A note on the Markov property for PII(S)

Matija Vidmar

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Setting. In what follows, let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathsf{P})$ be a filtered probability space, $d \in \mathbb{N}$ and $X = (X_t)_{t \geq 0}$ be an $\mathbb{F}$-adapted $\mathbb{R}^d$-valued process with independent increments (i.e. $X_t - X_u \perp \mathcal{F}_u$ whenever $t \geq u \geq 0$). For an $\mathbb{F}$-stopping time T, let (the relevant functions are taken, here and in the sequel, as appropriate, on the set $\{T < \infty\}$):

- $\mathcal{G}_T^\infty := \sigma(X_{T+u} : u \geq 0)$ be the future of X after T,
- $\mathcal{F}'_T := \{A \cap \{T < \infty\} : A \in \mathcal{F}_T\}$ be the past up to T on $\{T < \infty\}$,
- and $\Delta_T := \sigma(X_{T+u} - X_T : u \geq 0)$ be the incremental future after T, i.e. the σ-algebra of the increments after T.

Proposition 1 (Independent increments implies Markov). *Given any $t \geq 0$, Δ_t is independent of $\mathcal{F}_t$. Moreover, X is Markovian relative to the filtraton $\mathbb{F}$ in the sense that for any $t \geq 0$, $\mathcal{F}_t$ is independent of $\mathcal{G}_t^\infty$ conditionally on X_t; that is to say:*

$$\mathsf{P}(A \cap B | X_t) = \mathsf{P}(A|X_t)\mathsf{P}(B|X_t) \quad \mathsf{P}-\text{a.s.}$$

for any $A \in \mathcal{F}_t$ and $B \in \mathcal{G}_t^\infty$.

Assume now X is in addition càd and has stationary increments (in the sense that $X_{t+d} - X_t \sim X_d$ whenever $\{t, d\} \subset [0, \infty)$). For any $\mathbb{F}$-stopping time T with $\mathsf{P}(T < \infty) > 0$, Δ_T is independent of $\mathcal{F}'_T$ under the measure $\mathsf{P}' := \mathsf{P}(\cdot | T < \infty)$. Moreover, X satisfies the strong Markov property with respect to $\mathbb{F}$, i.e. for any stopping time, such as above, $\mathcal{F}'_T$ (with respect to which X_T is measurable) is independent of $\mathcal{G}_T^\infty$, conditionally on X_T under P'; that is to say:

$$\mathsf{P}'(A \cap B | X_T) = \mathsf{P}'(A|X_T)\mathsf{P}'(B|X_T) \quad \mathsf{P}'-\text{a.s.}$$

for any $A \in \mathcal{F}'_T$ and $B \in \mathcal{G}_T^\infty$.

In the latter case the process $(X_{T+u} - X_T)_{u \geq 0}$ under P' is actually identical in law with X under P on the space of càd $\mathbb{R}^d$-valued paths on $[0, \infty)$ with the σ-field of evaluation maps.

We shall use in the proof the properties of how independence disintegrates, resp. aggregates, over σ-algebras. These are easily seen to hold true: either directly or, resp., by a π/λ-argument.

Proof. For the independence of Δ_t and $\mathcal{F}_t$ note that given any finite sequence of times $0 \leq t = t_0 \leq t_1 \leq \cdots \leq t_n \leq t_{n+1}$ ($n \in \mathbb{N}_0$), one has that $X_{t_1} - X_t$ is independent of $\mathcal{F}_t$, $X_{t_2} - X_{t_1}$ is independent of $\sigma(\mathcal{F}_t, X_{t_1} - X_t) \subseteq \mathcal{F}_{t_1}$ and so on and so forth, finally $X_{t_{n+1}} - X_{t_n}$ is independent of $\sigma(\mathcal{F}_t, X_{t_1} - X_t, \ldots, X_{t_n} - X_{t_{n-1}})$. Thus $\mathcal{F}_t$, $X_{t_1} - X_t$, ..., $X_{t_{n+1}} - X_{t_n}$ are jointly independent (disintegration of independence). But then so are $\mathcal{F}_t$ and $\sigma(X_{t_1} - X_t, \ldots, X_{t_{n+1}} - X_{t_n})$ (aggregation of independence). Next, one has $(X_{t_1} - X_t, \ldots, X_{t_{n+1}} - X_t) = f \circ (X_{t_1} - X_t, \ldots, X_{t_{n+1}} - X_{t_n})$, where $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is given by $f(x_1, \ldots, x_n, x_{n+1}) = (x_1, x_2 + x_1, \ldots, x_{n+1} + x_n + \cdots x_1)$ and is certainly Borel. Thus $\sigma(X_{t_j} - X_t : 1 \leq j \leq n+1)$ is a sub-σ-algebra of $\sigma(X_{t_1} - X_t, \ldots, X_{t_{n+1}} - X_{t_n})$ and hence is independent of $\mathcal{F}_t$. But then Δ_t is indeed independent of $\mathcal{F}_t$ (aggregation of independence).

We now show the Markov property. Since $\mathcal{G}_t^\infty = \sigma(X_t) \vee \Delta_t$, by the "taking out what is known property" of conditional expectation and a π/λ-argument, it will be sufficient to maintain the independence of the past $\mathcal{F}_t$ and the incremental future Δ_t given the present X_t. But this follows immediately from the independence of $\mathcal{F}_t$ and Δ_t, since X_t is $\mathcal{F}_t$-measurable. Namely one applies the tower property (conditioning first on $\mathcal{F}_t$).

Assume now X is càd and has stationary increments. We shall show the independence of the incremental future Δ_T and the past $\mathcal{F}'_T$ under the conditional measure P'. The strong Markov property then obtains immediately, precisely as the Markov property did. Indeed, it will be enough to show the independence of the increment $X_{T+u} - X_T$ and $\mathcal{F}'_T$ under P' for each fixed $u \geq 0$ and $\mathbb{F}$-stopping time T. To this end let us first show the claim for the case when T takes values in the countable set $D \cup \{\infty\}$, $D \subset \mathbb{R}_+$. For $A \in \mathcal{F}'_T$ and $B \in \mathcal{B}(\mathbb{R}^d)$ one has (for the obvious reasons, in particular by stationary independent increments):

$$\begin{aligned}
\mathsf{P}'(A \cap \{X_{T+u} - X_T \in B\}) &= \sum_{d \in D} \mathsf{P}(A \cap \{T = d\} \cap \{X_{d+u} - X_d \in B\})/\mathsf{P}(T < \infty) \\
&= \sum_{d \in D} \mathsf{P}(A \cap \{T = d\})\mathsf{P}(X_{d+u} - X_d \in B)/\mathsf{P}(T < \infty) \\
&= \sum_{d \in D} \mathsf{P}(A \cap \{T = d\})\mathsf{P}(X_u \in B)/\mathsf{P}(T < \infty) \\
&= \mathsf{P}'(A)\mathsf{P}(X_u \in B) \\
&= \mathsf{P}'(A)\mathsf{P}'(X_{T+u} - X_T \in B).
\end{aligned}$$

Now, for a general T, approximate it by a nonincreasing sequence $(T_n)_{n \geq 0}$ of $\mathbb{F}$-stopping times having only countably many values and converging to T. We may assume $\{T_n < \infty\} = \{T < \infty\}$ for all $n \in \mathbb{N}_0$. Moreover, we have shown that for any $A \in \mathcal{F}'_T$ and bounded continuous $f : \mathbb{R}^d \to \mathbb{R}$, $\mathsf{E}'[\mathbb{1}_A f \circ (X_{T_n+u} - X_{T_n})] = \mathsf{P}'(A)\mathsf{E}'[f \circ (X_{T_n+u} - X_{T_n})]$. Let $n \to \infty$ and conclude, using the càd property, via dominated convergence (and then by the Functional Monotone Class Theorem).

The final remark is straightforward, from the above displayed computation. $\square$

References

[1] Cinlar, Erhan: *Probability and Stochastics*, Springer, 2011.