Problem 1 (Pólya urn model). Let an urn contain black and white balls, $B_0 \geq 1$ (resp. $W_0 \geq 1$) being the initial number of black (resp. white) balls (non-random). Draw from the urn a ball, uniformly at random, then return the ball so drawn, together with $a \in \mathbb{N}$ additional balls of the same colour. Repeat. Let $B_n$ (resp. $W_n$) denote the number of black (resp. white) balls in the urn after the $n$-th draw-and-replacement. Show that the sequence $\rho := (B_n/(B_n + W_n), n \geq 0)$ converges a.s. to a random variable $\rho_\infty$ and find the distribution of $\rho_\infty$!

Solution. Let for $n \in \mathbb{N}$, $X_n$ be an indicator random variable, 1 if a black ball was drawn on the $n$-th draw, 0 otherwise. Next allow $F_n := \sigma(X_i : 1 \leq i \leq n)$ to be the natural filtration of this sequence, $n \in \mathbb{N}_0$ ($F_0$ being trivial). We denote this filtration $F$. Note that conditionally on $F_n$, $X_{n+1}$ has distribution Ber($B_n/(B_n + W_n)$).

We show $\rho$ is an $F$-martingale. Indeed, certainly $\rho$ is adapted and integrable, and

$$E[\rho_{n+1}|F_n] = E\left[\frac{B_{n+1}}{B_{n+1} + W_{n+1}}|F_n\right]$$

$$= E\left[\frac{B_{n+1}}{B_0 + W_0 + (n + 1)a}|F_n\right]$$

$$= \frac{1}{B_0 + W_0 + (n + 1)a} E[B_n + aX_{n+1}|F_n]$$

$$= \frac{1}{B_0 + W_0 + (n + 1)a} \left(\frac{B_n + aB_n}{B_n + W_n + a}\right)$$

$$= \frac{B_0 + W_0 + (n + 1)a}{B_n + W_n} \frac{B_n}{B_n + W_n + a}$$

as required. Being a nonnegative supermartingale, $\rho \to \rho_\infty$, a.s. (and hence in distribution). It remains to find the law of $\rho_\infty$. In fact we know $\rho_\infty \in [0, 1]$ a.s. Observe that, while the sequence $X$ is not an independency, nevertheless the probability of drawing $k$ black balls in the first $n$ draws, $0 \leq k \leq n$, does not depend on the precise sequence of the colours so drawn (but only on their cumulative number). Hence:

$$P(k \text{ black balls in first } n \text{ draws}) = \binom{n}{k} \frac{B_0(B_0 + a) \cdots (B_0 + (k - 1)a)W_0(W_0 + a) \cdots (W_0 + (n - k - 1)a)}{(B_0 + W_0)(B_0 + W_0 + a) \cdots (B_0 + W_0 + (n - 1)a)},$$

i.e.

$$P(k \text{ black balls in first } n \text{ draws}) = \binom{n}{k} \frac{\beta(B_0/a + k, W_0/a + n - k)}{\beta(B_0/a, W_0/a)}.$$
It follows that

\[ \phi_{\rho_n}(t) = \mathbb{E}[e^{it\rho_n}] \]

\[ = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} e^{it\frac{B_0}{a} + k\frac{p}{n}} \int_0^1 p^k (1-p)^{n-k} \frac{p^{B_0/a-1}(1-p)^{W_0/a-1}}{\beta(B_0/a, W_0/a)} dp \]

\[ = e^{it\frac{B_0}{a} + \frac{p}{n}} \int_0^1 \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (p e^{it\frac{B_0}{a} + \frac{p}{n}})^k (1-p)^{n-k} \frac{p^{B_0/a-1}(1-p)^{W_0/a-1}}{\beta(B_0/a, W_0/a)} dp \]

\[ \rightarrow \int_0^1 e^{itp} \frac{p^{B_0/a-1}(1-p)^{W_0/a-1}}{\beta(B_0/a, W_0/a)} dp \]

by the dominated convergence theorem. Lévy’s continuity theorem then implies that the law of \( \rho_\infty \) is absolutely continuous w.r.t. Lebesgue and with Radon-Nikodým density:

\[ (p \mapsto \frac{p^{B_0/a-1}(1-p)^{W_0/a-1}}{\beta(B_0/a, W_0/a)} \mathbb{1}_{(0,1)}(p)) \cdot \]

Put in other words \( \rho_\infty \sim \text{Beta}(B_0/a, W_0/a) \). \( \blacksquare \)