Non-random overshoots and fluctuation theory for Lévy processes

Matija Vidmar

Paris, École Polytechnique

August 2012

Lévy processes and fluctuation theory

Lévy processes and fluctuation theory

Setting: filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathsf{P})$ under the standard assumptions supporting a Lévy process X.

Lévy processes and fluctuation theory

Setting: filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ under the standard assumptions supporting a Lévy process X.

Definition (Lévy process)

A continuous-time $\mathbb{F}$-adapted stochastic process X with state space $\mathbb{R}$ is a *Lévy process* on the stochastic basis $(\mathbb{F}, \mathsf{P})$, if it starts at 0 a.s.-P, is continuous in probability, $X_{t-s} \sim X_t - X_s \perp \mathcal{F}_s$ (stationary independent increments) and is càdlàg off a P-null set.

Lévy processes and fluctuation theory

Setting: filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$ under the standard assumptions supporting a Lévy process X.

Definition (Lévy process)

A continuous-time $\mathbb{F}$-adapted stochastic process X with state space $\mathbb{R}$ is a *Lévy process* on the stochastic basis $(\mathbb{F}, \mathsf{P})$, if it starts at 0 a.s.-P, is continuous in probability, $X_{t-s} \sim X_t - X_s \perp \mathcal{F}_s$ (stationary independent increments) and is càdlàg off a P-null set.

- Characterized by the Lévy triplet (σ^2, λ, μ), which features in the characteristic exponent $\mathsf{E}[e^{ipX_t}] = e^{t\Psi(p)}$ $(p \in \mathbb{R})$. Example: compound Poisson processes.
- Fluctuation theory: studies first passage times (above, or below a certain level), the running supremum and infimum processes, excursions from the maximum etc.
- Important results: Wiener-Hopf factorization, two-sided exit problem etc.

(cont.)

Definition (First passage times, supremum and infimum processes)

For $x \in \mathbb{R}$: $T_x := \inf\{t \geq 0 : X_t \geq x\}$ (resp. $\hat{T}_x := \inf\{t \geq 0 : X_t > x\}$) the *first entrance time* of X to $[x, \infty)$ (resp. (x, ∞)). For $t \geq 0$:

1. $\overline{X}_t := \sup\{X_s : s \in [0,t]\}$ $(t \geq 0)$ and $\underline{X} := -\overline{-X}$.
2. $\underline{G}_t := \sup\{s \in [0,t] : X_s = \underline{X}_s\}$ and (for CP processes) $\overline{G}^*_t := \inf\{s \in [0,t] : X_s = \overline{X}_t\}$.

- Overshoots ($x \geq 0$): $R_x := X(\hat{T}_x) - x$ on $\{\hat{T}_x < \infty\}$.
- Miscellaneous: $\mathbb{Z}_h := h\mathbb{Z}$.

Definition (Spectrally negative Lévy process)

A Lévy process is called *spectrally negative* if it has no positive jumps a.s.-P and does not have monotone paths.

Non-random position at the time of first passage — motivation

Non-random position at the time of first passage — motivation

- Fluctuation theory for spectrally negative processes is particularly explicit.

Non-random position at the time of first passage — motivation

- Fluctuation theory for spectrally negative processes is particularly explicit.
- Crucially, $X(T_x) = x$ a.s.-P on the event $\{T_x < \infty\}$.

Non-random position at the time of first passage — motivation

- Fluctuation theory for spectrally negative processes is particularly explicit.
- Crucially, $X(T_x) = x$ a.s.-P on the event $\{T_x < \infty\}$.
- Question: is there any other class of Lévy processes for which $X(T_x)$ conditionally on $\{T_x < \infty\}$ is a trivial (degenerate) random variable (for all $x > 0$)?

Non-random position at the time of first passage — motivation

- Fluctuation theory for spectrally negative processes is particularly explicit.
- Crucially, $X(T_x) = x$ a.s.-P on the event $\{T_x < \infty\}$.
- Question: is there any other class of Lévy processes for which $X(T_x)$ conditionally on $\{T_x < \infty\}$ is a trivial (degenerate) random variable (for all $x > 0$)?
- Answer: yes and we can characterize precisely the class of Lévy processes for which this is true.

Non-random position at the time of first passage — motivation

- Fluctuation theory for spectrally negative processes is particularly explicit.
- Crucially, $X(T_x) = x$ a.s.-P on the event $\{T_x < \infty\}$.
- Question: is there any other class of Lévy processes for which $X(T_x)$ conditionally on $\{T_x < \infty\}$ is a trivial (degenerate) random variable (for all $x > 0$)?
- Answer: yes and we can characterize precisely the class of Lévy processes for which this is true.
- Loosely speaking: for the overshoots of a Lévy process to be (conditionally on the process going above the level in question) almost surely constant quantities, it is both necessary and sufficient that *either* the process has no positive jumps (a.s.) *or* for some $h > 0$, it is compound Poisson, living on the lattice $\mathbb{Z}_h := h\mathbb{Z}$, and can only jump up by h.

Formal result

Formal result

Definition (Upwards-skip-free Lévy chain)

A Lévy process X is an *upwards-skip-free Lévy chain* if it is a compound Poisson process, and for some $h > 0$, $\mathrm{supp}(\lambda) \subset \mathbb{Z}_h$ and $\mathrm{supp}(\lambda|_{\mathcal{B}((0,\infty))}) = \{h\}$.

(Discrete time right-continuous random walk embedded into continuous time as a Lévy process.)

Definition (P-triviality)

A random variable R is said to be P-*trivial* on an event $A \in \mathcal{F}$ if there exists $r \in \mathbb{R}$ such that $R = r$ a.s.-P on A. R may only be defined on some $B \supset A$.

(i.e. R is a.s.-P constant conditionally on A.)

(cont.)

Theorem (Non-random position at first passage time)

The following are equivalent:

(a) *For some* $x > 0$, $X(T_x)$ *is* P*-trivial on* $\{T_x < \infty\}$.

(b) *For all* $x \in \mathbb{R}$, $X(T_x)$ *is* P*-trivial on* $\{T_x < \infty\}$.

(c) *For some* $x \geq 0$, $X(\hat{T}_x)$ *is* P*-trivial on* $\{\hat{T}_x < \infty\}$ *and a.s.-*P *positive thereon (in particular the latter obtains if* $x > 0$*).*

(d) *For all* $x \in \mathbb{R}$, $X(\hat{T}_x)$ *is* P*-trivial on* $\{\hat{T}_x < \infty\}$.

(e) *Either* X *has no positive jumps, a.s.-*P *or* X *is an upwards-skip-free Lévy chain.*

If so, then outside a P*-negligible set, for each* $x \in \mathbb{R}$, $X(T_x)$ *(resp.* $X(\hat{T}_x)$*) is constant on* $\{T_x < \infty\}$ *(resp.* $\{\hat{T}_x < \infty\}$*), i.e. the exceptional set in (b) (resp. (d)) can be chosen not to depend on* x.

Fluctuation theory for upwards-skip-free Lévy chains — miscellaneous results ...

Fluctuation theory for upwards-skip-free Lévy chains — miscellaneous results ...

...which are essentially (but not entirely) analogous to the spectrally negative case.

Fluctuation theory for upwards-skip-free Lévy chains — miscellaneous results ...

...which are essentially (but not entirely) analogous to the spectrally negative case.

- The Laplace exponent: $\mathsf{E}[e^{\beta X_t}] = \exp\{t\psi(\beta)\}$ for all $\beta \geq 0$.
- $\Phi(0)$ — largest root of ψ. $\psi : [\Phi(0), \infty) \to [0, \infty)$ is a continuous and increasing bijection. Φ is its inverse.

Let e_p be an $\mathrm{Exp}(p)$-random variable independent of X $(p > 0)$.

- The failure probability for the geometrically distributed $\overline{X}_{e_p}/h$ is $\exp\{-\Phi(p)h\}$.
- X drifts to ∞, oscillates or drifts to $-\infty$ according as $\psi'(0+)$ is positive, zero, or negative. In the latter case $\overline{X}_\infty/h$ has a geometric distribution with failure probability $\exp\{-\Phi(0)h\}$.

(cont.)

Recall: an excursion from the supremum is said to start the moment X leaves its running supremum and stop the moment it returns to it.

Table: Behaviour of X at large times and of its excursions away from the running supremum.

$\psi'(0+)$	Long-term behaviour	Excursion length
> 0	drifts to ∞	finite expectation
$= 0$	oscillates	a.s. finite with infinite expectation
< 0	drifts to $-\infty$	infinite with a positive probability

Wiener-Hopf factorization

Wiener-Hopf factorization

Theorem (Wiener-Hopf factorization for upwards skip-free Lévy chains)

For every $\alpha \geq 0$ and $\beta \geq 0$:

$$\mathrm{E}[\exp\{-\alpha \overline{G}^*_{e_p} - \beta \overline{X}_{e_p}\}] = \frac{1 - e^{-\Phi(p)h}}{1 - e^{-(\beta - \Phi(p+\alpha))h}}$$

and

$$\mathrm{E}[\exp\{-\alpha \underline{G}_{e_p} + \beta \underline{X}_{e_p}\}] = \frac{p}{p + \alpha - \psi(\beta)} \frac{1 - e^{(\beta - \Phi(p+\alpha))h}}{1 - e^{-\Phi(p)h}}.$$

The two-sided exit problem

The two-sided exit problem

- What is the probability of going above a certain level before going below another? What is the law of this first passage time on said event?

The two-sided exit problem

- What is the probability of going above a certain level before going below another? What is the law of this first passage time on said event?
- Answer lies in the so-called scale functions (both for spectrally negative processes as well as for upwards-skip-free Lévy chains). Many are known explicitly, but not nearly all.

The two-sided exit problem

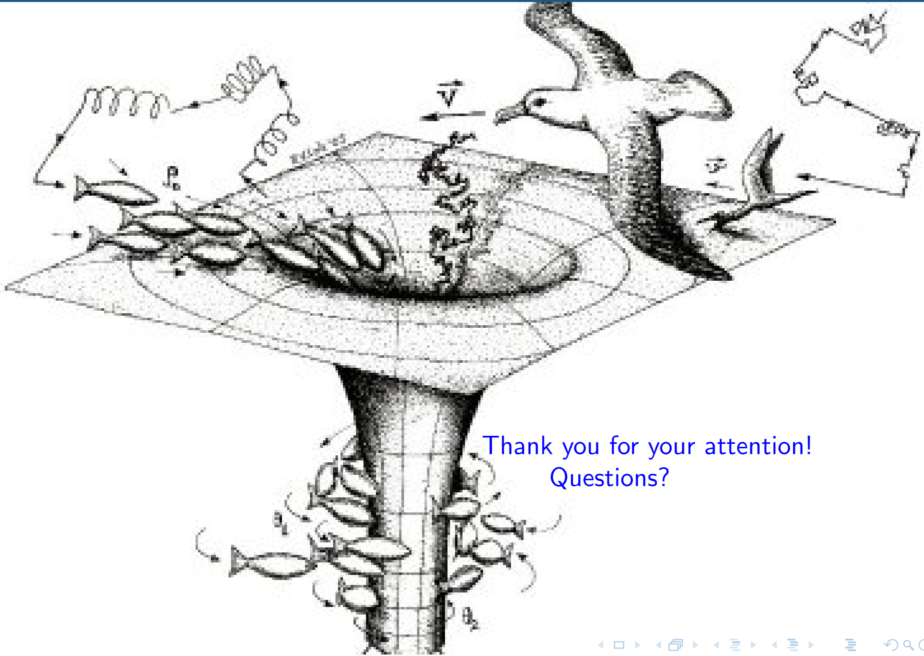
- What is the probability of going above a certain level before going below another? What is the law of this first passage time on said event?
- Answer lies in the so-called scale functions (both for spectrally negative processes as well as for upwards-skip-free Lévy chains). Many are known explicitly, but not nearly all.
- There exists a natural approximation (in law) of a spectrally negative process by an upwards-skip free Lévy chain.

The two-sided exit problem

- What is the probability of going above a certain level before going below another? What is the law of this first passage time on said event?
- Answer lies in the so-called scale functions (both for spectrally negative processes as well as for upwards-skip-free Lévy chains). Many are known explicitly, but not nearly all.
- There exists a natural approximation (in law) of a spectrally negative process by an upwards-skip free Lévy chain.
  - $\rightarrow$ Here calculation of scale functions is explicit straightforward recursion.

The two-sided exit problem

- What is the probability of going above a certain level before going below another? What is the law of this first passage time on said event?
- Answer lies in the so-called scale functions (both for spectrally negative processes as well as for upwards-skip-free Lévy chains). Many are known explicitly, but not nearly all.
- There exists a natural approximation (in law) of a spectrally negative process by an upwards-skip free Lévy chain.
  - → Here calculation of scale functions is explicit straightforward recursion.
- Convergence rates for scale functions (work in progress).



Thank you for your attention!
Questions?