \\ b = -c \\ 2b = -a \end{cases} \quad \begin{cases} a = 1 \\ b = -1/2 \\ c = 1/2 \end{cases}$$

The relation is thus verified

b) Dimension:

$$[f] = 1/T, \quad [m] = M, \quad [l] = L, \quad [T] = ML/T^2$$

The relationship should verify:

$$[f]^a [m]^b [l]^c [T]^d = 1$$

$$L^{c+d} M^{b+d} T^{-a-2d} = 1$$

This leads to the following system:

$$\begin{cases} c + d = 0 \\ b + d = 0 \\ -a - 2d = 0 \end{cases} \quad \begin{cases} d = -c \\ d = -b \\ 2d = -a \end{cases} \quad \begin{cases} a = 1 \\ b = 1/2 \\ c = 1/2 \\ d = -1/2 \end{cases}$$

The relation is thus verified

c) From the relation, we have : $K = \rho a^2 / C$

$$\text{so } [K] = [\rho][a]^2 = \frac{M}{L^3} \frac{L^2}{T^2} = \frac{M}{LT^2}$$

Question 2: Dimensional analysis

Water flows through a 2cm diameter pipe at 1.6m/s. Calculate the Reynolds number and find also the velocity required to give the same Reynolds number when the pipe is transporting air. For the water the kinematic viscosity was $1.31 \cdot 10^{-6} \text{ m}^2/\text{s}$ and the density was 1000 kg/m^3 . For air those quantities were $15.1 \cdot 10^{-6} \text{ m}^2/\text{s}$ and 1.19 kg/m^3 .

Kinematic viscosity is dynamic viscosity over density = $\nu = \mu/\rho$.

The Reynolds number is :

$$\text{Re} = \frac{\rho U d}{\mu} = \frac{U d}{\nu}$$

Reynolds number when carrying water:

$$\text{Re}_{\text{water}} = \frac{16 \times 0.02}{1.31 \times 10^{-6}} = 24427$$

To calculate Re_{air} we know:

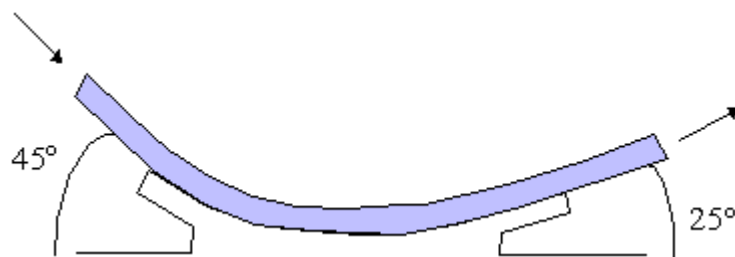
$$\text{Re}_{\text{air}} = \text{Re}_{\text{water}}$$

$$\Leftrightarrow \frac{U_{\text{air}} \times 0.02}{15 \times 10^{-6}} = 24427$$

$$\Leftrightarrow U_{\text{air}} = 18.44 \text{ m.s}^{-1}$$

Question 3: Momentum conservation

The figure below shows a smooth curved vane attached to a rigid foundation. The jet of water, rectangular in section, 75mm wide and 25mm thick, strike the vane with a velocity of 25m/s. Calculate the vertical and horizontal components of the force exerted on the vane and indicate in which direction these components act.



From the question: $a_1 = 1.875 \times 10^{-3} \text{ m}^2$, $u_1 = 25 \text{ m.s}^{-1}$

Since the jet section is constant $a_1 = a_2$, the velocity is also constant $u_1 = u_2$ (Mass conservation in the case of an incompressible flow)

The bulk flow is thus $Q = a_1 \cdot u_1 = a_2 \cdot u_2 = 0.0469 \text{ m}^3 \cdot \text{s}^{-1}$

Calculation of the total force using the momentum equation:

$$\vec{F} = \rho Q (\vec{V}_2 - \vec{V}_1)$$

Projecting this relation along x and y:

$$\begin{aligned}F_x &= \rho Q u (\cos 25 - \cos 45) \\ &= 1000 \times 0.0469 \times 25 \times (\cos 25 - \cos 45) \\ &= 233\text{N}\end{aligned}$$

$$\begin{aligned}F_y &= \rho Q u (\sin 25 - \sin 45) \\ &= 1000 \times 0.0469 \times 25 \times (\sin 25 - \sin 45) \\ &= 1325\text{N}\end{aligned}$$

Body force and pressure force are 0. F is thus equal to the reaction force applied on the fluid.

$$\vec{F} = \vec{F}_R + \vec{F}_B + \vec{F}_p = \vec{F}_R$$

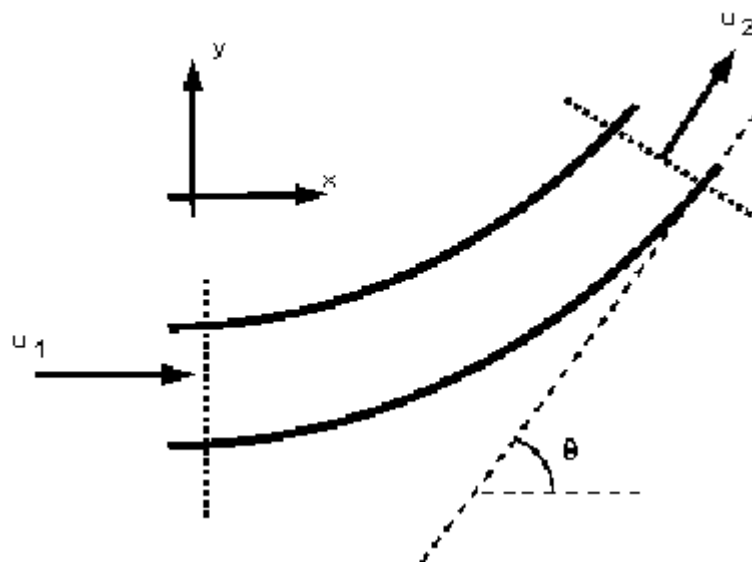
So force on vane:

$$\begin{aligned}R_x &= -F_x = -233\text{N} \\ R_y &= -F_y = -1325\text{N}\end{aligned}$$

R_x is directed toward the left and R_y toward the bottom.

Question 4: Momentum conservation

A 600mm diameter pipeline carries water under a head of 30m with a velocity of 3m/s. This water main is fitted with a horizontal bend which turns the axis of the pipeline through 75° (i.e. the internal angle at the bend is 105°). Calculate the resultant force on the bend and its angle to the horizontal.



From the question:

$$P_1 = P_2 = \rho gh = 1000 \times 9.81 \times 30 = 294.3 \text{ kN/m}^2$$

$$a_1 = a_2 = \pi (0.6/2)^2 = 0.283 \text{ m}^2$$

$$u_1 = u_2 = 3 \text{ m.s}^{-1}$$

The bulk flow is thus

$$Q = a_1 \cdot u_1 = a_2 \cdot u_2 = 0.848 \text{ m}^3 \cdot \text{s}^{-1}$$

Calculation of the total force using the momentum equation:

$$\vec{F} = \rho Q (\vec{V}_2 - \vec{V}_1)$$

Projecting this relation along x and y:

$$F_x = \rho Q u (\cos 75 - \cos 0)$$

$$= 1000 \times 0.848 \times 3 \times (\cos 75 - 1)$$

$$= -1886 \text{ N}$$

$$F_y = \rho Q u (\sin 75 - \sin 0)$$

$$= 1000 \times 0.848 \times 3 \times (\sin 75 - 0)$$

$$= 2457 \text{ N}$$

Calculation of the pressure force:

$$\vec{F}_p = \vec{F}_{p_1} + \vec{F}_{p_2}$$

Projecting this relation along x and y:

$$F_{px} = aP(\cos 0 - \cos 75)$$

$$= 0.283 \times 294.3 \times (1 - \cos 75)$$

$$= 62000 \text{ N}$$

$$F_{py} = aP(\sin 0 - \sin 75)$$

$$= 0.283 \times 294.3 \times (0 - \sin 75)$$

$$= -80376 \text{ N}$$

There is no body force in the x or y directions.

\vec{F} is thus equal to sum of the reaction force and the pressure force applied on the fluid.

$$\vec{F} = \vec{F}_R + \vec{F}_B + \vec{F}_p = \vec{F}_R + \vec{F}_p$$

$$\text{so } \vec{F}_R = \vec{F} - \vec{F}_p$$

So force on bend:

$$R_x = -F_{Rx} = -F_x + F_{px} = 1886 + 62000 = 63886 \text{ N}$$

$$R_y = -F_{Ry} = -F_y + F_{py} = -2457 - 80376 = -82833 \text{ N}$$

R_x is directed toward the right and R_y toward the bottom.

$$R = \sqrt{R_x^2 + R_y^2} = 104 \text{ kN}$$

$$\theta = \tan^{-1} \left(\frac{R_y}{R_x} \right) = -52^\circ$$

Question 5: Mass conservation + Bernoulli

A water clock is an axisymmetric vessel with a small exit pipe in the bottom. Find the shape for which the water level falls equal heights in equal intervals of time.

$$\text{Continuity equation: } \frac{d}{dt}(\rho v) + \dot{m}_{out} - \dot{m}_{in} = 0$$

There is no inflow and the fluid is incompressible.

Let 'a' be the cross-section of the exit pipe:

$$\rho \frac{d(v(t))}{dt} = -\dot{m}_{out} = -\rho a U(t),$$

$$v(t) = -a \int_0^t U(t) dt$$

Bernoulli equation between the upper free surface and the exit section:

$$\rho g h(t) + 0 = 0 + \frac{\rho U(t)^2}{2}$$

$$U(t) = \sqrt{2gh(t)}$$

So using the first relation:

$$v(t) = -a \int_0^t \sqrt{2gh(t)} dt$$

$$v(t) = -a\sqrt{2g} \int_0^t \sqrt{h(t)} dt$$

From the question, we know that the variation of h is linear in time:

$$h(t) = h_0 - xt$$

So

$$v(t) = -a\sqrt{2g} \int_0^t \sqrt{h_0 - xt} dt$$

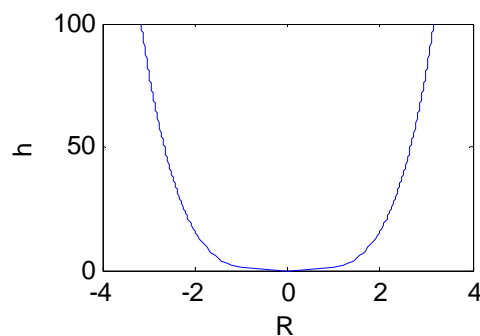
$$v(t) = \frac{2a}{3x} \sqrt{2g} (h_0 - xt)^{3/2}$$

$$v(t) = \frac{2a}{3x} \sqrt{2g} (h(t))^{3/2}$$

The volume of fluid can always be written as

$$v(t) = \int_0^{R(z)} \int_0^{2\pi} \int_0^{h(t)} r dr d\theta dz = \pi \int_0^{h(t)} R(z)^2 dz$$

$$\text{So } \int_0^{h(t)} R(z)^2 dz = \frac{2a}{3x\pi} \sqrt{2g} (h(t))^{3/2} \Rightarrow R(h(t))^2 = \frac{a}{x\pi} \sqrt{2g} (h(t))^{1/2}$$

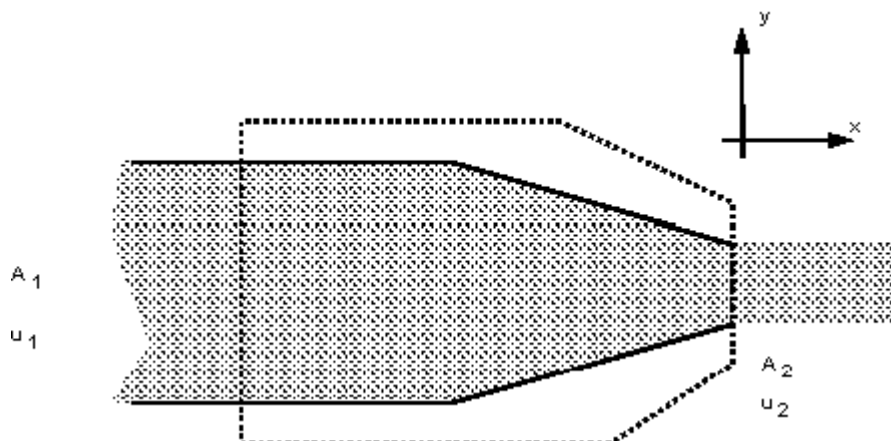


Question 6: Momentum conservation

Because the fluid is contracted at the nozzle forces are induced in the nozzle. Anything holding the nozzle (e.g. a fireman) must be strong enough to withstand these forces. Determine these forces. The analysis takes the following procedure:

- 1) Draw a control volume
- 2) Decide on co-ordinate axis system
- 3) Calculate the total force
- 4) Calculate the pressure force
- 5) Calculate the body force
- 6) Calculate the resultant force

1 & 2 Control volume and Co-ordinate axis have been done for you and are shown in the figure below.



Notice how this is a one dimensional system which greatly simplifies matters.

3) Calculation of the total force: $F = F_x = \rho Q (u_2 - u_1)$

From the continuity equation, the bulk flow is : $Q = A_1 u_1 = A_2 u_2$

$$\text{so } F = \rho Q^2 \left(\frac{1}{A_2} - \frac{1}{A_1} \right)$$

4) Calculation of the pressure force

$$\vec{F}_p = \vec{F}_{p_1} + \vec{F}_{p_2}$$

$$F_p = F_{p_x} = F_{p_1} - F_{p_2}$$

We use the Bernoulli equation to calculate the pressure

$$\frac{P_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{u_2^2}{2g} + z_2$$

The nozzle is horizontal ($z_1 = z_2$) and the pressure outside is atmospheric ($P_2 = 0$).

With continuity, it gives :

$$p_1 = \frac{\rho Q^2}{2} \left(\frac{1}{A_2^2} - \frac{1}{A_1^2} \right)$$

$$F_p = \frac{\rho Q^2}{2} \left(\frac{A_1^2}{A_2^2} - 1 \right)$$

5) Calculation of the body force

The only body force is the weight due to gravity in the y-direction - but we need not consider this as the only forces we are considering are in the x-direction.

6) Calculation of the resultant force

$$F = F_R + F_p$$

$$F_R = F - F_p = \rho Q^2 \left(\frac{1}{A_2} - \frac{1}{A_1} \right) - \frac{\rho Q^2}{2} \left(\frac{A_1^2}{A_2^2} - 1 \right)$$

$$F_R = \frac{\rho Q^2}{2} \left(1 - \frac{A_1^2}{A_2^2} + \frac{2}{A_2} - \frac{2}{A_1} \right)$$

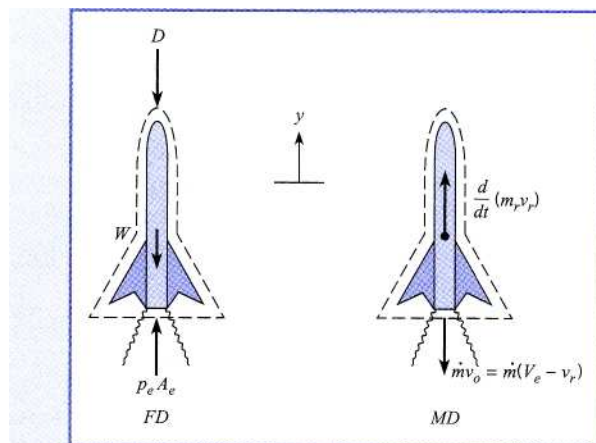
So the fireman must be able to resist the force of

$$R = -F_R = \frac{\rho Q^2}{2} \left(\frac{A_1^2}{A_2^2} - 1 - \frac{2}{A_2} + \frac{2}{A_1} \right)$$

This force increases when A_1 increase or A_2 decrease, which correspond to the expected behaviour.

Question 7: Momentum conservation

Consider a rocket of mass m_r traveling at a speed u_r as measured from the ground. Exhaust gases leave the engine nozzle (area A_e) at a speed U_e relative to the nozzle of the rocket, and with a pressure that is higher than local atmospheric pressure by an amount p_e . The aerodynamic drag force on the rocket is D . Derive an equation for the acceleration of the rocket.



We selected a control volume enclosing the rocket and moving with the rocket. Because a reference frame attached to the rocket would be noninertial, we chose a stationary reference frame fixed to the launch pad. We selected the y-direction momentum equation. The force diagram shows three forces: weight, aerodynamic drag force, and the pressure force acting at the nozzle exit. From the force diagram

$$\sum F_y = P_e A_e - D - W$$

The momentum diagram shows an accumulation term and an outflow term. The momentum accumulation term is not zero because the momentum of the rocket is changing with time as the rocket accelerates. The momentum of the rocket is the mass of the rocket times the velocity of the rocket, and the accumulation term is given by $(d/dt)(m_r u_r)$. This can be formally developed by using integration:

$$\frac{d}{dt} \int_{CV} u_y \rho dV = \frac{d}{dt} \left(u_r \int_{CV} \rho dV \right) = \frac{d}{dt} (m_r u_r)$$

In this development, we assumed that all parts of the rocket were traveling at the same speed u_r . The outward momentum flow is $\dot{m} u_o$, where u_o is relative to the stationary reference frame: $u_o = (U_e - u_r)$

$$\text{So : } \dot{m} u_o = \dot{m} (U_e - u_r)$$

From the momentum diagram

$$\frac{d}{dt} \int_{CV} u_y \rho dV + \sum_{cs} \dot{m}_o u_{o_y} - \sum_{cs} \dot{m}_i u_{i_y} = \frac{d}{dt} (m_r u_r) + [-\dot{m} (U_e - u_r)]$$

Substituting force and momentum terms into the momentum equation gives:

$$P_e A_e - D - W = \frac{d}{dt} (m_r u_r) + [-\dot{m} (U_e - u_r)]$$

The accumulation term can be expanded using the product rule for differentiation. This term may be evaluated by applying continuity to the control volume:

$$\frac{dm_r}{dt} = -\dot{m}$$

Combining the two previous equations gives:

$$\frac{d}{dt} (m_r u_r) = m_r \frac{d}{dt} (u_r) - \dot{m} u_r$$

Substituting this result into the momentum equation yields:

$$m_r \frac{d}{dt} (u_r) = \dot{m} U_e + P_e A_e - D - W$$

The mass flow rate exiting the rocket nozzle is: $\dot{m} = \rho_e A_e U_e$

$$\text{Combining the above two equations results in: } m_r \frac{d}{dt} (u_r) = \rho_e A_e U_e^2 + P_e A_e - D - W$$

The acceleration of the rocket (du_r/dt) depends on the instantaneous mass of the rocket and the 4 terms on the right side of the right side of this equation. The term $T = \rho_e A_e U_e^2 + P_e A_e$ is known as the thrust of the rocket motor. The exit pressure of the exhaust jet (P_e) is often expressed using absolute pressure, so the thrust is given by $T = \rho_e A_e U_e^2 + P_e A_e (P_e - P_0)$, where P_0 is the local atmospheric pressure, which will change with altitude as a rocket ascends.