Markov chain approximations for transition densities of Lévy processes

M. Vidmar (with A. Mijatović & S. Jacka)

1 Abstract

We consider the convergence of a continuous time Markov chain (CTMC) approximation X^h, $h>0$, to an $\mathbb{R}$-valued Lévy process X. The state space of X^h is an equidistant lattice and its Q-matrix is chosen to approximate the generator of X. Under a general sufficient condition for the existence of transition densities of X we establish sharp convergence rates of the normalised probability mass function of X^h to the probability density function of X.

2 Motivation

Discretization schemes for stochastic processes are relevant both *theoretically*, as they shed light on the nature of the underlying stochasticity, and *practically*, since they lend themselves well to numerical methods. Consequently there has been a plethora of publications devoted to the subject, see e.g. [1]. In particular, there is also a wealth of existing literature concerning approximations of Lévy processes in one form or another and a brief overview of simulation techniques is given by [3].

3 Setting

Let Ψ be the characteristic exponent corresponding to a Lévy process in law X with characteristic triplet $(\sigma^2,\lambda,\mu)_{\tilde{c}}$ — $\tilde{c}(y):=\mathbb{1}_{[-V,V]}(y)$ is the cut-off function and V is 1 or 0, the latter only if $\int_{[-1,1]}|x|d\lambda(x)<\infty$. Recall that X is then a Markov process with transition function $P_{t,T}(x,B):=\mathsf{P}(X_{T-t}\in B-x)$ $(0\le t\le T,\ x\in\mathbb{R}$ and $B\in\mathcal{B}(\mathbb{R}))$ and characteristic function $\phi_{X_t}(p)=\mathsf{E}[e^{ipX_t}]=\exp\{t\Psi(p)\}$ $(p\in\mathbb{R})$.

Fix $h>0$. Consider a CTMC $X^h=(X^h_t)_{t\ge0}$ approximating our Lévy process X (in law). We describe X^h as having state space $\mathbb{Z}_h$, initial state $X^h_0=0$, a.s. and an infinitesimal generator L^h given by a spatially homogeneous Q-matrix Q^h (i.e. $Q^h_{ss'}$ depends only on $s-s'$, for $\{s,s'\}\subset\mathbb{Z}_h$).

It remains to specify Q^h. To this end we discretise on $\mathbb{Z}_h$ the infinitesimal generator L of the Lévy process X, thus obtaining L^h. Recall that [4, p. 208]:

$$Lf(x)=\frac{\sigma^2}{2}f''(x)+\mu f'(x)+\int_{\mathbb{R}}\left[f(x+y)-f(x)-yf'(x)\mathbb{1}_{[-V,V]}(y)\right]d\lambda(y)$$

for $x\in\mathbb{R}$ and every twice continuously differentiable $f:\mathbb{R}\to\mathbb{R}$ with f,f',f'' vanishing at infinity. We thus have **discretisation scheme 1**, for $s\in\mathbb{Z}_h$ and $f:\mathbb{Z}_h\to\mathbb{R}$ vanishing at infinity:

$$L^hf(s)=\frac{1}{2}\left(\sigma^2+c^h_0\right)\frac{f(s+h)+f(s-h)-2f(s)}{h^2}+\left(\mu-\mu^h\right)\frac{f(s+h)-f(s-h)}{2h}+\sum_{s'\in\mathbb{Z}_h\backslash\{0\}}\left[f(s+s')-f(s)\right]c^h_{s'} \tag{1}$$

where we have introduced $c^h_s:=\lambda(A^h_s)$ with $A^h_s:=(s-h/2,s+h/2]$ if $s>0$ and $A^h_s:=[s-h/2,s+h/2)$ if $s<0$ $(s\in\mathbb{Z}_h\backslash\{0\})$; finally (letting $A^h_0:=[-h/2,h/2]$; $\delta^0\in\{0,1\}$):

$$c^h_0:=\delta^0\int_{A^h_0}y^2\mathbb{1}_{[-V,V]}(y)d\lambda(y)\qquad \mu^h:=\sum_{s\in\mathbb{Z}_h\backslash\{0\}}s\int_{A^h_s}\mathbb{1}_{[-V,V]}(y)d\lambda(y).$$

We reserve the right to set c^h_0 to 0 even if $V\neq0$, hence the presence of $\delta^0\in\{0,1\}$.

Note that Q^h has nonnegative off-diagonal entries for all h for which:

$$\frac{\sigma^2+c^h_0}{2h^2}+\frac{\mu-\mu^h}{2h}+c^h_h\ge0\qquad\text{and}\qquad\frac{\sigma^2+c^h_0}{2h^2}-\frac{\mu-\mu^h}{2h}+c^h_{-h}\ge0 \tag{2}$$

and in that case Q^h is a genuine regular Q-matrix. When $\sigma^2>0$, it is shown that (2) always obtains, at least for all sufficiently small h, but not in general. When $\sigma^2=0$ we shall thus employ an alternative discretisation for which the above condition holds vacuously. The changes are as follows.

First, (1) now reads (letting $\mathrm{sgn}:=\mathbb{1}_{(0,\infty)}-\mathbb{1}_{(-\infty,0)}$):

$$L^hf(s)=\frac{1}{2}\left(\sigma^2+c^h_0\right)\frac{f(s+h)+f(s-h)-2f(s)}{h^2}+\sum_{s'\in\mathbb{Z}_h\backslash\{0\}}[f(s+s'+h\mathrm{sgn}(s'))-f(s)]c^h_{s'}+(\mu-\mu^h)\left(\frac{f(s+h)-f(s)}{h}\mathbb{1}_{[0,\infty)}+\frac{f(s)-f(s-h)}{h}\mathbb{1}_{(-\infty,0]}\right)(\mu-\mu^h)$$

So, in place of $(\mu-\mu^h)\frac{f(s+h)-f(s-h)}{2h}$ we have rather $(\mu-\mu^h)\frac{f(s+h)-f(s)}{h}$ *or* $(\mu-\mu^h)\frac{f(s)-f(s-h)}{h}$ according as $\mu-\mu^h\ge0$ *or* $\mu-\mu^h\le0$. We have also modified slightly the final summation in L^h. This will turn out to be advantageous when considering the convergence of the transition kernel with $\sigma^2=0$ and is made solely for this purpose. Namely, it allows to control the decay of the characteristic functions of the process X and the approximating CTMCs at infinity, uniformly in the discretisation parameter h. It is possible that one could establish convergence rates even without this modification but our proof technique relies on it.

Second, we have:

$$\mu^h:=\sum_{s\in\mathbb{Z}_h\backslash\{0\}}(s+h\mathrm{sgn}(s))\int_{A^h_s}\mathbb{1}_{[-V,V]}(y)d\lambda(y).$$

Everything else remains unchanged. We shall refer to this as **discretisation scheme 2**.

4 Assumptions

The assumption alluded to in the abstract is the following:

Assumption 1. *Either $\sigma^2>0$ or Orey's condition holds:*

$$\exists\epsilon\in(0,2)\quad\textit{such that}\quad\liminf_{r\downarrow0}\frac{1}{r^{2-\epsilon}}\int_{[-r,r]}u^2d\lambda(u)>0.$$

The usage of the two schemes and the specification of δ^0 and V, depending on the nature of σ^2 and λ, is as summarized in the following table (for the definition of κ see (5)).

Lévy measure/diffusion part	$\sigma^2>0$ (scheme 1)	$\sigma^2=0$ (scheme 2)
$\lambda(\mathbb{R})<\infty$	$V=\delta^0=0$	$\times$
$\kappa(0)<\infty=\lambda(\mathbb{R})$	$V=1$ and $\delta^0=0$	
$\kappa(0)=\infty$	$V=\delta^0=1$	

5 Transition kernels

Under Assumption 1 for every $t>0$, $\phi_{X_t}\in L^1(\mathrm{Leb})$ where Leb is Lebesgue measure and (for $0\le t<T$, $\{x,y\}\subset\mathbb{R}$) the continuous transition density for X, from x at time t to y at time T, is:

$$p_{t,T}(x,y)=\frac{1}{2\pi}\int_{\mathbb{R}}\exp\{ip(x-y)\}\exp\{\Psi(p)(T-t)\}\,dp. \tag{3}$$

Similarly, with Ψ^h denoting the characteristic exponent of the compound Poisson process X^h (for $0\le t<T$, $y\in\mathbb{Z}_h$, $\mathsf{P}_{X^h_t}$-a.s. in $x\in\mathbb{Z}_h$) $P^h_{t,T}(x,y):=\mathsf{P}(X^h_T=y|X^h_t=x)$ is given by:

$$\frac{1}{h}P^h_{t,T}(x,y)=\frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}\exp\{ip(x-y)\}\exp\{\Psi^h(p)(T-t)\}dp. \tag{4}$$

Note that the RHS is defined even if $\mathsf{P}(X^h_t=x)=0$ and we let the LHS be this value when this is so.

6 Rates of convergence for the transition densities

For $0\le t<T$ and $x,y\in\mathbb{Z}_h$ define:

$$\Delta_{T-t}(h):=\sup_{\{x,y\}\subset\mathbb{Z}_h}D^h_{t,T}(x,y)\qquad\text{where}\qquad D^h_{t,T}(x,y):=\left|p_{t,T}(x,y)-\frac{1}{h}P^h_{t,T}(x,y)\right|$$

and we shall formulate the convergence rates in terms of the following key quantity:

$$\kappa(h):=\int_{[-1,1]\backslash[-h,h]}|x|d\lambda(x). \tag{5}$$

Theorem 1. *Under Assumption 1, whenever $s>0$, the convergence of $\Delta_s(h)$ is as summarized in the following table. In general convergence is no better than stipulated.*

$\Delta_s(h)$	$\lambda(\mathbb{R})=0$	$0<\lambda(\mathbb{R})<\infty$	$\kappa(0)<\infty=\lambda(\mathbb{R})$	$\kappa(0)=\infty$
$\sigma^2>0$	$O(h^2)$	$O(h)$	$O(h)$	$O(h\kappa(h/2))$
$\sigma^2=0$	$\times$	$\times$		

Remark 1. *X^h converges to X weakly in finite-dimensional distributions as $h\downarrow0$.*

The proof of Theorem 1 is in two steps:

(i) Establish the convergence rates of the characteristic exponents of X_t and X^h_t.

(ii) Apply the latter to study the convergence of the transition densities via (3) and (4).

For more detail on the proof, as well as a more nuanced statement of the results, we refer to the article [2].

Example: If $\lambda([-1,1]\backslash[-h,h])\sim1/h^{1+\alpha}$ for some $\alpha\in(0,1)$, then $\kappa(h)\sim h^{-\alpha}$ and the convergence of the normalized probability mass function to the transition density is by Theorem 1 of order $h^{1-\alpha}$. (We use the notation $f\sim g$ to mean $\lim_{h\downarrow0}f(h)/g(h)\in(0,\infty)$.)

7 Conclusion and extensions

CTMCs represent a much simpler class of processes for which many of the quantities are more easily calculated as for a general Lévy process. In particular, this is true of scale functions — where a spectrally negative Lévy process is approximated by an "upwards-skip-free Lévy chain" — and convergence rates for these are currently work in progress.

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References

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