

# PROPER TWO-SIDED EXITS OF A LÉVY PROCESS

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ABSTRACT. It is proved that the two-sided exits of a Lévy process are proper, i.e. are not a.s. equal to their one-sided counterparts, if and only if said process is not a subordinator or the negative of a subordinator. Furthermore, Lévy processes are characterized, for which the supports of the first exit times from bounded annuli, simultaneously on each of the two events corresponding to exit at the lower and the upper boundary, respectively are unbounded, contain 0, are equal to  $[0, \infty)$ .

## 1. INTRODUCTION

Two-sided exits of spectrally one-sided Lévy processes have been extensively studied, e.g. [3, Chapter VII] [10, Section 9.46] [7, Section 8.2], to which is added a great number of scientific papers. Less is known in the general case – though integral transforms of many relevant quantities still admit an analytic representation [6]. But the expressions entering these transforms are complicated, and in particular do not lend themselves easily to analysis. The study of the qualitative aspects of the two-sided exit problem, at least for the case of a general Lévy process, thus appears relevant.

Such study is clearly connected with that of the distributional properties of the running supremum  $\overline{|X|}$  of the absolute value  $|X|$  of a Lévy process  $X$ . However – by contrast to those of the supremum process  $\overline{X}$  of  $X$  itself, e.g. [8, 9, 5, 4] –, few such properties appear to have been analyzed in general. There are exceptions, e.g. [11, 2].

In a minor contribution to this area, the purpose of the present paper is to characterize those Lévy processes for which the supports of the ‘laws’ of the first exit times from bounded annuli, *simultaneously* on each of the two events corresponding to exit at the lower and the upper boundary, respectively are non-empty (Proposition 3), contain 0 (Proposition 4), are unbounded (Proposition 7), are equal to  $[0, \infty)$  (Corollary 9). Propositions 5 and 8 give a further analysis of the cases when, respectively, the second and third of the preceding properties fails, but the first does not.

In terms of practical relevance note that Lévy processes are often used to model the risk process of an insurance company [7, Paragraph 1.3.1 & Chapter 7], or their exponentials are used to model the price fluctuations of stocks [7, Paragraph 2.7.3]. Thus, for example, it may be useful to know whether or not (in both cases possibly before or after some time, or in each non-degenerate time interval) (i) an insurer with given initial capital will in fact go bankrupt or its capital will exceed some given level, each with a positive probability; or (ii) a perpetual two-sided barrier option will terminate with a positive probability on each of the two boundaries.

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## 2. SETTING AND NOTATION

Throughout we will let  $X$  be a Lévy process [10, Section 1] on the stochastic basis  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$  (it is assumed then that  $X$  is  $\mathbb{F}$ -adapted) satisfying the standard assumptions, with diffusion coefficient  $\sigma^2$ , Lévy measure  $\nu$  and, when  $\int 1 \wedge |x| \nu(dx) < \infty$ , drift  $\gamma_0$  [10, Section 8].

**Definition 1** (Two-sided exit times and their laws). Let  $\{a, b\} \subset (0, \infty)$ .

- (i) For a càdlàg path  $\omega$  mapping  $[0, \infty)$  into  $\mathbb{R}$ , vanishing at zero, we denote by  $T_{a,b}(\omega)$  the first entrance time of  $\omega$  into the set  $\mathbb{R} \setminus (-b, a)$  (i.e. the first exit time of  $\omega$  from  $(-b, a)$ ).
- (ii) We introduce the measures  $\lambda_{a,b}^+$  and  $\lambda_{a,b}^-$  on  $\mathcal{B}([0, \infty))$ , so that for  $A \in \mathcal{B}([0, \infty))$ ,  $\lambda_{a,b}^-(A) \stackrel{\text{def}}{=} \mathbf{P}(\{T_{a,b}(X) < \infty\} \cap \{X_{T_{a,b}(X)} \leq -b\} \cap \{T_{a,b}(X) \in A\})$  and  $\lambda_{a,b}^+(A) \stackrel{\text{def}}{=} \mathbf{P}(\{T_{a,b}(X) < \infty\} \cap \{X_{T_{a,b}(X)} \geq a\} \cap \{T_{a,b}(X) \in A\})$ .

We shall be concerned then with characterizing the pairs  $(\sigma^2, \nu)$  and, when  $\int 1 \wedge |x| \nu(dx) < \infty$ , further the drifts  $\gamma_0$ , under which the measures  $\lambda_{a,b}^\pm$  are non-vanishing (on each non-empty interval of the form  $[0, M)$ ,  $[m, \infty)$ , respectively  $[m, M)$  /that is contained in  $[0, \infty)/$ ).

**Definition 2** (Auxiliary notions/notation). For a time  $S : \Omega \rightarrow [0, \infty]$  we will call  $(X(S+t) - X(S))_{t \geq 0}$  (defined on  $\{S < \infty\}$ ) the incremental process of  $X$  after  $S$ . For a measure  $\rho$  on a topological space,  $\text{supp}(\rho)$  will be its support. Finally,  $a \wedge b := \min\{a, b\}$  (when  $\{a, b\} \subset [-\infty, +\infty]$ ): for measurable sets  $A$  and  $B$  and a measure  $\lambda$ ,  $\lambda A \wedge \lambda B > 0$  is thus shorthand for “ $\lambda(A) > 0$  and  $\lambda(B) > 0$ .”

## 3. RESULTS

Now the precise statements follow.

**Proposition 3.**  $\lambda_{a,b}^+[0, \infty) \wedge \lambda_{a,b}^-[0, \infty) > 0$  for some (then all)  $\{a, b\} \subset (0, \infty)$ , if and only if  $\sigma^2 > 0$ ; or  $\int |x| \wedge 1 \nu(dx) = \infty$ ; or  $\nu$  charges  $(-\infty, 0)$  and  $(0, \infty)$  both; or else  $\nu$  charges only (and does charge)  $(0, \infty)$  and  $\gamma_0 < 0$ , or  $\nu$  charges only (and does charge)  $(-\infty, 0)$  and  $\gamma_0 > 0$ ,

*i.e. if and only if neither  $X$  nor  $-X$  is a subordinator.*

*Proof.* For the last equivalence see [10, p. 137, Theorem 21.5]. The condition is clearly necessary.

Sufficiency. Let  $\{a, b\} \subset (0, \infty)$ . The condition implies  $X$  is not the zero process, so that  $\limsup_{\infty} X = \infty$  or  $\liminf_{-\infty} X = -\infty$  a.s. [10, p. 255, Proposition 37.10], and so a.s.  $T_{a,b}(X) < \infty$ . Suppose furthermore *per absurdum*, and then without loss of generality, that a.s.  $X_{T_{a,b}(X)} \geq a$ . Let  $X^{(0)} \stackrel{\text{def}}{=} X$ ,  $T^{(0)} \stackrel{\text{def}}{=} T_{a,b}(X)$ . Then by the strong Markov property [10, p. 278, Theorem 40.10] of Lévy processes, inductively, we would find that a.s. for all  $k \in \mathbb{N}_0$  the incremental process  $X^{(k+1)}$  of  $X$  after  $T^{(k)}$  would satisfy  $X^{(k+1)}(T^{(k+1)}) \geq a$ , where  $T^{(k+1)} \stackrel{\text{def}}{=} T_{a,b}(X^{(k+1)})$  would be equal in distribution to  $T^{(0)}$  and independent of  $\mathcal{F}_{T^{(k)}}$ . In particular, since by the right-continuity of the sample paths  $\mathbf{E}T^{(0)} > 0$  (indeed  $T^{(0)} > 0$  a.s.), and since  $(T^{(k)})_{k \in \mathbb{N}_0}$  is an iid sequence, it would

follow from the strong law of large numbers, that with probability one  $X$  would be  $> -b$  at all times. According to [10, p. 149, Theorem 24.7] this would only be possible if  $\sigma^2 = 0$ ,  $\int |x| \wedge 1 \nu(dx) < \infty$  with  $\nu$  charging only  $(0, \infty)$ . Then according to the assumed condition we would need to have  $\gamma_0 < 0$ , yielding a contradiction with [10, p. 151, Corollary 24.8], which necessitates the infimum of the support of  $X_t$  being  $\gamma_0 t$ , for all  $t \in [0, \infty)$ , in this case.  $\square$

In various subcases, this statement can be made more nuanced.

**Proposition 4.** *The condition that  $\lambda_{a,b}^+[0, M] \wedge \lambda_{a,b}^-[0, M] > 0$  for all  $M \in (0, \infty]$ , for some (then all)  $\{a, b\} \subset (0, \infty)$ , is equivalent to*

$$\nu \text{ charges } (-\infty, 0) \text{ and } (0, \infty) \text{ both; or } \sigma^2 > 0; \text{ or } \int 1 \wedge |x| \nu(dx) = \infty.$$

*Proof.* The condition is necessary. For, if  $\sigma^2 = 0$ ,  $\int 1 \wedge |x| < \infty$  and, say,  $\nu$  charges only  $(0, \infty)$ , then in order that  $X$  not have monotone paths, it will need to assume a strictly negative drift, but even then, according to the Lévy-Itô decomposition [1, Section 2.4], for given  $a$  and  $b$ ,  $M$  can clearly be chosen so small, that by time  $M$ , a.s.  $X$  can only have left  $(-b, a)$  at the upper boundary.

Sufficiency. The argument is similar as in the proof of the preceding proposition, so we forego explicating some of the details. Let  $\{a, b\} \subset (0, \infty)$ ,  $M \in (0, \infty)$ . Suppose *per absurdum*, and then without loss of generality, that  $X_{T_{a,b}(X)} \geq a$  a.s. on  $\{T_{a,b}(X) < M\}$ . Let  $X^{(0)} \stackrel{\text{def}}{=} X$ ,  $T^{(0)} \stackrel{\text{def}}{=} T_{a,b}(X)$ . By the strong Markov property of Lévy processes, inductively, we find that a.s. for all  $k \in \mathbb{N}_0$  the incremental process  $X^{(k+1)}$  of  $X$  after  $T^{(k)}$  satisfies  $X^{(k+1)}(T^{(k+1)}) \geq a$  on  $\{T^{(k+1)} < M\}$ , where  $T^{(k+1)} \stackrel{\text{def}}{=} T_{a,b}(X^{(k+1)})$  is equal in distribution to  $T^{(0)}$  and independent of  $\mathcal{F}_{T^{(k)}}$ . From the strong law of large numbers, it now follows, that with probability one  $X$  is  $> -b$  on  $[0, M)$ . But this is only possibly if  $\sigma^2 = 0$ ,  $\int |x| \wedge 1 \nu(dx) < \infty$  with  $\nu$  charging only  $(0, \infty)$ , a contradiction.  $\square$

The situation when  $X$  satisfies the condition of Proposition 3 but not that of Proposition 4 can (up to the trivial transformation  $X \rightarrow -X$ ) easily be described as follows (in particular, in Proposition 4, we cannot change the qualification “for all  $M \in (0, \infty]$ , for some (then all)  $\{a, b\} \subset (0, \infty)$ ” to “for some (then all)  $M \in (0, \infty]$  and  $\{a, b\} \subset (0, \infty)$ ”):

**Proposition 5.** *Suppose  $\int 1 \wedge |x| \nu(dx) < \infty$ ,  $\sigma^2 = 0$  and  $\nu$  charges only (and does charge)  $(-\infty, 0)$ , finally  $\gamma_0 > 0$ . Let furthermore  $M \in (0, \infty]$ ,  $\{a, b\} \subset (0, \infty)$ . Then  $\lambda_{a,b}^+[0, M] \wedge \lambda_{a,b}^-[0, M] > 0$ , if and only if  $a/\gamma_0 < M$ .*

*Proof.* The condition is clearly necessary. Sufficiency. In view of Proposition 3, we may assume  $M < \infty$ . If  $\nu$  is finite, note that there is  $\epsilon \in (0, \infty)$  with  $\nu(-\infty, -\epsilon) > 0$ , and then the desired conclusion follows from the fact that with positive probability the process  $X$  will have  $\lceil (a+b)/\epsilon \rceil$  many jumps of size  $< -\epsilon$  before  $a/\gamma_0$ , going below  $-b$  (but not above  $a$ ) strictly before time  $M$ , but with positive probability will also not have a jump up to, reaching level  $a$  at, time  $a/\gamma_0 < M$ . If  $\nu$  is infinite, it follows simply from the fact that the support of the jump part of  $X$  will be  $(-\infty, 0]$  at all strictly positive times [10, p. 152, Theorem 24.10]: thus, with a positive probability, at time

$a/\gamma_0 + (M - a/\gamma_0)/2$ , say, the jump part will not have gone below the level  $-((M\gamma_0 - a)/2) \wedge b$ , and hence  $X$  not below the level  $-b$ , but  $X$  will of course have gone above the level  $a$  by this time; in the same vein, with a positive probability, at time  $a/(2\gamma_0)$ , say, the jump part of  $X$  will have gone below  $-(b + a/2)$ , and so  $X$  below the level  $-b$ , but  $X$  will not have gone above level  $a$  by that time.  $\square$

For the next few statements let us borrow the following result from [2, Proposition 1.1] (that itself refers to [11]):

**Proposition 6.** *Let  $\overline{|X|}$  be the running supremum process of the absolute value process of  $X$ . Then for some  $M \in (0, \infty)$  and then all  $\epsilon \in (0, \infty)$   $\mathbb{P}(\overline{|X|}_M < \epsilon) > 0$ , if and only if*

$$\int 1 \wedge |x| \nu(dx) < \infty \text{ and } (\gamma_0 = 0) \vee (\gamma_0 > 0 \wedge 0 \in \text{supp}(\nu|_{(-\infty, 0]}) \vee (\gamma_0 < 0 \wedge 0 \in \text{supp}(\nu|_{[0, \infty)})); \text{ or } \sigma^2 > 0; \text{ or } \int 1 \wedge |x| \nu(dx) = \infty.$$

When so, then  $\mathbb{P}(\overline{|X|}_M < \epsilon) > 0$  whenever  $\{M, \epsilon\} \subset (0, \infty)$ .

**Proposition 7.** *The condition that  $\lambda_{a,b}^+[m, \infty) \wedge \lambda_{a,b}^-[m, \infty) > 0$  whenever  $\{a, b\} \subset (0, \infty)$  and  $m \in [0, \infty)$ , is equivalent to*

$$\int 1 \wedge |x| \nu(dx) < \infty \text{ and } ((\gamma_0 = 0) \wedge (\nu \text{ charges } (0, \infty) \text{ and } (-\infty, 0) \text{ both})) \vee ((\gamma_0 > 0) \wedge (0 \in \text{supp}(\nu|_{(-\infty, 0]}))) \vee ((\gamma_0 < 0) \wedge (0 \in \text{supp}(\nu|_{[0, \infty)}))); \text{ or } \sigma^2 > 0; \text{ or } \int 1 \wedge |x| \nu(dx) = \infty.$$

*Proof.* Since  $\lambda_{a,b}^+[m, \infty) \wedge \lambda_{a,b}^-[m, \infty) > 0$  whenever  $\{a, b\} \subset (0, \infty)$  and  $m \in (0, \infty)$ , implies  $\mathbb{P}(\overline{|X|}_M < \epsilon) > 0$  for all  $\epsilon \in (0, \infty)$  and  $M \in (0, \infty)$ , necessity of the condition is clear from Propositions 3 and 6.

Sufficiency. Let  $\{a, b\} \subset (0, \infty)$ ,  $m \in [0, \infty)$ . Let  $A$  be the event that the process  $X$  will stay within the annulus  $(-b, a)$  up to and including time  $m$ . According to Proposition 6,  $\mathbb{P}(A) > 0$ . Let  $B_a$  (respectively  $B_b$ ) be the event that the incremental process of  $X$  after  $m$  will exit  $(-b - X_m, a - X_m)$  in finite time at a level that is  $\geq a - X_m$  (respectively  $\leq -b - X_m$ ). By the Markov property, from Proposition 3, and via a standard manipulation of conditional expectations, we obtain that  $\mathbb{P}(B_a | \mathcal{F}_m) > 0$  a.s. on  $A$ , hence  $\mathbb{P}(A \cap B_a) = \mathbb{E}[\mathbb{P}(B_a | \mathcal{F}_m); A] > 0$  and likewise for  $A \cap B_b$ .  $\square$

Again the situation when  $X$  satisfies the condition of Proposition 3 but not that of Proposition 7 can (up to the trivial transformation  $X \rightarrow -X$ ) rather easily be made more precise (in particular, in Proposition 7, we cannot change the qualification “whenever  $\{a, b\} \subset (0, \infty)$  and  $m \in [0, \infty)$ ” to “for some (then all)  $\{a, b\} \subset (0, \infty)$ ,  $m \in [0, \infty)$ ” or even to “for all  $m \in [0, \infty)$ , for some (then all)  $\{a, b\} \subset (0, \infty)$ ”):

**Proposition 8.** *Suppose  $\int 1 \wedge |x| \nu(dx) < \infty$ ,  $\sigma^2 = 0$ ,  $\gamma_0 > 0$ ,  $0 \notin \text{supp}(\nu|_{(-\infty, 0]})$ , but  $\nu(-\infty, 0) \neq 0$ . Let  $W := -\text{supp}(\nu|_{(-\infty, 0]})$ . Let furthermore  $m \in [0, \infty)$ ,  $\{a, b\} \subset (0, \infty)$ . Then  $\lambda_{a,b}^+[m, \infty) \wedge \lambda_{a,b}^-[m, \infty) > 0$ , if and only if  $m\gamma_0 < a$  or  $a + b > W$ .*

*Proof.* Note  $\mathbb{1}_{(-\infty,0)} \cdot \nu$  is finite.

The condition is necessary, since its falsity necessitates  $X$  a.s. never exiting  $(-b, a)$  for the first time, after and inclusive of time  $m$ , at the lower boundary.

Sufficiency. Under the assumed condition, with a positive probability,  $X$  will not exit  $(-b, a)$  up to time (inclusive of)  $m$ , which fact we can argue as follows.

Assume first  $a + b > W$ . According to the Lévy-Itô decomposition we can decompose  $X$  into the independent sum of a pure drift process with drift coefficient  $\gamma_0$ , a pure-jump subordinator  $Y$  and a compound Poisson process  $Z$  with Lévy measure  $\mathbb{1}_{(-\infty,0)} \cdot \nu$ . Let  $\{\delta, \gamma\} \in (0, \infty)$ ,  $K \in \mathbb{N}$ . With a positive probability  $Y$  will not increase by more than  $\delta$  by time (inclusive of)  $m$ . Independently  $Z$  will not have a jump during the time interval  $[0, a/\gamma_0 - \gamma]$ , and then  $\lceil m/(W/\gamma_0) \rceil$  successive times will behave as follows: have precisely one jump into  $(-W - \delta, -W)$  during the next (left-open, right-closed) interval of length  $\gamma/K$ , and then will not jump for the next (left-open, right-closed) interval of time of length  $W/\gamma_0$ . It is clear that thanks to  $W < a + b$ ,  $\gamma$  can be chosen small enough, and then  $\delta$  small enough and  $K$  large enough, that on the described event of positive probability  $X$  remains in  $(-b, a)$  up to time (inclusive of)  $m$ .

Now assume  $m\gamma_0 < a$ . With a positive probability  $X$  will have no negative jump up to time  $m$  inclusive, whilst independently and with a positive probability the part of  $X$  consisting of the positive jumps of  $X$  only, will be found in  $[0, a - m\gamma_0)$  at time  $m$ .

In either case  $X$  remains inside the annulus  $(-b, a)$  up to time (inclusive of)  $m$  with a positive probability. Moreover, by the Markov property and from Proposition 3, on this event, conditionally on  $\mathcal{F}_m$ , the incremental process of  $X$  after  $m$  will exit  $(-b - X_m, a - X_m)$  in finite time with a positive probability at a level  $\geq a - X_m$  and with a positive probability also at a level  $\leq -b - X_m$  (cf. proof of Proposition 7). This demonstrates that in fact  $\lambda_{a,b}^+[m, \infty) \wedge \lambda_{a,b}^-[m, \infty) > 0$ .  $\square$

**Corollary 9.** *The condition that  $\lambda_{a,b}^+[m, M) \wedge \lambda_{a,b}^-[m, M) > 0$ , whenever  $\{a, b\} \subset (0, \infty)$ ,  $m \in [0, \infty)$ ,  $M \in (0, +\infty]$ ,  $m < M$ , is equivalent to*

$$\int 1 \wedge |x| \nu(dx) < \infty \text{ and } \nu \text{ charges } (0, \infty) \text{ and } (-\infty, 0) \text{ both and } (\gamma_0 = 0) \vee ((\gamma_0 > 0) \wedge (0 \in \text{supp}(\nu|_{(-\infty,0]}))) \vee ((\gamma_0 < 0) \wedge (0 \in \text{supp}(\nu|_{[0,\infty)}))); \text{ or } \sigma^2 > 0; \text{ or } \int 1 \wedge |x| \nu(dx) = \infty.$$

*Proof.* One combines Propositions 4 and 6, using the Markov property in the sufficiency part (similarly as it was used in the proof of Proposition 7).  $\square$

It does not appear the situation in which  $X$  satisfies the condition of Proposition 3 but not of Corollary 9 can be readily described (in a concise manner).

*Remark 10.* One verifies at once that no two sets of conditions, characterizing the situations present in Propositions 3, 4, and 7, and in Corollary 9 are equivalent in general.

## REFERENCES

1. D. Applebaum, *Lévy processes and stochastic calculus*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2009.
2. F. Aurzada and S. Dereich, *Small deviations of general Lévy processes*, The Annals of Probability **37** (2009), no. 5, 2066–2092.
3. J. Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1996.
4. L. Chaumont, *On the law of the supremum of Lévy processes*, The Annals of Probability **41** (2013), no. 3A, 1191–1217.
5. R. A. Doney and A. E. Kyprianou, *Overshoots and undershoots of Lévy processes*, The Annals of Applied Probability **16** (2006), no. 1, 91–106.
6. V. Kandakov, T. Kadankova, N. Veraverbeke, and N. Kartaskov, *Two-sided exit problem for general Lévy processes. Theory and application.*, Tech. report, IAP Statistics Network.
7. A. E. Kyprianou, *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer-Verlag, Berlin Heidelberg, 2006.
8. E. A. Pecherskii and B. A. Rogozin, *On joint distributions of random variables associated with fluctuations of a process with independent increments*, Theory of Probability & Its Applications **14** (1969), no. 3, 410–423.
9. B. A. Rogozin, *On distributions of functionals related to boundary problems for processes with independent increments*, Theory of Probability & Its Applications **11** (1966), no. 4, 580–591.
10. K. I. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge studies in advanced mathematics, Cambridge University Press, Cambridge, 1999.
11. T. Simon, *Sur les petites déviations d'un processus de Lévy*, Potential Analysis **14** (2001), no. 2, 155–173.

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