SHEET 5

& checked notes for Rob. Henry. $\boxed{3}$ ✓
(see exercises — many omitted)

$\boxed{1}$ ✓ $\boxed{2}$ $\{\emptyset, B_1, B_2, B_3, B_1 \cup B_2, B_2 \cup B_3, B_1 \cup B_3, \Omega\}$ $\boxed{4}$ $\Omega = \{0,1\} \times \{0,1\}$ Heads Tails $\{$coin 1, coin 2

$\mathcal{F} = 2^{\Omega}$, $P(\{(i,j)\}) = \frac{1}{4}$ (discrete space), $E_H = \{(0,0),(0,1),(1,0)\}$,

$E_T = \{(1,1),(0,1),(1,0)\}$ $P(E_H \cap E_T) = P(\underbrace{\{(0,1),(1,0)\}}_{E_H \cap E_T}) = \frac{1}{2} \neq \frac{3}{4}\cdot\frac{3}{4} = P(E_H)P(E_T)$

classical playing cards

pack / deck → 52 cards

suits: heart ♡, club ♣, spade ♠, diamond ◇ (13 each) (joker)

court cards/face cards: jack, queen, king, ace, 2–9.

$\boxed{5}$ $\binom{52}{5}$; $\binom{13}{4} \times \binom{39}{1}$ (order not important)

$\boxed{6}$ $\binom{13}{3} \times \binom{13}{2} \Big/ \binom{52}{5}$ (all hands equally likely)

$\boxed{8}$ $\overbrace{x \mid x \mid x \ldots \mid x\ x\ x}^{n+r \text{ crosses}}$

m − 1 borders

elem a_n gets $\mu(a_n)$ = # crosses here − 1

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$\boxed{7}$ $P_n(n+r)$ – if boxes indistinguishable
$1 \le m_1 \le m_2 \le \ldots \le m_n$ (reduce by 1).

crosses is $n+r$

$x \ x \mid x \ldots x$
$n-1$ borders

$\binom{n+r-1}{n-1}$ – if boxes distinguishable
↳ COMBINATIONS WITH REPETITION

$\boxed{9}$ first put at least one red between every blue one

BRB BRB

$r-(b-1)$ left red balls distribute among $b+1$ boxes
→ 0 or more each

$\binom{b+1+r-(b-1)-1}{b+1-1}$ – ways

$= \binom{r+1}{b}$ ⟷ different way of looking at it

R R ... R R

$r+1$ spaces
choose b spaces

$\boxed{10}$ each (permutation) ordering equally likely

$\binom{40}{13} \times 13! \times 39! \Big/ 52!$ now order is important.

cards actually distinguishable.

as in $\boxed{9}$: 13 blue balls (hearts)
39 red balls (all others)
distribute hearts among the rest!

12 $\dfrac{5+5}{6\times 5} = \dfrac{10}{30} = 1/3$ (vice versa)

the first one shows a 6 does, the 2nd one not

(PoS)

(PoB)

13 C : 0.7 → 0.2 faulty

B : 0.3 → 0.1

(a) $0.7 \times 0.2 = \underline{0.14}$

(b) $0.7\times0.2 + 0.3\times0.1 = \underline{0.17}$

in other words, cond. prob.

11 $S \ni a$ choose.

$\{a\}$ is in the partition ⇒ one-to-one & onto partitioning $S\setminus\{a\}$ into $k-1$ sets → # is $S(n-1,k-1)$

$\bigcup \binom{p}{1}\times k\binom{p}{}$ partit of $S\setminus\{a\}$ into k ...

it is not! a one-to-one & onto "$S(n-1,k)\times k$" partition $S\setminus\{a\}$ into k — where to add a (MR)

SHEET 6

2 $\{\{(H,H)\}, \{(T,T)\}, \{(H,T),(T,H)\}, \{(H,H),(T,T)\},$ $\{(H,T),(T,H),(H,H)\}, \{(H,T),(T,H),(T,T)\}, \Omega, \emptyset\}$.

X measurable: $(\Omega,\mathcal{F}) \to (\mathbb{R},\mathcal{B}(\mathbb{R}))$ iff $X^{-1}(A)\in\mathcal{F}$ $\forall A\in\mathcal{B}(\mathbb{R})$

generated by partition

if Ω answers and the more values $x_1, x_2, \ldots$ $X^{-1}(\{x_i\})$ ...

2^3

X discrete

$EX = 1\cdot\frac{1}{4} + (-1)\frac{1}{4} + 0\cdot\frac{1}{2} = 0$.

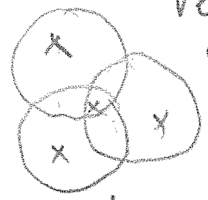
3 ✓ **4** ✓ $E[X]$, $E[X(X-1)]$ ...

5 ✓✓ $EH = E\, 1_1^H + \cdots + 1_n^H = n(p\cdot 1 + (1-p)\cdot 0) = np$

$EH^2 = E\sum_{i,j} 1_i^H 1_j^H = pn + n(n-1)p^2$

$\operatorname{Var} H = np(1-p)$

1 Venn diagram



1/4 each

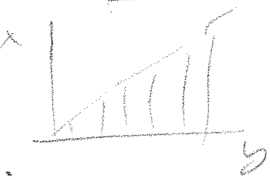
6 ✓ **7** $c\displaystyle\int_0^\infty \frac{dx}{(1+x)^3} = 1 \iff c\cdot\frac{1}{2} = 1 \qquad c = 2$

$\int f(x)dx = 1$

$m(f) = 2\displaystyle\int_0^\infty \frac{x\,dx}{(1+x)^3} = \int_0^\infty \frac{dx}{(1+x)^2} = 1$

$\left(\dfrac{x}{(1+x)^2}\right)' = \dfrac{1}{(1+x)^2} - \dfrac{2x}{(1+x)^3}$

$F(x) = 2\displaystyle\int_0^x \frac{du}{(1+u)^3} = \begin{cases} 1-(1+x)^{-2} & x\ge 0 \\ 0 & x<0 \end{cases}$.



8 $EY = \displaystyle\int_0^\infty y\, f_Y(y)\,dy = \int_0^\infty\int_0^y dx\, f_Y(y)\,dy$

PDF

$= \displaystyle\int_0^\infty dx \int_x^\infty dy\, f_Y(y) = \int_0^\infty dx\, P(Y>x)$

9 $Y = f\circ X \qquad f_Y = f_X\circ F^{-1}\,(F^{-1})' = \dfrac{1}{F'\circ F^{-1}}$ (conditions)

$\dfrac{X-\mu}{\sigma}\sim N(0,\sigma^2) \qquad Y = e^{X-\mu}e^{\mu}$

$f_Y(y) = \dfrac{1}{\sqrt{2\pi}\,\sigma y}\, e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}$

$EY = Ee^X$
$= e^{\mu}Ee^{X-\mu}$ $N(0,\sigma^2)$
$= e^{\mu+\sigma^2/2}$ (charact. func.)

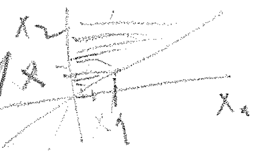
completing the squares

10 ✓

SHEET 7

[1] $P\left(\frac{X_1}{X_2} < x\right) = P(X_1 < x X_2) = \int_0^\infty dx_1 \int_{(1/x)x_1}^\infty dx_2\, e^{-x_1} e^{-x_2}$

$x > 0$



$x \geq 0$

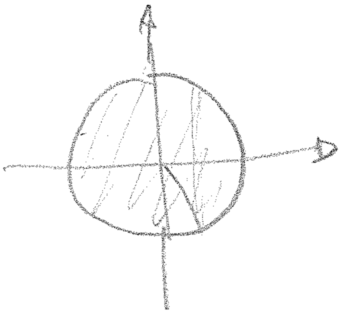
$= \int_0^\infty dx_1 e^{-x_1} e^{-x_1/x} = \frac{1}{1+\frac{1}{x}}$

$0, x < 0.$

$\therefore\ f_{\frac{X_1}{X_2}}(x) = \frac{1}{x^2(1+\frac{1}{x})^2} = \frac{1}{(1+x)^2}, x > 0$

$0, x < 0$

[2]



$c = \frac{1}{\pi}$ X & Y are not independent

(X close to ± 1 implies Y close to 0)

Formally $f_X(x) = \frac{2\sqrt{1-x^2}}{\pi}$ $f_Y(y) = \frac{2\sqrt{1-y^2}}{\pi}$

$\times \neq f_{(X,Y)}(x,y)$

etc. etc.

$D = \sqrt{X^2+Y^2}$ $P(D \leq r) = \frac{\pi r^2}{\pi} = r^2$ $0 \leq r \leq 1$

$\therefore f_D(d) = 2d$

[3] $N \sim Poi(\beta)$

$P(N = n \mid N \text{ is even})$ n odd $\to 0$

n even $\frac{\beta^n e^{-\beta}}{n!}$ $\sum_{k=0}^\infty \frac{\beta^{2k} e^{-\beta}}{(2k)!} = e^{-\beta} ch\beta$ $\frac{e^\beta + e^{-\beta}}{2}$

$\frac{e^\beta - e^{-\beta}}{2}$

$\mathbb{E}[N \mid N \text{ is even}] = \sum_{n \text{ even}} (\text{n}) \frac{\beta^n}{n!\, ch\beta} = \frac{\beta}{ch\beta} sh\beta = \beta\, th\beta < \beta$

[4] $C_0 = (\beta ch\beta + sh\beta)e^{-\beta}$

[5] p.g.f. $\gamma_X(z) = \mathbb{E}[z^X] = \sum_{k=0}^\infty z^k P(X = k)$ $R_X \subset \mathbb{N} \cup \{0\}$ a.s.

$X \sim Bin(n, p)$ $\sum_{k=0}^n z^k \binom{n}{k} p^k (1-p)^{n-k} = (pz + (1-p))^n = (1 + p(z-1))^n$

$\gamma_{X+Y} = \gamma_X \gamma_Y$ if X & Y indep. $\therefore X + Y \sim Bin(n+m, p)$

heads in n # heads in m

6 $e^{\theta(t-1)} = e^{-\theta} \sum \frac{\theta^k t^k}{k!}$ $\therefore X \sim Poi(\theta)$

7 m.g.f. $\phi_X(t) = E[e^{tX}]$

$$\theta \int_0^\infty e^{tx} e^{-\theta x} dx = \frac{\theta}{\theta - t} \quad t < \theta$$

8 $\int_0^\infty e^{tx} \left[-\lambda e^{-\lambda x} + \lambda^2 x e^{-\lambda x} \right] dx = \frac{\lambda^2}{(\lambda-t)^2} \int_0^\infty x e^{-(\lambda - t)x} dx \, (\lambda - t)^2$

$t < \lambda$

$(xe^{-hx})' = e^{-hx} - hxe^{-hx}$ $\quad = \underline{\left(\frac{\lambda}{\lambda - t}\right)^2}$

$$\frac{d}{dt} \left(\frac{1}{1 - \frac{t}{\lambda}}\right)^2 \Big|_{t=0}$$

$$= \frac{2}{\lambda}$$

sheet 8

$EX^2 = \mathrm{Var}X$ if X zero

$P(X^2 \geq c^2) \leq \frac{EX^2}{c^2}$

1 $P(X \geq c) \leq \frac{EX}{c}$ $\quad X \geq 0$

$$P(X \geq a) = P(X + b \geq a + b) \leq P((X+b)^2 \geq (a+b)^2) \leq \frac{E(X+b)^2}{(a+b)^2} = \frac{\sigma^2 + b^2}{(a+b)^2}$$

$\frac{\sigma^2}{a^2}$

naively: $\frac{2b(a+b)^2 - 2(\sigma^2+b^2)(a+b)}{(a+b)^4} = 0$

$b(a+b) = \sigma^2 + b^2$

$ba = \sigma^2$

$b = \frac{\sigma^2}{a}$

$$\frac{\sigma^2 + b^2}{(a+b)^2} = \frac{b}{a+b} = \frac{\sigma^2}{\sigma^2 + a^2}$$

$$\therefore P(X \geq a) \leq \frac{\sigma^2}{\sigma^2 + a^2}$$

$\frac{\sigma^2}{\sigma^2 + a^2} \leq \frac{\sigma^2 + b^2}{(a+b)^2}$ $\quad \sigma^2(a^2 + 2ab + b^2) \leq (\sigma^2 + b^2)(\sigma^2 + a^2)$

$2ab\sigma^2 \leq \sigma^4 + a^2b^2$

$0 \leq a^2b^2 + \sigma^4 - 2ab\sigma^2$

$= (ab - \sigma^2)^2$

$D = \sigma^4 a^2 - 4a^2\sigma^4 = 0$

② $P(W \geq 12) = P(\cancel{\ldots}) \leq \frac{10}{12} = 5/6$

$P(W-10 \geq 2) \leq 4/(4+4) = 1/2$ (much better)

③ $\mathbb{E}[e^{tX}] \geq e^{ta} P(X \geq a)$.

$X \sim Poi(\lambda) \Rightarrow \sum_{k=0}^{\infty} e^{tk} \frac{e^{-\lambda}\lambda^k}{k!} = e^{\lambda(e^t-1)}$

$P(X \geq a) \leq e^{-ta} e^{\lambda(e^t-1)} = e^{\lambda(e^t-1)-ta}$

$\lambda \geq a$ $\quad \underbrace{\lambda e^t - a}_{\geq 0} = 0 \quad t^* = \ln(a/\lambda) \vee 0$.

so minimum attained at $t=0$.

if $k > \lambda$

$P(X \geq k) \leq \frac{\lambda^k}{k^k} e^{k-\lambda}$

④ (i) $X \approx N(5000,$

$\frac{X-5000}{\sqrt{10{,}000\left(\frac{1}{2}\right)^2 \frac{1}{2} \times 2}} = \frac{X-5000}{25} \sim N(0,1)$

(a) $P\left(\frac{X-5000}{25} > 40\right) \approx 1-\Phi(40)$

(b) $P\left(\frac{X-5000}{25} \leq 4\right) \approx \Phi(4)$

(c) $\cancel{1-}\,2\Phi(-2)$

(d) $1-\Phi(-4)$

⑤ $P(Poi(n) \leq n) = P(\overbrace{Poi(n)}^{\sum^n Poi(1)} - n \leq 0) \xrightarrow{n\to\infty} \Phi(0) = \frac{1}{2}$.