Functional Monotone Class Theorem

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The next few results deal with monotone classes of functions.

Proposition 1. *Let $\mathcal{H}$ be a class of $\overline{\mathbb{R}}$-valued functions on Ω for which:*

(i) $\mathbb{1}_\Omega \in \mathcal{H}$;

(ii) $\mathcal{H}_b := \{f \in \mathcal{H} : f \text{ is bounded}\}$ *is a vector space over* $\mathbb{R}$;

(iii) $\mathcal{H}_+ := \{f \in \mathcal{H} : f \geq 0\}$ *(resp.* $\mathcal{H}_- := \{f \in \mathcal{H} : f \leq 0\}$*) is closed under nondecreasing (resp. nonincreasing) limits.*

Suppose furthermore $\mathcal{C}$ is a π-system generating the σ-algebra $\mathcal{B}$ on Ω and such that $\mathbb{1}_C \in \mathcal{H}$ for all $C \in \mathcal{C}$. Then all bounded ($:= \mathcal{B}_b$), nonnegative ($:= \mathcal{B}_+$) and all nonpositive ($:= \mathcal{B}_-$) $\mathcal{B}$-measurable $\overline{\mathbb{R}}$-valued functions are in $\mathcal{H}$. In order that $\mathcal{B}_b$ be such, it is sufficient that in (iii) $\mathcal{H}$ should be closed under nondecreasing limits converging uniformly to a bounded function. Similarly if one needs only $\mathcal{B}_+$ to be such, then in (ii) one may work only with nonnegative bounded functions and have them be closed under nonnegative linear combinations and in (iii) the condition on $\mathcal{H}_-$ is not necessary (vice versa for the case of $\mathcal{B}_+$).

Proof. The claim follows at once from the fact that any nonnegative (resp. nonpositive) function is a nondecreasing (resp. nonincreasing) limit of nonnegative (resp. nonpositive) simple functions (uniformly so on any set on which the function is bounded) and that any bounded function is the difference of its bounded nonnegative and its bounded nonpositive part. So it is enough to show that every indicator on an element of $\mathcal{E}$ is in $\mathcal{H}$. But the set of all E for which this is true (by assumptions) is a Dynkin system containing the π-system $\mathcal{C}$. Enough said. □

Theorem 1 (**Monotone class theorem for functions**). *Let $\mathcal{K}$ be a collection of bounded $\mathbb{R}$-valued functions on Ω closed under multiplication (i.e. $\{f, g\} \subset \mathcal{K} \Rightarrow fg \in \mathcal{K}$) and let $\mathcal{B} := \sigma(\mathcal{K})$ be the smallest σ-algebra w.r.t. which all elements of $\mathcal{K}$ are measurable. Suppose $\mathcal{H} \supset \mathcal{K}$ is a vector space over $\mathbb{R}$ of bounded $\mathbb{R}$-valued functions, containing $\mathbb{1}_\Omega$ and closed under uniform limits of nonnegative functions nondecreasing to a bounded function. Then $\mathcal{B}_b \subset \mathcal{H}$, i.e. $\mathcal{H}$ contains all bounded $\mathcal{B}$-measurable $\mathbb{R}$-valued functions.*

Proof. By the proposition above it is enough to show the claim for indicators of a π-system generating $\mathcal{B}$. Let $\mathcal{A}_0$ denote the algebra generated by $\mathcal{K}$. Since $\mathcal{K}$ is already closed under multiplication, $\mathcal{A}_0$ is simply the linear span of $\mathcal{K}$ (adding possibly the constant function $\mathbb{1}_\Omega$). Consequently $\mathcal{A}_0 \subset \mathcal{H}$.

Lemma 1. *If $\mathcal{H}$ is as stated, then $\mathcal{H}$ is closed for $\|\cdot\|_{\sup}$, i.e. it is closed under uniform convergence.*

Proof. Suppose $(f_n)_{n\geq 1} \subset \mathcal{H}$ and $f_n \to f$ pointwise uniformly on Ω as $n \to \infty$. By passing to a subsequence if necessary, one can arrange that $\|f_{n+1} - f_n\|_{\sup} \leq 2^{-n}$, $n \geq 1$. Define $g_n := f_n - 2^{1-n} + \|f_1\|_{\sup} + 1$, $n \geq 1$. Then $g_n \in \mathcal{H}$ and $\|g_n\|_{\sup} \leq \|f_n\|_{\sup} + \|f_1\|_{\sup} + 1$, $n \geq 1$ so the sequence $(g_n)_{n\geq 1}$ is uniformly bounded. Also $g_{n+1} - g_n = f_{n+1} - f_n + 2^{-n} \geq 0$, $n \geq 1$ and $g_1 = f_1 + \|f_1\| \geq 0$. It follows that $\lim_{n\to\infty} g_n = \lim_{n\to\infty} f_n + \|f_1\|_{\sup} + 1 \in \mathcal{H}$ and thus $\lim_{n\to\infty} f_n \in \mathcal{H}$. □

This lemma thus tells us that the uniform closure $\mathcal{A}$ of $\mathcal{A}_0$ is again a subset of $\mathcal{H}$ and it is trivial to check that it is itself in turn an algebra. Referring to Weierstrass' Theorem, let p_n, $n \geq 1$ be a sequence of polynomials converging uniformly to $(s \mapsto |s|)$ on $[-1, 1]$. Then if $0 \neq f \in \mathcal{A}$, $p_n \circ (f/\|f\|_{\sup}) \to |f|/\|f\|_{\sup}$ uniformly. Hence, since $\mathcal{A}$ is an algebra closed for uniform convergence, one has $|f| \in \mathcal{A}$. It follows, moreover, that for $\{f, g\} \subset \mathcal{A}$, $f \vee g = [|f - g| + f + g]/2$ and $f \wedge g = [f + g - |f - g|]/2$ are elements of $\mathcal{A}$ in turn. Take now $\{f_1, \ldots, f_m\} \subset \mathcal{A}$ and $(b_1, \ldots, b_m)$ a sequence of real numbers. Then the function

$$g_n := \prod_{k=1}^{m} [n(f_k - b_k)^+] \wedge 1$$

is in $\mathcal{A}$ for each natural n and $g_n \uparrow \prod_{k=1}^m \mathbb{1}_{\{f_k > b_k\}}$ pointwise (and boundedly), as $n \to \infty$, and hence the latter function is in $\mathcal{H}$. Enough said. □

As an application of the latter consider:

Proposition 2. *Two laws on $\mathbb{R}$ sharing their characteristic functions, coincide.*

Proof. If μ and ν are two such laws, then clearly the class $\mathcal{H}$ of bounded $\mathbb{R}$-valued functions f, for which $\int f d\mu = \int f d\nu$, is a vector space, containing the constants, and closed under nondecreasing limits of nonnegative functions converging to a bounded function (monotone convergence). Moreover it contains finite linear combinations of the functions $\sin(a\cdot)$, $\cos(a\cdot)$, $a \in \mathbb{R}$, whose totality is closed under multiplication (trigonometry). Finally, the latter generate the Borel σ-field (Fourier analysis). $\square$