Exercise Sheet 1, ST213

Let Ω be a sample space. In the sequel (unless otherwise indicated) algebra will mean algebra on Ω. Prove the following statements:

1) The power set $\mathcal{P}(\Omega)$, that is the set of precisely all subsets of Ω, is an algebra.

2) $\{\emptyset, \Omega\}$ is an algebra (the trivial algebra).

3) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two algebras. Then $\mathcal{A}_1 \cap \mathcal{A}_2$ is an algebra. If $(\mathcal{A}_j)_{j \in \mathcal{J}}$ is a family of algebras, then $\bigcap_{j \in \mathcal{J}} \mathcal{A}_j$ is an algebra.

4) Let $\mathcal{C}$ be a set of subsets of Ω, i.e. $\mathcal{C} \subseteq \mathcal{P}(\Omega)$. Then

$$a(\mathcal{C}) := \bigcap_{\mathcal{A} \text{ algebra}, \, \mathcal{C} \subseteq \mathcal{A}} \mathcal{A}$$

is an algebra, the *algebra generated by* $\mathcal{C}$.

5) If $A \subseteq \Omega$ and $\mathcal{C} = \{A\}$, then $a(\mathcal{C}) = \{\emptyset, A, A^c, \Omega\}$.

6) If for some $n \in \mathbb{N}$, $\mathcal{C} = \{A_1, \ldots, A_n\}$ where $A_i \subseteq \Omega$ for $1 \leq i \leq n$, $A_i \cap A_j = \emptyset$ for $1 \leq i, j \leq n$ with $i \neq j$, and $A_1 \cup \cdots \cup A_n = \Omega$; then:

$$a(\mathcal{C}) = \{\emptyset\} \cup \{A_{i_1} \cup \cdots \cup A_{i_m} : 1 \leq i_k \leq n; \, k = 1, 2, \ldots, m; \; m = 1, 2, \ldots, n\}.$$

In other words if $\mathcal{R} \subseteq \mathcal{P}(\Omega)$ is a finite partition of Ω then $a(\mathcal{R})$ consists of the empty set and the collection of all finite unions of elements in $\mathcal{R}$. Recall that $\mathcal{R}$ is a partition of Ω if and only if $\mathcal{R} \subseteq \mathcal{P}(\Omega)$; $\cup_{R \in \mathcal{R}} R = \Omega$; $\emptyset \notin \mathcal{R}$; and $V, U \in \mathcal{R}$, $V \neq U$, implies that $V \cap U = \emptyset$.

7) Let $\Omega = \mathbb{R}$. Write out explicitly $a(\{(1, 2], [2, 3)\})$!

8) Let $\mathcal{A}$ be an algebra of subsets of $\mathbb{R}$ and $X : \Omega \to \mathbb{R}$. Then $\{X^{-1}(A) : A \in \mathcal{A}\}$ is an algebra, where, for $A \in \mathcal{A}$, $X^{-1}(A) = \{X \in A\} = \{\omega \in \Omega : X(\omega) \in A\}$.