MATHEMATICS PREPARATION WORKBOOK

FP16 Mathematics & Statistics
FP17 Economics
FP18 Computer Science

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Contents

Expanding brackets and simplifying expressions

Key points

- When you expand one set of brackets you must multiply everything inside the bracket by what is outside.
- When you expand two linear expressions, each with two terms of the form $ax + b$, where $a \neq 0$ and $b \neq 0$, you create four terms. Two of these can usually be simplified by collecting like terms.

Examples

Example 1 Expand $4(3x - 2)$

$4(3x - 2) = 12x - 8$	Multiply everything inside the bracket by the 4 outside the bracket

Example 2 Expand and simplify $3(x + 5) - 4(2x + 3)$

$3(x + 5) - 4(2x + 3)$ $= 3x + 15 - 8x - 12$ $= 3 - 5x$	**1** Expand each set of brackets separately by multiplying $(x + 5)$ by 3 and $(2x + 3)$ by -4 **2** Simplify by collecting like terms: $3x - 8x = -5x$ and $15 - 12 = 3$

Example 3 Expand and simplify $(x + 3)(x + 2)$

$(x + 3)(x + 2)$ $= x(x + 2) + 3(x + 2)$ $= x^2 + 2x + 3x + 6$ $= x^2 + 5x + 6$	**1** Expand the brackets by multiplying $(x + 2)$ by x and $(x + 2)$ by 3 **2** Simplify by collecting like terms: $2x + 3x = 5x$

Example 4 Expand and simplify $(x - 5)(2x + 3)$

$(x - 5)(2x + 3)$ $= x(2x + 3) - 5(2x + 3)$ $= 2x^2 + 3x - 10x - 15$ $= 2x^2 - 7x - 15$	**1** Expand the brackets by multiplying $(2x + 3)$ by x and $(2x + 3)$ by -5 **2** Simplify by collecting like terms: $3x - 10x = -7x$

Practice

Watch out!

When multiplying (or dividing) positive and negative numbers, if the signs are the same the answer is ‘+’; if the signs are different the answer is ‘–’.

1 Expand.

a $3(2x - 1)$ **b** $-2(5pq + 4q^2)$

2 Expand and simplify.

a $7(3x + 5) + 6(2x - 8)$ **b** $8(5p - 2) - 3(4p + 9)$

3 Expand.

a $3x(4x + 8)$ **b** $4k(5k^2 - 12)$

4 Expand and simplify.

a $3(y^2 - 8) - 4(y^2 - 5)$ **b** $2x(x + 5) + 3x(x - 7)$

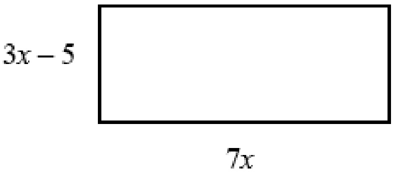
5 Expand and simplify.

a $13 - 2(m + 7)$ **b** $5p(p^2 + 6p) - 9p(2p - 3)$

6 The diagram shows a rectangle.

Write down an expression, in terms of x, for the area of the rectangle.

Show that the area of the rectangle can be written as $21x^2 - 35x$



7 Expand and simplify.

a $(x + 4)(x + 5)$ **b** $(x + 7)(x + 3)$

c $(x + 7)(x - 2)$ **d** $(x + 5)(x - 5)$

e $(5x - 3)(2x - 5)$ **f** $(3x - 2)(7 + 4x)$

g $(2x - 7)^2$ **h** $(4x - 3y)^2$

8 Expand and simplify.

a $\left(x + \frac{1}{x}\right)\left(x - \frac{2}{x}\right)$ **b** $\left(x + \frac{1}{x}\right)^2$

Surds and rationalising the denominator

Key points

- A surd is the square root of a number that is not a square number, for example $\sqrt{2}, \sqrt{3}, \sqrt{5},$ etc.
- Surds can be used to give the exact value for an answer.
- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
- To rationalise the denominator means to remove the surd from the denominator of a fraction.
- To rationalise $\frac{a}{\sqrt{b}}$ you multiply the numerator and denominator by the surd $\sqrt{b}$
- To rationalise $\frac{a}{b+\sqrt{c}}$ you multiply the numerator and denominator by $b-\sqrt{c}$

Examples

Example 1 Simplify $\sqrt{50}$

$\sqrt{50} = \sqrt{25 \times 2}$	**1** Choose two numbers that are factors of 50. One of the factors must be a square number
$= \sqrt{25} \times \sqrt{2}$	**2** Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
$= 5 \times \sqrt{2}$	**3** Use $\sqrt{25} = 5$
$= 5\sqrt{2}$	

Example 2 Simplify $\sqrt{147} - 2\sqrt{12}$

$\sqrt{147} - 2\sqrt{12}$ $= \sqrt{49 \times 3} - 2\sqrt{4 \times 3}$	**1** Simplify $\sqrt{147}$ and $2\sqrt{12}$. Choose two numbers that are factors of 147 and two numbers that are factors of 12. One of each pair of factors must be a square number
$= \sqrt{49} \times \sqrt{3} - 2\sqrt{4} \times \sqrt{3}$	**2** Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
$= 7 \times \sqrt{3} - 2 \times 2 \times \sqrt{3}$	**3** Use $\sqrt{49} = 7$ and $\sqrt{4} = 2$
$= 7\sqrt{3} - 4\sqrt{3}$ $= 3\sqrt{3}$	**4** Collect like terms

Example 3 Simplify $(\sqrt{7}+\sqrt{2})(\sqrt{7}-\sqrt{2})$

$(\sqrt{7}+\sqrt{2})(\sqrt{7}-\sqrt{2})$ $= \sqrt{49}-\sqrt{7}\sqrt{2}+\sqrt{2}\sqrt{7}-\sqrt{4}$	**1** Expand the brackets. A common mistake here is to write $(\sqrt{7})^2 = 49$
$= 7 - 2$ $= 5$	**2** Collect like terms: $-\sqrt{7}\sqrt{2}+\sqrt{2}\sqrt{7}$ $= -\sqrt{7}\sqrt{2}+\sqrt{7}\sqrt{2} = 0$

Example 4 Rationalise $\frac{1}{\sqrt{3}}$

$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$	**1** Multiply the numerator and denominator by $\sqrt{3}$
$= \frac{1\times\sqrt{3}}{\sqrt{9}}$	**2** Use $\sqrt{9} = 3$
$= \frac{\sqrt{3}}{3}$	

Example 5 Rationalise and simplify $\frac{\sqrt{2}}{\sqrt{12}}$

$\frac{\sqrt{2}}{\sqrt{12}} = \frac{\sqrt{2}}{\sqrt{12}}\times\frac{\sqrt{12}}{\sqrt{12}}$	**1** Multiply the numerator and denominator by $\sqrt{12}$
$= \frac{\sqrt{2}\times\sqrt{4\times3}}{12}$	**2** Simplify $\sqrt{12}$ in the numerator. Choose two numbers that are factors of 12. One of the factors must be a square number
$= \frac{2\sqrt{2}\sqrt{3}}{12}$	**3** Use the rule $\sqrt{ab} = \sqrt{a}\times\sqrt{b}$ **4** Use $\sqrt{4} = 2$
$= \frac{\sqrt{2}\sqrt{3}}{6}$	**5** Simplify the fraction: $\frac{2}{12}$ simplifies to $\frac{1}{6}$

Example 6 Rationalise and simplify $\dfrac{3}{2+\sqrt{5}}$

$\dfrac{3}{2+\sqrt{5}} = \dfrac{3}{2+\sqrt{5}} \times \dfrac{2-\sqrt{5}}{2-\sqrt{5}}$	**1** Multiply the numerator and denominator by $2-\sqrt{5}$
$= \dfrac{3\left(2-\sqrt{5}\right)}{\left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right)}$	**2** Expand the brackets
$= \dfrac{6-3\sqrt{5}}{4+2\sqrt{5}-2\sqrt{5}-5}$	**3** Simplify the fraction
$= \dfrac{6-3\sqrt{5}}{-1}$	**4** Divide the numerator by −1 Remember to change the sign of all terms when dividing by −1
$= 3\sqrt{5}-6$	

Practice

1 Simplify.

a $\sqrt{45}$ **b** $\sqrt{125}$

c $\sqrt{48}$ **d** $\sqrt{175}$

Hint

One of the two numbers you choose at the start must be a square number.

Watch out!

Check you have chosen the highest square number at the start.

2 Simplify.

a $\sqrt{72}+\sqrt{162}$ **b** $\sqrt{45}-2\sqrt{5}$

c $\sqrt{50}-\sqrt{8}$ **d** $\sqrt{75}-\sqrt{48}$

3 Expand and simplify.

a $(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$ **b** $(3+\sqrt{3})(5-\sqrt{12})$

c $(4-\sqrt{5})(\sqrt{45}+2)$ **d** $(5+\sqrt{2})(6-\sqrt{8})$

4 Rationalise and simplify, if possible.

a $\dfrac{2}{\sqrt{7}}$ **b** $\dfrac{2}{\sqrt{8}}$

c $\dfrac{\sqrt{8}}{\sqrt{24}}$ **d** $\dfrac{\sqrt{5}}{\sqrt{45}}$

5 Rationalise and simplify.

a $\dfrac{1}{3-\sqrt{5}}$ **b** $\dfrac{2}{4+\sqrt{3}}$

6 Expand and simplify $\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)$

7 Rationalise and simplify, if possible.

a $\dfrac{1}{\sqrt{9}-\sqrt{8}}$ **b** $\dfrac{1}{\sqrt{x}-\sqrt{y}}$

Rules of indices

Key points

- $a^m \times a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $a^0 = 1$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$ i.e. the *n*th root of *a*
- $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$
- $a^{-m} = \frac{1}{a^m}$
- The square root of a number produces two solutions, e.g. $\sqrt{16} = \pm 4$.

Examples

Example 1 Evaluate 10^0

$10^0 = 1$	Any value raised to the power of zero is equal to 1

Example 2 Evaluate $9^{\frac{1}{2}}$

$9^{\frac{1}{2}} = \sqrt{9}$ $= 3$	Use the rule $a^{\frac{1}{n}} = \sqrt[n]{a}$

Example 3 Evaluate $27^{\frac{2}{3}}$

$27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2$ $= 3^2$ $= 9$	**1** Use the rule $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ **2** Use $\sqrt[3]{27} = 3$

Example 4 Evaluate 4^{-2}

$4^{-2} = \frac{1}{4^2}$ $= \frac{1}{16}$	**1** Use the rule $a^{-m} = \frac{1}{a^m}$ **2** Use $4^2 = 16$

Example 5 Simplify $\dfrac{6x^5}{2x^2}$

$\dfrac{6x^5}{2x^2} = 3x^3$	$6 \div 2 = 3$ and use the rule $\dfrac{a^m}{a^n} = a^{m-n}$ to give $\dfrac{x^5}{x^2} = x^{5-2} = x^3$

Example 6 Simplify $\dfrac{x^3 \times x^5}{x^4}$

$\dfrac{x^3 \times x^5}{x^4} = \dfrac{x^{3+5}}{x^4} = \dfrac{x^8}{x^4}$ $= x^{8-4} = x^4$	**1** Use the rule $a^m \times a^n = a^{m+n}$ **2** Use the rule $\dfrac{a^m}{a^n} = a^{m-n}$

Example 7 Write $\dfrac{1}{3x}$ as a single power of x

$\dfrac{1}{3x} = \dfrac{1}{3}x^{-1}$	Use the rule $\dfrac{1}{a^m} = a^{-m}$, note that the fraction $\dfrac{1}{3}$ remains unchanged

Example 8 Write $\dfrac{4}{\sqrt{x}}$ as a single power of x

$\dfrac{4}{\sqrt{x}} = \dfrac{4}{x^{\frac{1}{2}}}$ $= 4x^{-\frac{1}{2}}$	**1** Use the rule $a^{\frac{1}{n}} = \sqrt[n]{a}$ **2** Use the rule $\dfrac{1}{a^m} = a^{-m}$

Practice

1 Evaluate.

a 14^0 **b** 3^0

2 Evaluate.

a $49^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$

3 Evaluate.

a $25^{\frac{3}{2}}$ **b** $8^{\frac{5}{3}}$

4 Evaluate.

a 5^{-2} **b** 4^{-3}

5 Simplify.

a $\dfrac{3x^2 \times x^3}{2x^2}$ **b** $\dfrac{10x^5}{2x^2 \times x}$

c $\dfrac{\left(2x^2\right)^3}{4x^0}$ **d** $\dfrac{x^{\frac{1}{2}} \times x^{\frac{3}{2}}}{x^{-2} \times x^3}$

Watch out!

Remember that any value raised to the power of zero is 1. This is the rule $a^0 = 1$.

6 Evaluate.

a $4^{-\frac{1}{2}}$ **b** $27^{-\frac{2}{3}}$ **c** $9^{-\frac{1}{2}} \times 2^3$

7 Write the following as a single power of x.

a $\dfrac{1}{x}$ **b** $\dfrac{1}{x^7}$ **c** $\sqrt[4]{x}$

8 Write the following without negative or fractional powers.

a $x^{\frac{2}{5}}$ **b** $x^{-\frac{1}{2}}$ **c** $x^{-\frac{3}{4}}$

9 Write the following in the form ax^n.

a $5\sqrt{x}$ **b** $\dfrac{2}{x^3}$ **c** $\dfrac{1}{3x^4}$

10 Write as sums of powers of x.

a $\dfrac{x^5 + 1}{x^2}$ **b** $x^2\left(x + \dfrac{1}{x}\right)$ **c** $x^{-4}\left(x^2 + \dfrac{1}{x^3}\right)$

Factorising expressions

Key points

- Factorising an expression is the opposite of expanding the brackets.
- A quadratic expression is in the form $ax^2 + bx + c$, where $a \neq 0$.
- To factorise a quadratic equation find two numbers whose sum is b and whose product is ac.
- An expression in the form $x^2 - y^2$ is called the difference of two squares. It factorises to $(x - y)(x + y)$.

Examples

Example 1 Factorise $15x^2y^3 + 9x^4y$

$15x^2y^3 + 9x^4y = 3x^2y(5y^2 + 3x^2)$	The highest common factor is $3x^2y$. So take $3x^2y$ outside the brackets and then divide each term by $3x^2y$ to find the terms in the brackets

Example 2 Factorise $4x^2 - 25y^2$

$4x^2 - 25y^2 = (2x + 5y)(2x - 5y)$	This is the difference of two squares as the two terms can be written as $(2x)^2$ and $(5y)^2$

Example 3 Factorise $x^2 + 3x - 10$

$b = 3$, $ac = -10$	**1** Work out the two factors of $ac = -10$ which add to give $b = 3$ (5 and −2)
So $x^2 + 3x - 10 = x^2 + 5x - 2x - 10$	**2** Rewrite the b term ($3x$) using these two factors
$= x(x + 5) - 2(x + 5)$	**3** Factorise the first two terms and the last two terms
$= (x + 5)(x - 2)$	**4** $(x + 5)$ is a factor of both terms

Example 4 Factorise $6x^2 - 11x - 10$

$b = -11$, $ac = -60$	**1** Work out the two factors of $ac = -60$ which add to give $b = -11$ (−15 and 4)
So $6x^2 - 11x - 10 = 6x^2 - 15x + 4x - 10$	**2** Rewrite the b term ($-11x$) using these two factors
$= 3x(2x - 5) + 2(2x - 5)$	**3** Factorise the first two terms and the last two terms
$= (2x - 5)(3x + 2)$	**4** $(2x - 5)$ is a factor of both terms

Example 5 Simplify $\dfrac{x^2 - 4x - 21}{2x^2 + 9x + 9}$

$\dfrac{x^2 - 4x - 21}{2x^2 + 9x + 9}$	**1** Factorise the numerator and the denominator
For the numerator: $b = -4$, $ac = -21$	**2** Work out the two factors of $ac = -21$ which add to give $b = -4$ (−7 and 3)
So $x^2 - 4x - 21 = x^2 - 7x + 3x - 21$	**3** Rewrite the b term ($-4x$) using these two factors
$= x(x - 7) + 3(x - 7)$	**4** Factorise the first two terms and the last two terms
$= (x - 7)(x + 3)$	**5** $(x - 7)$ is a factor of both terms
For the denominator: $b = 9$, $ac = 18$	**6** Work out the two factors of $ac = 18$ which add to give $b = 9$ (6 and 3)
So $2x^2 + 9x + 9 = 2x^2 + 6x + 3x + 9$	**7** Rewrite the b term ($9x$) using these two factors
$= 2x(x + 3) + 3(x + 3)$	**8** Factorise the first two terms and the last two terms
$= (x + 3)(2x + 3)$	**9** $(x + 3)$ is a factor of both terms
So $\dfrac{x^2 - 4x - 21}{2x^2 + 9x + 9} = \dfrac{(x - 7)(x + 3)}{(x + 3)(2x + 3)}$ $= \dfrac{x - 7}{2x + 3}$	**10** $(x + 3)$ is a factor of both the numerator and denominator so cancels out as a value divided by itself is 1

Practice

Hint

Take the highest common factor outside the bracket.

1 Factorise.

a $6x^4y^3 - 10x^3y^4$ **b** $21a^3b^5 + 35a^5b^2$

c $25x^2y^2 - 10x^3y^2 + 15x^2y^3$

2 Factorise

a $x^2 + 7x + 12$ **b** $x^2 + 5x - 14$

c $x^2 - 11x + 30$ **d** $x^2 - 5x - 24$

3 Factorise

a $36x^2 - 49y^2$ **b** $4x^2 - 81y^2$

c $18a^2 - 200b^2c^2$

4 Factorise

a $2x^2 + x -3$ **b** $6x^2 + 17x + 5$

c $2x^2 + 7x + 3$ **d** $9x^2 - 15x + 4$

5 Simplify the algebraic fractions.

a $\frac{2x^2+4x}{x^2-x}$ **b** $\frac{x^2+3x}{x^2+2x-3}$

c $\frac{x^2-2x-8}{x^2-4x}$ **d** $\frac{x^2-5x}{x^2-25}$

6 Simplify

a $\frac{9x^2-16}{3x^2+17x-28}$ **b** $\frac{2x^2-7x-15}{3x^2-17x+10}$

c $\frac{4-25x^2}{10x^2-11x-6}$ **d** $\frac{6x^2-x-1}{2x^2+7x-4}$

7 Simplify $\frac{(x+2)^2+3(x+2)^2}{x^2-4}$

Completing the square

Key points

- Completing the square for a quadratic rearranges $ax^2 + bx + c$ into the form $p(x + q)^2 + r$
- If $a \neq 1$, then factorise using a as a common factor.

Examples

Example 1 Complete the square for the quadratic expression $x^2 + 6x - 2$

$x^2 + 6x - 2$	**1** Write $x^2 + bx + c$ in the form
$= (x + 3)^2 - 9 - 2$	$\left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$
$= (x + 3)^2 - 11$	**2** Simplify

Example 2 Write $2x^2 - 5x + 1$ in the form $p(x + q)^2 + r$

$2x^2 - 5x + 1$	**1** Before completing the square write $ax^2 + bx + c$ in the form $a\left(x^2 + \frac{b}{a}x\right) + c$
$= 2\left(x^2 - \frac{5}{2}x\right) + 1$	**2** Now complete the square by writing $x^2 - \frac{5}{2}x$ in the form
$= 2\left[\left(x - \frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2\right] + 1$	$\left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$
$= 2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8} + 1$	**3** Expand the square brackets – don't forget to multiply $\left(\frac{5}{4}\right)^2$ by the factor of 2
$= 2\left(x - \frac{5}{4}\right)^2 - \frac{17}{8}$	**4** Simplify

Practice

1 Write the following quadratic expressions in the form $(x + p)^2 + q$

a $x^2 + 4x + 3$ **b** $x^2 - 10x - 3$

c $x^2 - 8x$ **d** $x^2 + 6x$

2 Write the following quadratic expressions in the form $p(x + q)^2 + r$

a $2x^2 - 8x - 16$ **b** $4x^2 - 8x - 16$

c $3x^2 + 12x - 9$ **d** $2x^2 + 6x - 8$

3 Complete the square.

a $2x^2 + 3x + 6$ **b** $3x^2 - 2x$

c $5x^2 + 3x$ **d** $3x^2 + 5x + 3$

4 Write $(25x^2 + 30x + 12)$ in the form $(ax + b)^2 + c$.

Solving quadratic equations by factorisation

Key points

- A quadratic equation is an equation in the form $ax^2 + bx + c = 0$ where $a \neq 0$.
- To factorise a quadratic equation find two numbers whose sum is b and whose products is ac.
- When the product of two numbers is 0, then at least one of the numbers must be 0.
- If a quadratic can be solved it will have two solutions (these may be equal).

Examples

Example 1 Solve $5x^2 = 15x$

$5x^2 = 15x$ $5x^2 - 15x = 0$	**1** Rearrange the equation so that all of the terms are on one side of the equation and it is equal to zero. Do not divide both sides by x as this would lose the solution $x = 0$.
$5x(x - 3) = 0$	**2** Factorise the quadratic equation. $5x$ is a common factor.
So $5x = 0$ or $(x - 3) = 0$	**3** When two values multiply to make zero, at least one of the values must be zero.
Therefore $x = 0$ or $x = 3$	**4** Solve these two equations.

Example 2 Solve $x^2 + 7x + 12 = 0$

$x^2 + 7x + 12 = 0$ $b = 7, ac = 12$	**1** Factorise the quadratic equation. Work out the two factors of $ac = 12$ which add to give you $b = 7$. (4 and 3)
$x^2 + 4x + 3x + 12 = 0$	**2** Rewrite the b term ($7x$) using these two factors.
$x(x + 4) + 3(x + 4) = 0$	**3** Factorise the first two terms and the last two terms.
$(x + 4)(x + 3) = 0$	**4** $(x + 4)$ is a factor of both terms.
So $(x + 4) = 0$ or $(x + 3) = 0$	**5** When two values multiply to make zero, at least one of the values must be zero.
Therefore $x = -4$ or $x = -3$	**6** Solve these two equations.

Example 3 Solve $9x^2 - 16 = 0$

$9x^2 - 16 = 0$ $(3x + 4)(3x - 4) = 0$ So $(3x + 4) = 0$ or $(3x - 4) = 0$ $x = -\frac{4}{3}$ or $x = \frac{4}{3}$	**1** Factorise the quadratic equation. This is the difference of two squares as the two terms are $(3x)^2$ and $(4)^2$. **2** When two values multiply to make zero, at least one of the values must be zero. **3** Solve these two equations.

Example 4 Solve $2x^2 - 5x - 12 = 0$

$b = -5$, $ac = -24$ So $2x^2 - 8x + 3x - 12 = 0$ $2x(x - 4) + 3(x - 4) = 0$ $(x - 4)(2x + 3) = 0$ So $(x - 4) = 0$ or $(2x + 3) = 0$ $x = 4$ or $x = -\frac{3}{2}$	**1** Factorise the quadratic equation. Work out the two factors of $ac = -24$ which add to give you $b = -5$. (-8 and 3) **2** Rewrite the b term ($-5x$) using these two factors. **3** Factorise the first two terms and the last two terms. **4** $(x - 4)$ is a factor of both terms. **5** When two values multiply to make zero, at least one of the values must be zero. **6** Solve these two equations.

Practice

1 Solve

a $6x^2 + 4x = 0$ **b** $28x^2 - 21x = 0$

c $x^2 + 7x + 10 = 0$ **d** $x^2 - 5x + 6 = 0$

e $x^2 - 3x - 4 = 0$ **f** $x^2 + 3x - 10 = 0$

2 Solve

a $x^2 - 3x = 10$ **b** $x^2 - 3 = 2x$

c $x(3x + 1) = x^2 + 15$ **d** $3x(x - 1) = 2(x + 1)$

Hint

Get all terms onto one side of the equation.

Solving quadratic equations by completing the square

Key points

- Completing the square lets you write a quadratic equation in the form $p(x + q)^2 + r = 0$.

Examples

Example 1 Solve $x^2 + 6x + 4 = 0$. Give your solutions in surd form.

$x^2 + 6x + 4 = 0$	**1** Write $x^2 + bx + c = 0$ in the form
$(x + 3)^2 - 9 + 4 = 0$	$\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c=0$
$(x + 3)^2 - 5 = 0$	**2** Simplify.
$(x + 3)^2 = 5$	**3** Rearrange the equation to work out x. First, add 5 to both sides.
$x + 3 = \pm\sqrt{5}$	**4** Square root both sides. Remember that the square root of a value gives two answers.
$x = \pm\sqrt{5} - 3$	**5** Subtract 3 from both sides to solve the equation.
So $x = -\sqrt{5} - 3$ or $x = \sqrt{5} - 3$	**6** Write down both solutions.

Example 2 Solve $2x^2 - 7x + 4 = 0$. Give your solutions in surd form.

$2x^2 - 7x + 4 = 0$	**1** Before completing the square write $ax^2 + bx + c$ in the form
$2\left(x^2-\frac{7}{2}x\right)+4=0$	$a\left(x^2+\frac{b}{a}x\right)+c$
$2\left[\left(x-\frac{7}{4}\right)^2-\left(\frac{7}{4}\right)^2\right]+4=0$	**2** Now complete the square by writing $x^2-\frac{7}{2}x$ in the form $\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2$
$2\left(x-\frac{7}{4}\right)^2-\frac{49}{8}+4=0$	**3** Expand the square brackets.
$2\left(x-\frac{7}{4}\right)^2-\frac{17}{8}=0$	**4** Simplify.
$2\left(x-\frac{7}{4}\right)^2=\frac{17}{8}$	**5** Rearrange the equation to work out x. First, add $\frac{17}{8}$ to both sides.

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$\left(x-\frac{7}{4}\right)^2=\frac{17}{16}$	**6** Divide both sides by 2.
$x-\frac{7}{4}=\pm\frac{\sqrt{17}}{4}$	**7** Square root both sides. Remember that the square root of a value gives two answers.
$x=\pm\frac{\sqrt{17}}{4}+\frac{7}{4}$	**8** Add $\frac{7}{4}$ to both sides.
So $x=\frac{7}{4}-\frac{\sqrt{17}}{4}$ or $x=\frac{7}{4}+\frac{\sqrt{17}}{4}$	**9** Write down both the solutions.

Practice

1 Solve by completing the square.

a $x^2 - 4x - 3 = 0$

b $x^2 - 10x + 4 = 0$

c $x^2 + 8x - 5 = 0$

d $x^2 - 2x - 6 = 0$

2 Solve by completing the square.

a $(x - 4)(x + 2) = 5$

b $2x^2 + 6x - 7 = 0$

c $x^2 - 5x + 3 = 0$

Hint

Get all terms onto one side of the equation.

Solving quadratic equations by using the formula

Key points

- Any quadratic equation of the form $ax^2 + bx + c = 0$ can be solved using the formula $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
- If $b^2 - 4ac$ is negative then the quadratic equation does not have any real solutions.
- It is useful to write down the formula before substituting the values for a, b and c.

Examples

Example 1 Solve $x^2 + 6x + 4 = 0$. Give your solutions in surd form.

$a = 1$, $b = 6$, $c = 4$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	**1** Identify a, b and c and write down the formula. Remember that $-b \pm \sqrt{b^2 - 4ac}$ is all over $2a$, not just part of it.
$x = \dfrac{-6 \pm \sqrt{6^2 - 4(1)(4)}}{2(1)}$	**2** Substitute $a = 1$, $b = 6$, $c = 4$ into the formula.
$x = \dfrac{-6 \pm \sqrt{20}}{2}$	**3** Simplify. The denominator is 2, but this is only because $a = 1$. The denominator will not always be 2.
$x = \dfrac{-6 \pm 2\sqrt{5}}{2}$	**4** Simplify $\sqrt{20}$. $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$
$x = -3 \pm \sqrt{5}$	**5** Simplify by dividing numerator and denominator by 2.
So $x = -3 - \sqrt{5}$ or $x = \sqrt{5} - 3$	**6** Write down both the solutions.

Example 2 Solve $3x^2 - 7x - 2 = 0$. Give your solutions in surd form.

$a = 3$, $b = -7$, $c = -2$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	**1** Identify a, b and c, making sure you get the signs right and write down the formula. Remember that $-b \pm \sqrt{b^2 - 4ac}$ is all over $2a$, not just part of it.
$x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-2)}}{2(3)}$	**2** Substitute $a = 3$, $b = -7$, $c = -2$ into the formula.
$x = \dfrac{7 \pm \sqrt{73}}{6}$	**3** Simplify. The denominator is 6 when $a = 3$. A common mistake is to always write a denominator of 2.
So $x = \dfrac{7 - \sqrt{73}}{6}$ or $x = \dfrac{7 + \sqrt{73}}{6}$	**4** Write down both the solutions.

Practice

1 Solve, giving your solutions in surd form.

a $3x^2 + 6x + 2 = 0$ **b** $2x^2 - 4x - 7 = 0$

2 Solve the equation $x^2 - 7x + 2 = 0$

Give your solutions in the form $\dfrac{a \pm \sqrt{b}}{c}$, where a, b and c are integers.

3 Solve $10x^2 + 3x + 3 = 5$

Give your solution in surd form.

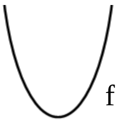
Hint

Get all terms onto one side of the equation.

Sketching quadratic graphs

Key points

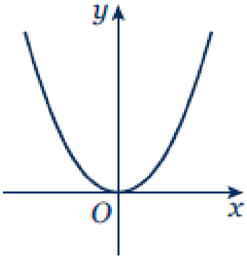
- The graph of the quadratic function $y = ax^2 + bx + c$, where $a \neq 0$, is a curve called a parabola.
- Parabolas have a line of symmetry and a shape as shown.



- To sketch the graph of a function, find the points where the graph intersects the axes.
- To find where the curve intersects the y-axis substitute $x = 0$ into the function.
- To find where the curve intersects the x-axis substitute $y = 0$ into the function.
- At the turning points of a graph the gradient of the curve is 0 and any tangents to the curve at these points are horizontal.
- To find the coordinates of the maximum or minimum point (turning points) of a quadratic curve (parabola) you can use the completed square form of the function.

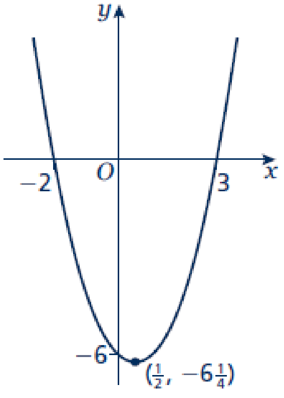
Examples

Example 1 Sketch the graph of $y = x^2$.

	The graph of $y = x^2$ is a parabola. When $x = 0$, $y = 0$. $a = 1$ which is greater than zero, so the graph has the shape: 

Example 2 Sketch the graph of $y = x^2 - x - 6$.

When $x = 0$, $y = 0^2 - 0 - 6 = -6$ So the graph intersects the y-axis at (0, −6)	**1** Find where the graph intersects the y-axis by substituting $x = 0$.
When $y = 0$, $x^2 - x - 6 = 0$	**2** Find where the graph intersects the x-axis by substituting $y = 0$.
$(x + 2)(x - 3) = 0$	**3** Solve the equation by factorising.
$x = -2$ or $x = 3$	**4** Solve $(x + 2) = 0$ and $(x - 3) = 0$.
So, the graph intersects the x-axis at (−2, 0) and (3, 0)	**5** $a = 1$ which is greater than zero, so the graph has the shape:  *(continued on next page)*
$x^2 - x - 6 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - 6$ $= \left(x - \frac{1}{2}\right)^2 - \frac{25}{4}$	**6** To find the turning point, complete the square.

When $\left(x-\frac{1}{2}\right)^2=0$, $x=\frac{1}{2}$ and $y=-\frac{25}{4}$, so the turning point is at the point $\left(\frac{1}{2},-\frac{25}{4}\right)$  	**7** The turning point is the minimum value for this expression and occurs when the term in the bracket is equal to zero.

Practice

1 Sketch the graph of $y = -x^2$.

2 Sketch each graph, labelling where the curve crosses the axes.

a $y = (x + 2)(x - 1)$ **b** $y = x(x - 3)$

3 Sketch each graph, labelling where the curve crosses the axes.

a $y = x^2 - x - 6$ **b** $y = x^2 - 5x + 4$ **c** $y = x^2 - 4$

4 Sketch each graph. Label where the curve crosses the axes and write down the coordinates of the turning point.

a $y = x^2 - 5x + 6$ **b** $y = -x^2 + 7x - 12$

5 Sketch the graph of $y = x^2 + 2x + 1$. Label where the curve crosses the axes and write down the equation of the line of symmetry.

Solving linear simultaneous equations using the elimination method

Key points

- Two equations are simultaneous when they are both true at the same time.
- Solving simultaneous linear equations in two unknowns involves finding the value of each unknown which works for both equations.
- Make sure that the coefficient of one of the unknowns is the same in both equations.
- Eliminate this equal unknown by either subtracting or adding the two equations.

Examples

Example 1 Solve the simultaneous equations $3x + y = 5$ and $x + y = 1$

$3x + y = 5$ $- \quad x + y = 1$ $2x \quad = 4$ So $x = 2$	**1** Subtract the second equation from the first equation to eliminate the y term.
Using $x + y = 1$ $2 + y = 1$ So $y = -1$	**2** To find the value of y, substitute $x = 2$ into one of the original equations.
Check: equation 1: $3 \times 2 + (-1) = 5$ YES equation 2: $2 + (-1) = 1$ YES	**3** Substitute the values of x and y into both equations to check your answers.

Example 2 Solve $x + 2y = 13$ and $5x - 2y = 5$ simultaneously.

$x + 2y = 13$ $+ \quad 5x - 2y = 5$ $6x \quad = 18$ So $x = 3$	**1** Add the two equations together to eliminate the y term.
Using $x + 2y = 13$ $3 + 2y = 13$ So $y = 5$	**2** To find the value of y, substitute $x = 3$ into one of the original equations.
Check: equation 1: $3 + 2 \times 5 = 13$ YES equation 2: $5 \times 3 - 2 \times 5 = 5$ YES	**3** Substitute the values of x and y into both equations to check your answers.

Example 3 Solve $2x + 3y = 2$ and $5x + 4y = 12$ simultaneously.

$(2x + 3y = 2) \times 4 \rightarrow \quad 8x + 12y = 8$ $(5x + 4y = 12) \times 3 \rightarrow \quad \underline{15x + 12y = 36}$ $7x = 28$ So $x = 4$	**1** Multiply the first equation by 4 and the second equation by 3 to make the coefficient of y the same for both equations. Then subtract the first equation from the second equation to eliminate the y term.
Using $2x + 3y = 2$ $2 \times 4 + 3y = 2$ So $y = -2$	**2** To find the value of y, substitute $x = 4$ into one of the original equations.
Check: equation 1: $2 \times 4 + 3 \times (-2) = 2$ YES equation 2: $5 \times 4 + 4 \times (-2) = 12$ YES	**3** Substitute the values of x and y into both equations to check your answers.

Practice

Solve these simultaneously equations.

1 $4x + y = 8$
$x + y = 5$

2 $3x + y = 7$
$3x + 2y = 5$

3 $4x + y = 3$
$3x - y = 11$

4 $3x + 4y = 7$
$x - 4y = 5$

Solving linear simultaneous equations using the substitution method

Key points

- The subsitution method is the method most commonly used for A level. This is because it is the method used to solve linear and quadratic simultaneous equations.

Examples

Example 1 Solve the simultaneous equations $y = 2x + 1$ and $5x + 3y = 14$

$5x + 3(2x + 1) = 14$	**1** Substitute $2x + 1$ for y into the second equation.
$5x + 6x + 3 = 14$ $11x + 3 = 14$	**2** Expand the brackets and simplify.
$11x = 11$ So $x = 1$	**3** Work out the value of x.
Using $y = 2x + 1$ $y = 2 \times 1 + 1$ So $y = 3$	**4** To find the value of y, substitute $x = 1$ into one of the original equations.
Check: equation 1: $3 = 2 \times 1 + 1$ YES equation 2: $5 \times 1 + 3 \times 3 = 14$ YES	**5** Substitute the values of x and y into both equations to check your answers.

Example 2 Solve $2x - y = 16$ and $4x + 3y = -3$ simultaneously.

$y = 2x - 16$	**1** Rearrange the first equation.
$4x + 3(2x - 16) = -3$	**2** Substitute $2x - 16$ for y into the second equation.
$4x + 6x - 48 = -3$ $10x - 48 = -3$	**3** Expand the brackets and simplify.
$10x = 45$ So $x = 4\frac{1}{2}$	**4** Work out the value of x.
Using $y = 2x - 16$ $y = 2 \times 4\frac{1}{2} - 16$ So $y = -7$	**5** To find the value of y, substitute $x = 4\frac{1}{2}$ into one of the original equations.
Check: equation 1: $2 \times 4\frac{1}{2} - (-7) = 16$ YES equation 2: $4 \times 4\frac{1}{2} + 3 \times (-7) = -3$ YES	**6** Substitute the values of x and y into both equations to check your answers.

Practice

Solve these simultaneous equations.

1 $y = x - 4$
$2x + 5y = 43$

2 $y = 2x - 3$
$5x - 3y = 11$

3 $2y = 4x + 5$
$9x + 5y = 22$

4 $2x = y - 2$
$8x - 5y = -11$

5 Solve the simultaneous equations $3x + 5y - 20 = 0$ and $2(x + y) = \frac{3(y - x)}{4}$.

Solving linear and quadratic simultaneous equations

Key points

- Make one of the unknowns the subject of the linear equation (rearranging where necessary).
- Use the linear equation to substitute into the quadratic equation.
- There are usually two pairs of solutions.

Examples

Example 1 Solve the simultaneous equations $y = x + 1$ and $x^2 + y^2 = 13$

$x^2 + (x + 1)^2 = 13$	**1** Substitute $x + 1$ for y into the second equation.
$x^2 + x^2 + x + x + 1 = 13$ $2x^2 + 2x + 1 = 13$	**2** Expand the brackets and simplify.
$2x^2 + 2x - 12 = 0$ $(2x - 4)(x + 3) = 0$	**3** Factorise the quadratic equation.
So $x = 2$ or $x = -3$	**4** Work out the values of x.
Using $y = x + 1$ When $x = 2$, $y = 2 + 1 = 3$ When $x = -3$, $y = -3 + 1 = -2$	**5** To find the value of y, substitute both values of x into one of the original equations.
So the solutions are $x = 2,\ y = 3$ and $x = -3,\ y = -2$	
Check: equation 1: $3 = 2 + 1$ YES and $-2 = -3 + 1$ YES equation 2: $2^2 + 3^2 = 13$ YES and $(-3)^2 + (-2)^2 = 13$ YES	**6** Substitute both pairs of values of x and y into both equations to check your answers.

Example 2 Solve $2x + 3y = 5$ and $2y^2 + xy = 12$ simultaneously.

$x = \frac{5-3y}{2}$	**1** Rearrange the first equation.
$2y^2 + \left(\frac{5-3y}{2}\right)y = 12$	**2** Substitute $\frac{5-3y}{2}$ for x into the second equation. Notice how it is easier to substitute for x than for y.
$2y^2 + \frac{5y-3y^2}{2} = 12$ $4y^2 + 5y - 3y^2 = 24$ $y^2 + 5y - 24 = 0$	**3** Expand the brackets and simplify.
$(y + 8)(y - 3) = 0$	**4** Factorise the quadratic equation.
So $y = -8$ or $y = 3$	**5** Work out the values of y.
Using $2x + 3y = 5$ When $y = -8$, $2x + 3 \times (-8) = 5$, $x = 14.5$ When $y = 3$, $2x + 3 \times 3 = 5$, $x = -2$	**6** To find the value of x, substitute both values of y into one of the original equations.
So the solutions are $x = 14.5$, $y = -8$ and $x = -2$, $y = 3$	
Check: equation 1: $2 \times 14.5 + 3 \times (-8) = 5$ YES and $2 \times (-2) + 3 \times 3 = 5$ YES equation 2: $2\times(-8)^2 + 14.5\times(-8) = 12$ YES and $2 \times (3)^2 + (-2) \times 3 = 12$ YES	**7** Substitute both pairs of values of x and y into both equations to check your answers.

Practice

Solve these simultaneous equations.

1 $y = 2x + 1$
$x^2 + y^2 = 10$

2 $y = 6 - x$
$x^2 + y^2 = 20$

3 $y = 3x - 5$
$y = x^2 - 2x + 1$

4 $y = x - 5$
$y = x^2 - 5x - 12$

5 $y = 2x$
$y^2 - xy = 8$

6 $2x + y = 11$
$xy = 15$

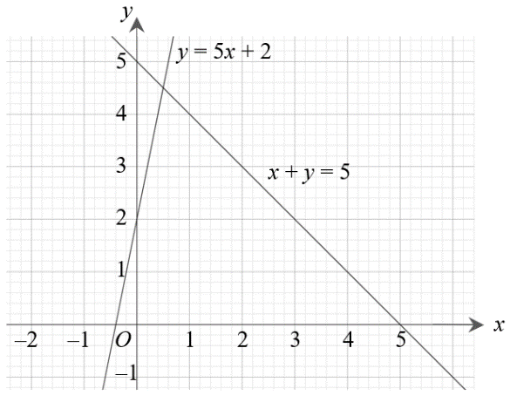
Solving simultaneous equations graphically

Key points

- You can solve any pair of simultaneous equations by drawing the graph of both equations and finding the point/points of intersection.

Examples

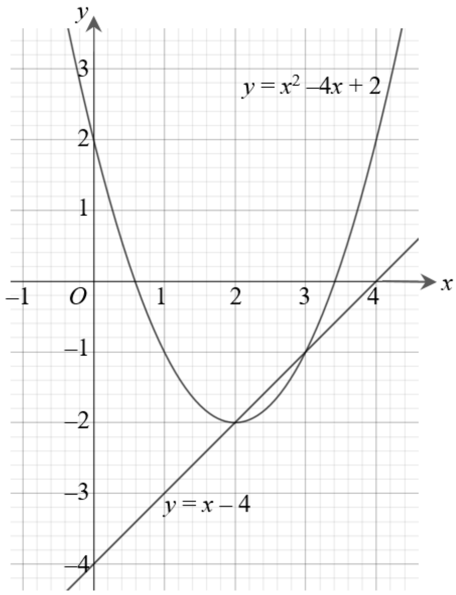
Example 1 Solve the simultaneous equations $y = 5x + 2$ and $x + y = 5$ graphically.

$y = 5 - x$	**1** Rearrange the equation $x + y = 5$ to make y the subject.
$y = 5 - x$ has gradient –1 and y-intercept 5. $y = 5x + 2$ has gradient 5 and y-intercept 2. 	**2** Plot both graphs on the same grid using the gradients and y-intercepts.
Lines intersect at $x = 0.5, y = 4.5$	**3** The solutions of the simultaneous equations are the point of intersection.
Check: First equation $y = 5x + 2$: $4.5 = 5 \times 0.5 + 2$ YES Second equation $x + y = 5$: $0.5 + 4.5 = 5$ YES	**4** Check your solutions by substituting the values into both equations.

Example 2 Solve the simultaneous equations $y = x - 4$ and $y = x^2 - 4x + 2$ graphically.

x	0	1	2	3	4
y	2	–1	–2	–1	2

1 Construct a table of values and calculate the points for the quadratic equation.



2 Plot the graph.

3 Plot the linear graph on the same grid using the gradient and y-intercept.
$y = x – 4$ has gradient 1 and y-intercept –4.

The line and curve intersect at
$x = 3, y = -1$ and $x = 2, y = -2$

Check:
First equation $y = x - 4$:
$-1 = 3 - 4$ YES
$-2 = 2 - 4$ YES
Second equation $y = x^2 - 4x + 2$:
$-1 = 3^2 - 4 \times 3 + 2$ YES
$-2 = 2^2 - 4 \times 2 + 2$ YES

4 The solutions of the simultaneous equations are the points of intersection.

5 Check your solutions by substituting the values into both equations.

Practice

1 Solve these pairs of simultaneous equations graphically.
- **a** $y = 3x - 1$ and $y = x + 3$
- **b** $y = x - 5$ and $y = 7 - 5x$

2 Solve these pairs of simultaneous equations graphically.
- **a** $x + y = 0$ and $y = 2x + 6$
- **b** $4x + 2y = 3$ and $y = 3x - 1$

Hint

Rearrange the equation to make y the subject.

3 Solve these pairs of simultaneous equations graphically.
- **a** $y = x - 1$ and $y = x^2 - 4x + 3$
- **b** $y = 1 - 3x$ and $y = x^2 - 3x - 3$

4 Solve the simultaneous equations $x + y = 1$ and $x^2 + y^2 = 25$ graphically.

Linear inequalities

Key points

- Solving linear inequalities uses similar methods to those for solving linear equations.
- When you multiply or divide an inequality by a negative number you need to reverse the inequality sign, e.g. < becomes >.

Examples

Example 1 Solve $-8 \leq 4x < 16$

$-8 \leq 4x < 16$ $-2 \leq x < 4$	Divide all three terms by 4.

Example 2 Solve $4 \leq 5x < 10$

$4 \leq 5x < 10$ $\frac{4}{5} \leq x < 2$	Divide all three terms by 5.

Example 3 Solve $2x - 5 < 7$

$2x - 5 < 7$ $2x < 12$ $x < 6$	**1** Add 5 to both sides. **2** Divide both sides by 2.

Example 4 Solve $2 - 5x \geq -8$

$2 - 5x \geq -8$ $-5x \geq -10$ $x \leq 2$	**1** Subtract 2 from both sides. **2** Divide both sides by −5. Remember to reverse the inequality when dividing by a negative number.

Example 5 Solve $4(x - 2) > 3(9 - x)$

$4(x - 2) > 3(9 - x)$ $4x - 8 > 27 - 3x$ $7x - 8 > 27$ $7x > 35$ $x > 5$	**1** Expand the brackets. **2** Add $3x$ to both sides. **3** Add 8 to both sides. **4** Divide both sides by 7.

Practice

1 Solve these inequalities.

a $4x > 16$ **b** $5x - 7 \leq 3$ **c** $1 \geq 3x + 4$

2 Solve these inequalities.

a $\frac{x}{5} < -4$ **b** $10 \geq 2x + 3$ **c** $7 - 3x > -5$

3 Solve

a $2 - 4x \geq 18$ **b** $3 \leq 7x + 10 < 45$ **c** $6 - 2x \geq 4$

4 Solve these inequalities.

a $3t + 1 < t + 6$ **b** $2(3n - 1) \geq n + 5$

5 Find the set of values of x for which $2x + 1 > 11$ and $4x - 2 > 16 - 2x$.

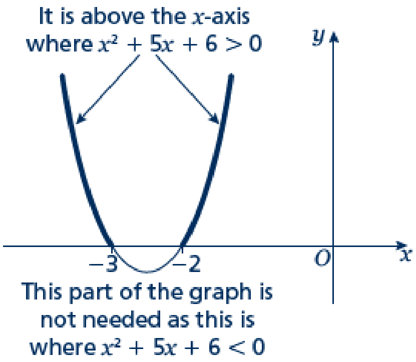
Quadratic inequalities

Key points

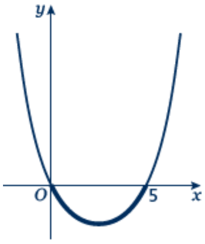
- First replace the inequality sign by = and solve the quadratic equation.
- Sketch the graph of the quadratic function.
- Use the graph to find the values which satisfy the quadratic inequality.

Examples

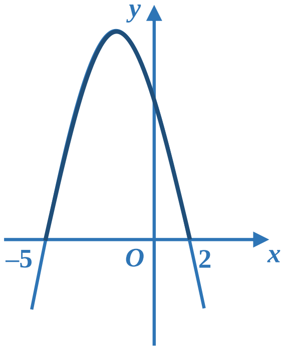
Example 1 Find the set of values of x which satisfy $x^2 + 5x + 6 > 0$

$x^2 + 5x + 6 = 0$ $(x + 3)(x + 2) = 0$ $x = -3$ or $x = -2$	**1** Solve the quadratic equation by factorising.
	**2** Sketch the graph of $y = (x + 3)(x + 2)$ **3** Identify on the graph where $x^2 + 5x + 6 > 0$, i.e. where $y > 0$
$x < -3$ or $x > -2$	**4** Write down the values which satisfy the inequality $x^2 + 5x + 6 > 0$

Example 2 Find the set of values of x which satisfy $x^2 - 5x \leq 0$

$x^2 - 5x = 0$ $x(x - 5) = 0$ $x = 0$ or $x = 5$	**1** Solve the quadratic equation by factorising.
	**2** Sketch the graph of $y = x(x - 5)$ **3** Identify on the graph where $x^2 - 5x \leq 0$, i.e. where $y \leq 0$
$0 \leq x \leq 5$	**4** Write down the values which satisfy the inequality $x^2 - 5x \leq 0$

Example 3 Find the set of values of x which satisfy $-x^2 - 3x + 10 \geq 0$

$-x^2 - 3x + 10 = 0$ $(-x + 2)(x + 5) = 0$ $x = 2$ or $x = -5$	**1** Solve the quadratic equation by factorising.
	**2** Sketch the graph of $y = (-x + 2)(x + 5) = 0$ **3** Identify on the graph where $-x^2 - 3x + 10 \geq 0$, i.e. where $y \geq 0$
$-5 \leq x \leq 2$	**3** Write down the values which satisfy the inequality $-x^2 - 3x + 10 \geq 0$

Practice

1 Find the set of values of x for which $(x + 7)(x - 4) \leq 0$

2 Find the set of values of x for which $2x^2 -7x + 3 < 0$

3 Find the set of values of x for which $12 + x - x^2 \geq 0$

4 Find the set of values which satisfy the following inequalities.

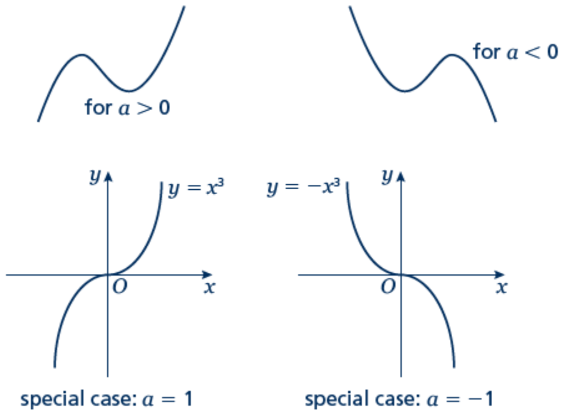
$x(2x - 9) < -10$

$6x^2 \geq 15 + x$

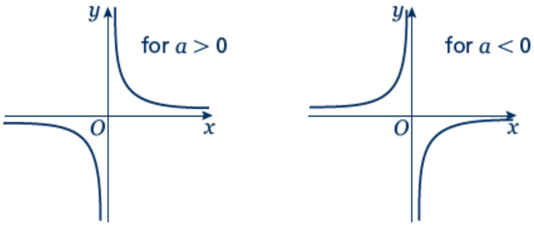
Sketching cubic and reciprocal graphs

Key points

- The graph of a cubic function, which can be written in the form $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$, has one of the shapes shown here.



- The graph of a reciprocal function of the form $y = \frac{a}{x}$ has one of the shapes shown here.

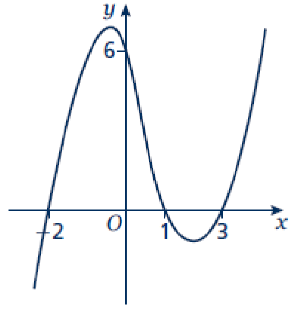


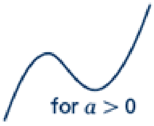
- To sketch the graph of a function, find the points where the graph intersects the axes.
- To find where the curve intersects the y-axis substitute $x = 0$ into the function.
- To find where the curve intersects the x-axis substitute $y = 0$ into the function.
- Where appropriate, mark and label the asymptotes on the graph.
- Asymptotes are lines (usually horizontal or vertical) which the curve gets closer to but never touches or crosses. Asymptotes usually occur with reciprocal functions. For example, the asymptotes for the graph of $y = \frac{a}{x}$ are the two axes (the lines $y = 0$ and $x = 0$).
- At the turning points of a graph the gradient of the curve is 0 and any tangents to the curve at these points are horizontal.
- A double root is when two of the solutions are equal. For example $(x - 3)^2(x + 2)$ has a double root at $x = 3$.
- When there is a double root, this is one of the turning points of a cubic function.

Examples

Example 1 Sketch the graph of $y = (x - 3)(x - 1)(x + 2)$

<table>
<tr><td colspan="2">To sketch a cubic curve find intersects with both axes and use the key points above for the correct shape.</td></tr>
<tr><td>When $x = 0$, $y = (0 - 3)(0 - 1)(0 + 2)$
$= (-3) \times (-1) \times 2 = 6$
The graph intersects the y-axis at $(0, 6)$

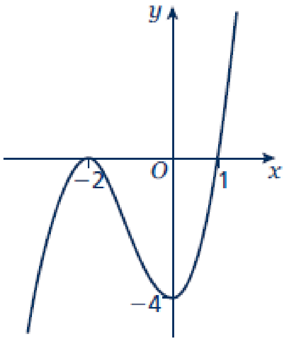
When $y = 0$, $(x - 3)(x - 1)(x + 2) = 0$
So $x = 3$, $x = 1$ or $x = -2$
The graph intersects the x-axis at $(-2, 0)$, $(1, 0)$ and $(3, 0)$
 </td><td>1 Find where the graph intersects the axes by substituting $x = 0$ and $y = 0$. Make sure you get the coordinates the right way around, (x, y).
2 Solve the equation by solving $x - 3 = 0$, $x - 1 = 0$ and $x + 2 = 0$

3 Sketch the graph.
$a = 1 > 0$ so the graph has the shape:
 </td></tr>
</table>

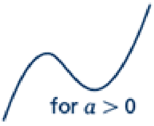
Example 2 Sketch the graph of $y = (x + 2)^2(x - 1)$

<table>
<tr><td colspan="2">To sketch a cubic curve find intersects with both axes and use the key points above for the correct shape.</td></tr>
<tr><td>When $x = 0$, $y = (0 + 2)^2(0 - 1)$
$= 2^2 \times (-1) = -4$
The graph intersects the y-axis at $(0, -4)$

When $y = 0$, $(x + 2)^2(x - 1) = 0$
So $x = -2$ or $x = 1$

$(-2, 0)$ is a turning point as $x = -2$ is a double root.
The graph crosses the x-axis at $(1, 0)$
 </td><td>1 Find where the graph intersects the axes by substituting $x = 0$ and $y = 0$.

2 Solve the equation by solving $x + 2 = 0$ and $x - 1 = 0$

3 $a = 1 > 0$ so the graph has the shape:
 </td></tr>
</table>

Practice

1 Here are six equations.

A $y = \frac{5}{x}$ **B** $y = x^2 + 3x - 10$ **C** $y = x^3 + 3x^2$

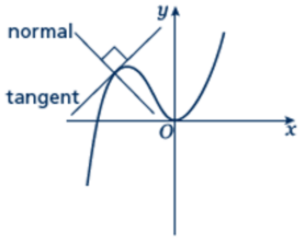
D $y = 1 - 3x^2 - x^3$ **E** $y = x^3 - 3x^2 - 1$ **F** $x + y = 5$

Hint

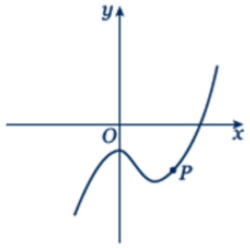
Find where each of the cubic equations cross the y-axis.

Here are six graphs.

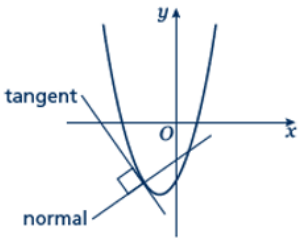
i



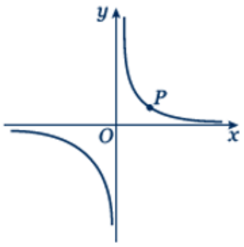
ii



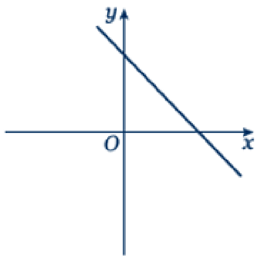
iii



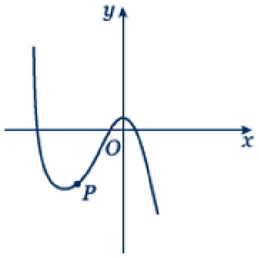
iv



v



vi



a Match each graph to its equation.

b Copy the graphs ii, iv and vi and draw the tangent and normal each at point P.

Sketch the following graphs

2 $y = 2x^3$

3 $y = (x + 1)(x + 4)(x - 3)$

4 $y = (x - 3)^2(x + 1)$

5 $y = \frac{3}{x}$

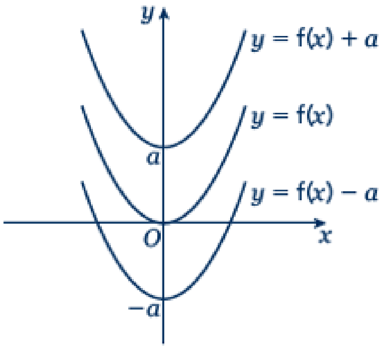
6 Sketch the graph of $y = \frac{1}{x+2}$

Translating graphs

Key points

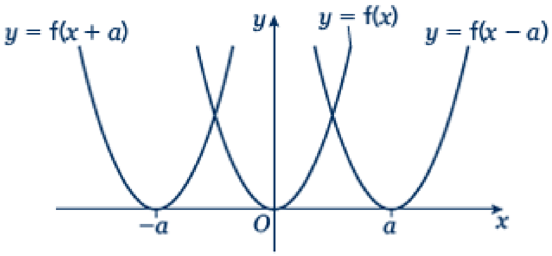
- The transformation $y = f(x) \pm a$ is a translation of $y = f(x)$ parallel to the y-axis; it is a vertical translation.

  As shown on the graph,
  - $y = f(x) + a$ translates $y = f(x)$ up
  - $y = f(x) - a$ translates $y = f(x)$ down.



- The transformation $y = f(x \pm a)$ is a translation of $y = f(x)$ parallel to the x-axis; it is a horizontal translation.

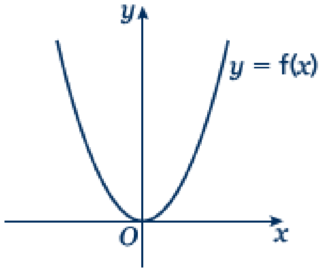
  As shown on the graph,
  - $y = f(x + a)$ translates $y = f(x)$ to the left
  - $y = f(x - a)$ translates $y = f(x)$ to the right.

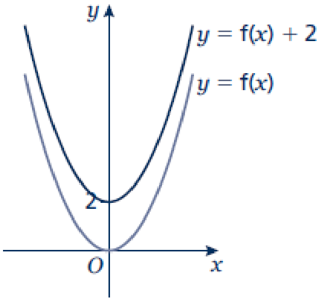


Examples

Example 1 The graph shows the function $y = f(x)$.

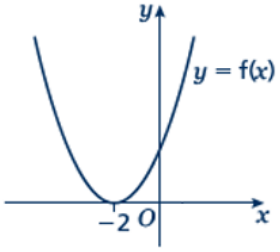
Sketch the graph of $y = f(x) + 2$.

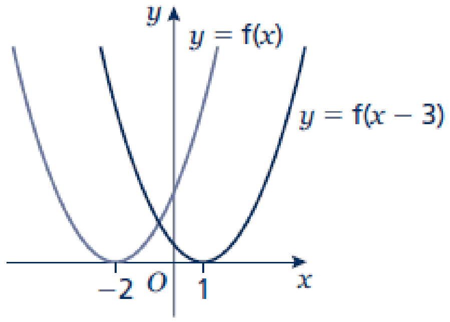


	For the function $y = f(x) + 2$ translate the function $y = f(x)$ 2 units up.

Example 2 The graph shows the function $y = \mathrm{f}(x)$.

Sketch the graph of $y = \mathrm{f}(x - 3)$.

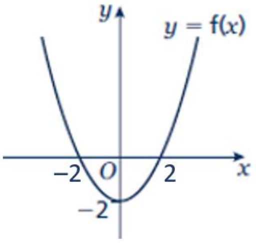




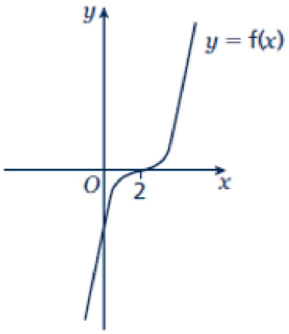
For the function $y = \mathrm{f}(x - 3)$ translate the function $y = \mathrm{f}(x)$ 3 units right.

Practice

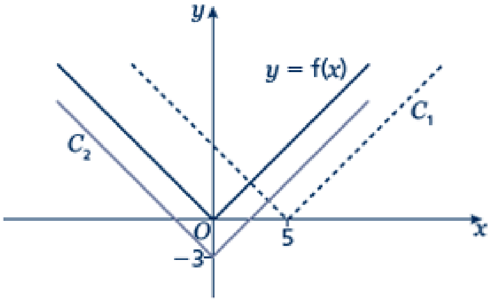
1 The graph shows the function $y = \mathrm{f}(x)$.
Copy the graph and on the same axes sketch and label the graphs of $y = \mathrm{f}(x) + 4$ and $y = \mathrm{f}(x + 2)$.



2 The graph shows the function $y = \mathrm{f}(x)$.
Copy the graph and on the same axes sketch and label the graphs of $y = \mathrm{f}(x + 3)$ and $y = \mathrm{f}(x) - 3$.



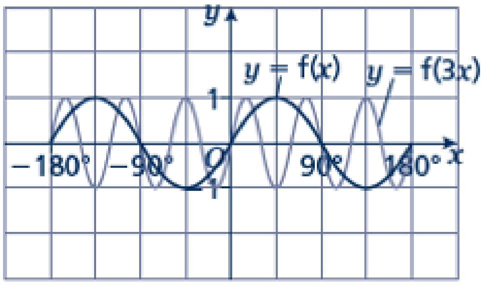
3 The graph shows the function $y = \mathrm{f}(x)$ and two transformations of $y = \mathrm{f}(x)$, labelled C_1 and C_2.
Write down the equations of the translated curves C_1 and C_2 in function form.



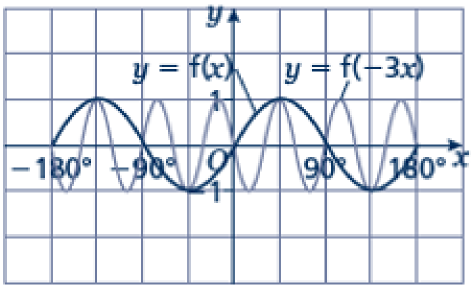
Stretching graphs

Key points

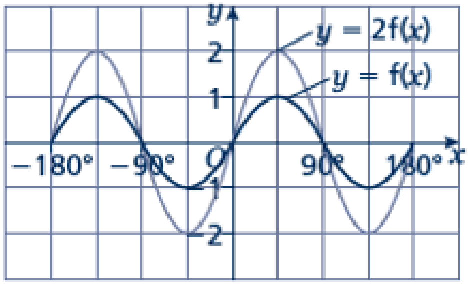
- The transformation $y = \mathrm{f}(ax)$ is a horizontal stretch of $y = \mathrm{f}(x)$ with scale factor $\frac{1}{a}$ parallel to the x-axis.



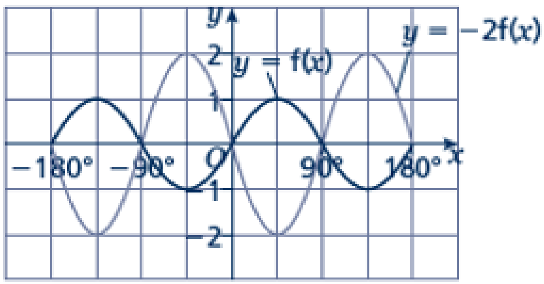
- The transformation $y = \mathrm{f}(–ax)$ is a horizontal stretch of $y = \mathrm{f}(x)$ with scale factor $\frac{1}{a}$ parallel to the x-axis and then a reflection in the y-axis.



- The transformation $y = a\mathrm{f}(x)$ is a vertical stretch of $y = \mathrm{f}(x)$ with scale factor a parallel to the y-axis.



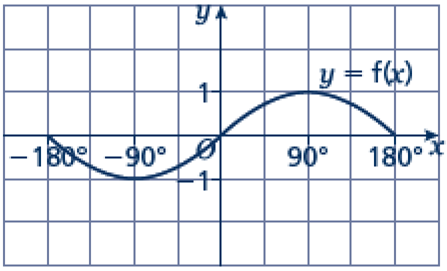
- The transformation $y = –a\mathrm{f}(x)$ is a vertical stretch of $y = \mathrm{f}(x)$ with scale factor a parallel to the y-axis and then a reflection in the x-axis.

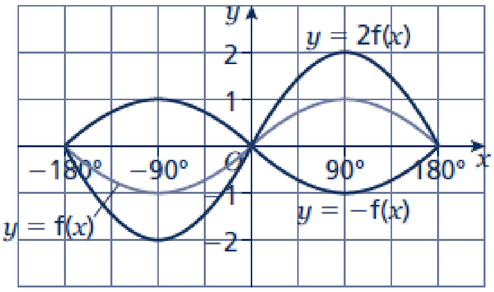


Examples

Example 1 The graph shows the function $y = \mathrm{f}(x)$.

Sketch and label the graphs of
$y = 2\mathrm{f}(x)$ and $y = -\mathrm{f}(x)$.



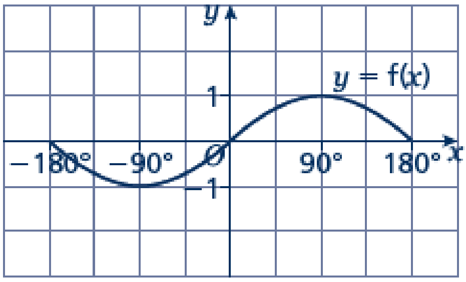


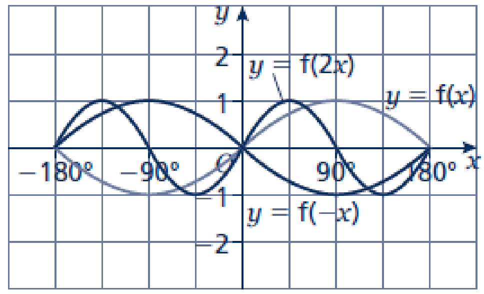
The function $y = 2\mathrm{f}(x)$ is a vertical stretch of $y = \mathrm{f}(x)$ with scale factor 2 parallel to the y-axis.

The function $y = -\mathrm{f}(x)$ is a reflection of $y = \mathrm{f}(x)$ in the x-axis.

Example 2 The graph shows the function $y = \mathrm{f}(x)$.

Sketch and label the graphs of
$y = \mathrm{f}(2x)$ and $y = \mathrm{f}(-x)$.



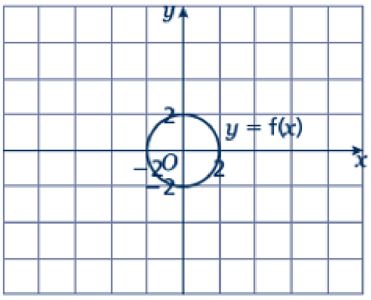


The function $y = \mathrm{f}(2x)$ is a horizontal stretch of $y = \mathrm{f}(x)$ with scale factor $\frac{1}{2}$ parallel to the x-axis.

The function $y = \mathrm{f}(-x)$ is a reflection of $y = \mathrm{f}(x)$ in the y-axis.

Practice

1 The graph shows the function $y = \mathrm{f}(x)$.

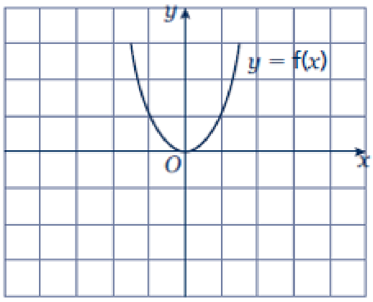


a Copy the graph and on the same axes sketch and label the graph of $y = 3\mathrm{f}(x)$.

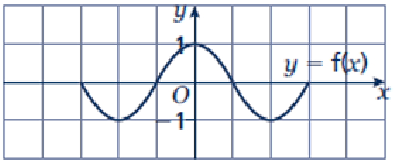
b Make another copy of the graph and on the same axes sketch and label the graph of $y = \mathrm{f}(2x)$.

2 The graph shows the function $y = \mathrm{f}(x)$. Copy the graph and, on the same axes, sketch and label the graphs of

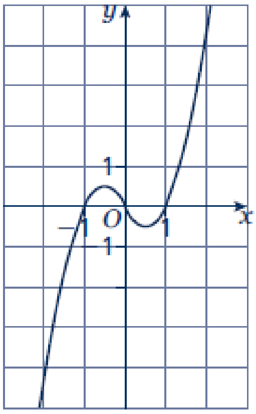
$y = -\mathrm{f}(x)$ and $y = \mathrm{f}\left(\frac{1}{2}x\right)$.



3 The graph shows the function $y = \mathrm{f}(x)$. Copy the graph and, on the same axes, sketch the graph of $y = -\mathrm{f}(2x)$.



4 The graph shows the function $y = \mathrm{f}(x)$.



a Sketch the graph of $y = -\mathrm{f}(x)$.

b Sketch the graph of $y = 2\mathrm{f}(x)$.

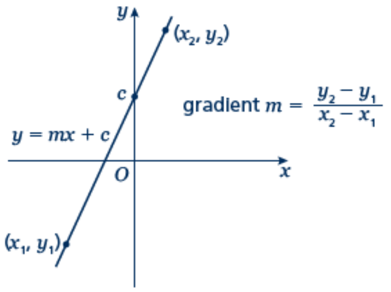
5 **a** Sketch and label the graph of $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = (x - 1)(x + 1)$.

b On the same axes, sketch and label the graphs of $y = \mathrm{f}(x) - 2$ and $y = \mathrm{f}(x + 2)$.

Straight line graphs

Key points

- A straight line has the equation $y = mx + c$, where m is the gradient and c is the y-intercept (where $x = 0$).
- The equation of a straight line can be written in the form $ax + by + c = 0$, where a, b and c are integers.
- When given the coordinates (x_1, y_1) and (x_2, y_2) of two points on a line the gradient is calculated using the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$



Examples

Example 1 A straight line has gradient $-\frac{1}{2}$ and y-intercept 3.

Write the equation of the line in the form $ax + by + c = 0$.

$m = -\frac{1}{2}$ and $c = 3$ So $y = -\frac{1}{2}x + 3$	**1** A straight line has equation $y = mx + c$. Substitute the gradient and y-intercept given in the question into this equation.
$\frac{1}{2}x + y - 3 = 0$	**2** Rearrange the equation so all the terms are on one side and 0 is on the other side.
$x + 2y - 6 = 0$	**3** Multiply both sides by 2 to eliminate the denominator.

Example 2 Find the gradient and the y-intercept of the line with the equation $3y - 2x + 4 = 0$.

$3y - 2x + 4 = 0$ $3y = 2x - 4$	**1** Make y the subject of the equation.
$y = \frac{2}{3}x - \frac{4}{3}$	**2** Divide all the terms by three to get the equation in the form $y = \ldots$
Gradient $= m = \frac{2}{3}$ y-intercept $= c = -\frac{4}{3}$	**3** In the form $y = mx + c$, the gradient is m and the y-intercept is c.

Example 3 Find the equation of the line which passes through the point (5, 13) and has gradient 3.

$m = 3$ $y = 3x + c$	**1** Substitute the gradient given in the question into the equation of a straight line $y = mx + c$.
$13 = 3 \times 5 + c$	**2** Substitute the coordinates $x = 5$ and $y = 13$ into the equation.
$13 = 15 + c$ $c = -2$	**3** Simplify and solve the equation.
$y = 3x - 2$	**4** Substitute $c = -2$ into the equation $y = 3x + c$

Example 4 Find the equation of the line passing through the points with coordinates (2, 4) and (8, 7).

$x_1 = 2,\ x_2 = 8,\ y_1 = 4$ and $y_2 = 7$ $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-4}{8-2} = \frac{3}{6} = \frac{1}{2}$	**1** Substitute the coordinates into the equation $m = \frac{y_2 - y_1}{x_2 - x_1}$ to work out the gradient of the line.
$y = \frac{1}{2}x + c$	**2** Substitute the gradient into the equation of a straight line $y = mx + c$.
$4 = \frac{1}{2} \times 2 + c$	**3** Substitute the coordinates of either point into the equation.
$c = 3$	**4** Simplify and solve the equation.
$y = \frac{1}{2}x + 3$	**5** Substitute $c = 3$ into the equation $y = \frac{1}{2}x + c$

Practice

1 Find the gradient and the y-intercept of the following equations.

a $y = 3x + 5$ **b** $y = -\frac{1}{2}x - 7$

c $2y = 4x - 3$ **d** $x + y = 5$

Hint
Rearrange the equations to the form $y = mx + c$

2 Find, in the form $ax + by + c = 0$ where a, b and c are integers, an equation for each of the lines with the following gradients and y-intercepts.

a gradient $-\frac{1}{2}$, y-intercept –7 **b** gradient 2, y-intercept 0

3 Write an equation for the line which passes through the point (6, 3) and has gradient $-\frac{2}{3}$

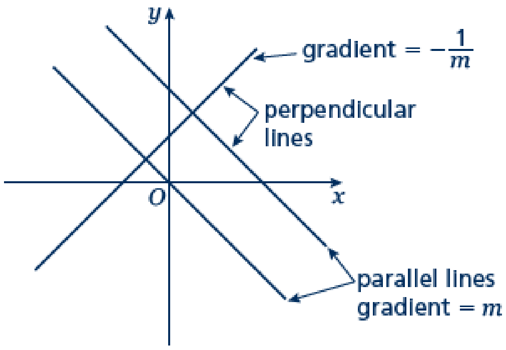
4 Write an equation for the line passing through each of the following pairs of points.

a (4, 5), (10, 17) **b** (0, 6), (–4, 8)

Parallel and perpendicular lines

Key points

- When lines are parallel they have the same gradient.
- A line perpendicular to the line with equation $y = mx + c$ has gradient $-\frac{1}{m}$.



Examples

Example 1 Find the equation of the line parallel to $y = 2x + 4$ which passes through the point (4, 9).

$y = 2x + 4$ $m = 2$	**1** As the lines are parallel they have the same gradient.
$y = 2x + c$	**2** Substitute $m = 2$ into the equation of a straight line $y = mx + c$.
$9 = 2 \times 4 + c$	**3** Substitute the coordinates into the equation $y = 2x + c$
$9 = 8 + c$ $c = 1$	**4** Simplify and solve the equation.
$y = 2x + 1$	**5** Substitute $c = 1$ into the equation $y = 2x + c$

Example 2 Find the equation of the line perpendicular to $y = 2x - 3$ which passes through the point (−2, 5).

$y = 2x - 3$ $m = 2$ $-\frac{1}{m} = -\frac{1}{2}$	**1** As the lines are perpendicular, the gradient of the perpendicular line is $-\frac{1}{m}$.
$y = -\frac{1}{2}x + c$	**2** Substitute $m = -\frac{1}{2}$ into $y = mx + c$.
$5 = -\frac{1}{2} \times (-2) + c$	**3** Substitute the coordinates (–2, 5) into the equation $y = -\frac{1}{2}x + c$
$5 = 1 + c$ $c = 4$	**4** Simplify and solve the equation.
$y = -\frac{1}{2}x + 4$	**5** Substitute $c = 4$ into $y = -\frac{1}{2}x + c$.

Example 3 A line passes through the points (0, 5) and (9, −1).
Find the equation of the line which is perpendicular to the line and passes through its midpoint.

$x_1 = 0$, $x_2 = 9$, $y_1 = 5$ and $y_2 = -1$ $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-5}{9-0}$ $= \frac{-6}{9} = -\frac{2}{3}$	**1** Substitute the coordinates into the equation $m = \frac{y_2 - y_1}{x_2 - x_1}$ to work out the gradient of the line.
$-\frac{1}{m} = \frac{3}{2}$	**2** As the lines are perpendicular, the gradient of the perpendicular line is $-\frac{1}{m}$.
$y = \frac{3}{2}x + c$	**3** Substitute the gradient into the equation $y = mx + c$.
Midpoint = $\left(\frac{0+9}{2}, \frac{5+(-1)}{2}\right) = \left(\frac{9}{2}, 2\right)$	**4** Work out the coordinates of the midpoint of the line.
$2 = \frac{3}{2} \times \frac{9}{2} + c$	**5** Substitute the coordinates of the midpoint into the equation.
$c = -\frac{19}{4}$	**6** Simplify and solve the equation.
$y = \frac{3}{2}x - \frac{19}{4}$	**7** Substitute $c = -\frac{19}{4}$ into the equation $y = \frac{3}{2}x + c$.

Practice

1 Find the equation of the line parallel to each of the given lines and which passes through each of the given points.

a $y = 3x + 1$ (3, 2) **b** $y = 3 - 2x$ (1, 3)

Hint
If $m = \frac{a}{b}$ then the negative reciprocal $-\frac{1}{m} = -\frac{b}{a}$

2 Find the equation of the line perpendicular to $y = \frac{1}{2}x - 3$ which passes through the point (–5, 3).

3 Find the equation of the line perpendicular to each of the given lines and which passes through each of the given points.

a $y = 2x - 6$ (4, 0) **b** $y = -\frac{1}{3}x + \frac{1}{2}$ (2, 13)

4 In each case find an equation for the line passing through the origin which is also perpendicular to the line joining the two points given.

a (4, 3), (–2, –9) **b** (0, 3), (–10, 8)

5 Work out whether these pairs of lines are parallel, perpendicular or neither.

a	$y = 2x + 3$ $y = 2x - 7$	**b**	$y = 3x$ $2x + y - 3 = 0$	**c**	$y = 4x - 3$ $4y + x = 2$

6 The straight line $\mathbf{L_1}$ passes through the points A and B with coordinates $(-4, 4)$ and $(2, 1)$, respectively.

a Find the equation of $\mathbf{L_1}$ in the form $ax + by + c = 0$

The line $\mathbf{L_2}$ is parallel to the line $\mathbf{L_1}$ and passes through the point C with coordinates $(-8, 3)$.

b Find the equation of $\mathbf{L_2}$ in the form $ax + by + c = 0$

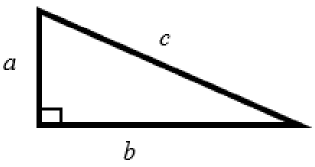
The line $\mathbf{L_3}$ is perpendicular to the line $\mathbf{L_1}$ and passes through the origin.

c Find an equation of $\mathbf{L_3}$

Pythagoras’ theorem

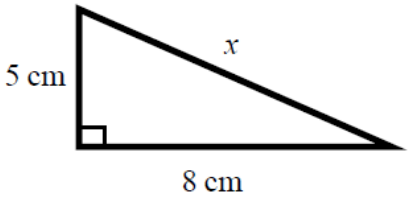
Key points

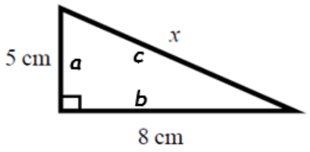
- In a right-angled triangle the longest side is called the hypotenuse.
- Pythagoras’ theorem states that for a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.
  $c^2 = a^2 + b^2$



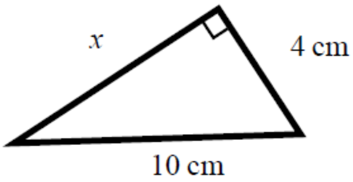
Examples

Example 1 Calculate the length of the hypotenuse.
Give your answer to 3 significant figures.



$c^2 = a^2 + b^2$  	**1** Always start by stating the formula for Pythagoras’ theorem and labelling the hypotenuse *c* and the other two sides *a* and *b*.
$x^2 = 5^2 + 8^2$ $x^2 = 25 + 64$ $x^2 = 89$ $x = \sqrt{89}$ $x = 9.433\ 981\ 13...$ $x = 9.43$ cm	**2** Substitute the values of *a*, *b* and *c* into the formula for Pythagoras' theorem. **3** Use a calculator to find the square root. **4** Round your answer to 3 significant figures and write the units with your answer.

Example 2 Calculate the length x.
Give your answer in surd form.

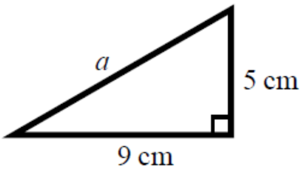


$c^2 = a^2 + b^2$	**1** Always start by stating the formula for Pythagoras' theorem.
$10^2 = x^2 + 4^2$ $100 = x^2 + 16$ $x^2 = 84$	**2** Substitute the values of a, b and c into the formula for Pythagoras' theorem.
$x = \sqrt{84}$ $x = 2\sqrt{21}$ cm	**3** Simplify the surd where possible and write the units in your answer.

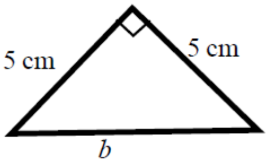
Practice

1 Work out the length of the unknown side in each triangle.
Give your answers correct to 3 significant figures.

a

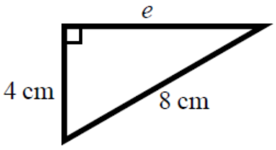


b

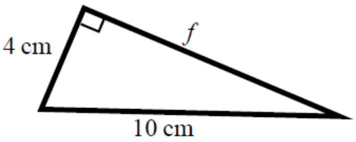


2 Work out the length of the unknown side in each triangle.
Give your answers in surd form.

a

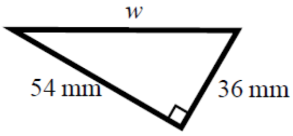


b

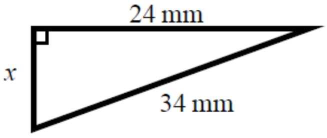


3 Work out the length of the unknown side in each triangle.
Give your answers in surd form.

a



b



4 A rectangle has length 84 mm and width 45 mm.
Calculate the length of the diagonal of the rectangle.
Give your answer correct to 3 significant figures.

Hint

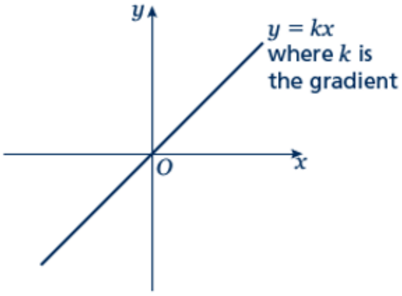
Draw a diagram using the information given in the question.

5 A yacht is 40 km due North of a lighthouse.
A rescue boat is 50 km due East of the same lighthouse.
Work out the distance between the yacht and the rescue boat.
Give your answer correct to 3 significant figures.

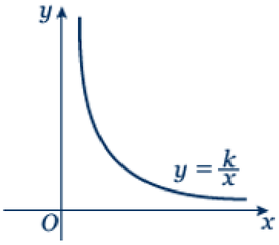
Proportion

Key points

- Two quantities are in direct proportion when, as one quantity increases, the other increases at the same rate. Their ratio remains the same.
- 'y is directly proportional to x' is written as $y \propto x$.
If $y \propto x$ then $y = kx$, where k is a constant.
- When x is directly proportional to y, the graph is a straight line passing through the origin.



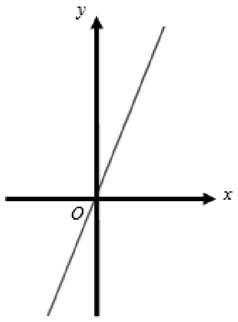
- Two quantities are in inverse proportion when, as one quantity increases, the other decreases at the same rate.
- 'y is inversely proportional to x' is written as $y \propto \frac{1}{x}$.
If $y \propto \frac{1}{x}$ then $y = \frac{k}{x}$, where k is a constant.
- When x is inversely proportional to y the graph is the same shape as the graph of $y = \frac{1}{x}$



Examples

Example 1 y is directly proportional to x.
When $y = 16$, $x = 5$.
a Find x when $y = 30$.
b Sketch the graph of the formula.

a $y \propto x$	**1** Write y is directly proportional to x, using the symbol $\propto$.
$y = kx$	**2** Write the equation using k.
$16 = k \times 5$	**3** Substitute $y = 16$ and $x = 5$ into $y = kx$.
$k = 3.2$	**4** Solve the equation to find k.
$y = 3.2x$	**5** Substitute the value of k back into the equation $y = kx$.
When $y = 30$, $30 = 3.2 \times x$ $x = 9.375$	**6** Substitute $y = 30$ into $y = 3.2x$ and solve to find x when $y = 30$.

b 	**7** The graph of $y = 3.2x$ is a straight line passing through (0, 0) with a gradient of 3.2.

Example 2 y is directly proportional to x^2.
When $x = 3$, $y = 45$.
a Find y when $x = 5$.
b Find x when $y = 20$.

a $y \propto x^2$	**1** Write y is directly proportional to x^2, using the symbol $\propto$.
$y = kx^2$	**2** Write the equation using k.
$45 = k \times 3^2$	**3** Substitute $y = 45$ and $x = 3$ into $y = kx^2$.
$k = 5$	**4** Solve the equation to find k.
$y = 5x^2$	**5** Substitute the value of k back into the equation $y = kx^2$.
When $x = 5$, $y = 5 \times 5^2$ $y = 125$	**6** Substitute $x = 5$ into $y = 5x^2$ and solve to find y when $x = 5$.
b $20 = 5 \times x^2$ $x^2 = 4$ $x = \pm 2$	**7** Substitute $y = 20$ into $y = 5x^2$ and solve to find x when $y = 4$.

Example 3 P is inversely proportional to Q.
When $P = 100$, $Q = 10$.
Find Q when $P = 20$.

$P \propto \frac{1}{Q}$	**1** Write P is inversely proportional to Q, using the symbol $\propto$.
$P = \frac{k}{Q}$	**2** Write the equation using k.
$100 = \frac{k}{10}$	**3** Substitute $P = 100$ and $Q = 10$.
$k = 1000$	**4** Solve the equation to find k.
$P = \frac{1000}{Q}$	**5** Substitute the value of k into $P = \frac{k}{Q}$
$20 = \frac{1000}{Q}$ $Q = \frac{1000}{20} = 50$	**6** Substitute $P = 20$ into $P = \frac{1000}{Q}$ and solve to find Q when $P = 20$.

Practice

1 Paul gets paid an hourly rate. The amount of pay (£P) is directly proportional to the number of hours (h) he works.
When he works 8 hours he is paid £56.
If Paul works for 11 hours, how much is he paid?

Hint

Substitute the values given for P and h into the formula to calculate k.

2 x is directly proportional to y.
$x = 35$ when $y = 5$.

a Find a formula for x in terms of y.

b Sketch the graph of the formula.

c Find x when $y = 13$.

d Find y when $x = 63$.

3 y is directly proportional to the square of x.
$x = 2$ when $y = 10$.

a Find a formula for y in terms of x.

b Sketch the graph of the formula.

c Find x when $y = 90$.

4 B is directly proportional to the square root of C.
$C = 25$ when $B = 10$.

a Find B when $C = 64$.

b Find C when $B = 20$.

5 m is proportional to the cube of n.
$m = 54$ when $n = 3$.
Find n when $m = 250$.

6 s is inversely proportional to t.

a Given that $s = 2$ when $t = 2$, find a formula for s in terms of t.

b Sketch the graph of the formula.

c Find t when $s = 1$.

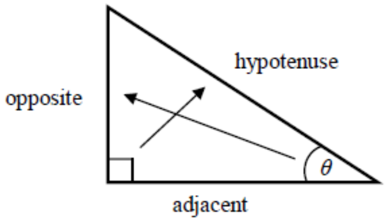
7 y is inversely proportional to x^2.
$y = 4$ when $x = 2$.
Find y when $x = 4$.

8 y is inversely proportional to the square root of x.
$x = 25$ when $y = 1$.
Find x when $y = 5$.

Trigonometry in right-angled triangles

Key points

- In a right-angled triangle:
  - the side opposite the right angle is called the hypotenuse
  - the side opposite the angle θ is called the opposite
  - the side next to the angle θ is called the adjacent.



- In a right-angled triangle:
  - the ratio of the opposite side to the hypotenuse is the sine of angle θ, $\sin\theta = \frac{\text{opp}}{\text{hyp}}$
  - the ratio of the adjacent side to the hypotenuse is the cosine of angle θ, $\cos\theta = \frac{\text{adj}}{\text{hyp}}$
  - the ratio of the opposite side to the adjacent side is the tangent of angle θ, $\tan\theta = \frac{\text{opp}}{\text{adj}}$

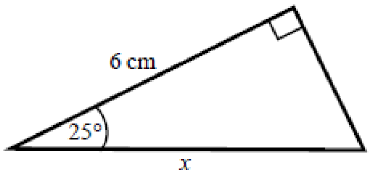
- If the lengths of two sides of a right-angled triangle are given, you can find a missing angle using the inverse trigonometric functions: $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$.

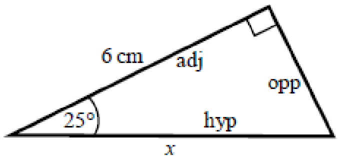
- The sine, cosine and tangent of some angles may be written exactly.

	0	**30°**	**45°**	**60°**	**90°**
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	

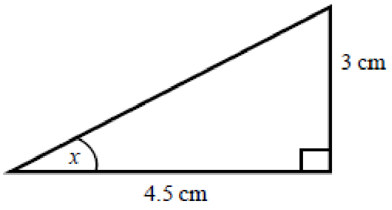
Examples

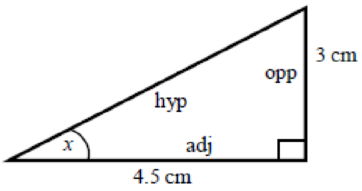
Example 1 Calculate the length of side x.
Give your answer correct to 3 significant figures.



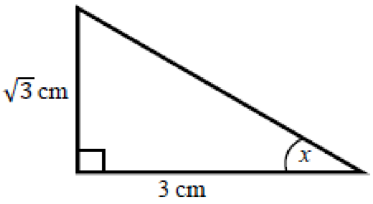
	**1** Always start by labelling the sides.
$\cos\theta = \frac{\text{adj}}{\text{hyp}}$	**2** You are given the adjacent and the hypotenuse so use the cosine ratio.
$\cos 25° = \frac{6}{x}$	**3** Substitute the sides and angle into the cosine ratio.
$x = \frac{6}{\cos 25°}$	**4** Rearrange to make x the subject.
$x = 6.620\ 267\ 5...$	**5** Use your calculator to work out 6 ÷ cos 25°.
$x = 6.62$ cm	**6** Round your answer to 3 significant figures and write the units in your answer.

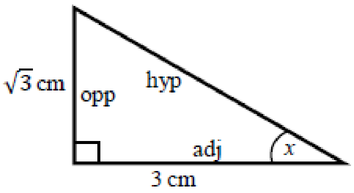
Example 2 Calculate the size of angle x.
Give your answer correct to 3 significant figures.



	**1** Always start by labelling the sides.
$\tan\theta = \frac{\text{opp}}{\text{adj}}$	**2** You are given the opposite and the adjacent so use the tangent ratio.
$\tan x = \frac{3}{4.5}$	**3** Substitute the sides and angle into the tangent ratio.
$x = \tan^{-1}\left(\frac{3}{4.5}\right)$	**4** Use $\tan^{-1}$ to find the angle.
$x = 33.690\ 067\ 5...$	**5** Use your calculator to work out $\tan^{-1}(3 \div 4.5)$.
$x = 33.7°$	**6** Round your answer to 3 significant figures and write the units in your answer.

Example 3 Calculate the exact size of angle *x*.

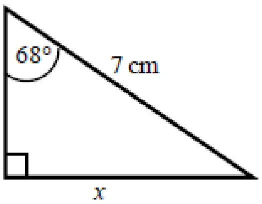


	**1** Always start by labelling the sides.
$\tan\theta = \frac{\text{opp}}{\text{adj}}$	**2** You are given the opposite and the adjacent so use the tangent ratio.
$\tan x = \frac{\sqrt{3}}{3}$	**3** Substitute the sides and angle into the tangent ratio.
$x = 30°$	**4** Use the table from the key points to find the angle.

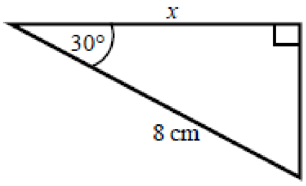
Practice

1 Calculate the length of the unknown side in each triangle.
Give your answers correct to 3 significant figures.

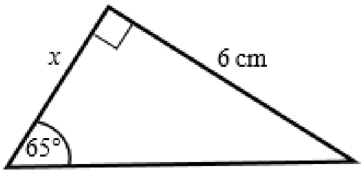
a



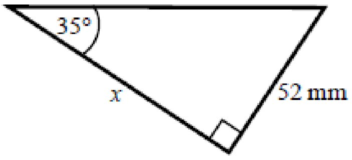
b



c

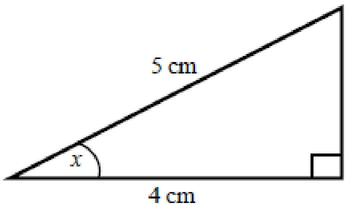


d

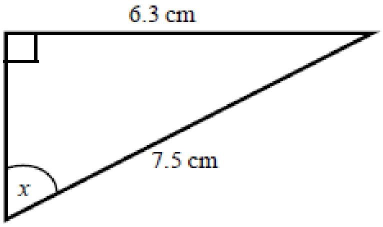


2 Calculate the size of angle *x* in each triangle.
Give your answers correct to 1 decimal place.

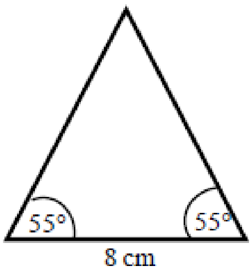
a



b



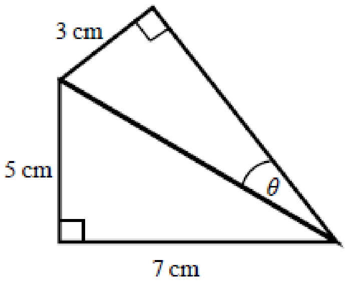
3 Work out the height of the isosceles triangle.
Give your answer correct to 3 significant figures.



Hint:

Split the triangle into two right-angled triangles.

4 Calculate the size of angle θ.
Give your answer correct to 1 decimal place.

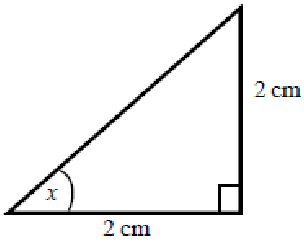


Hint:

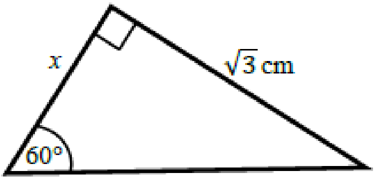
First work out the length of the common side to both triangles, leaving your answer in surd form.

5 Find the exact value of x in each triangle.

a



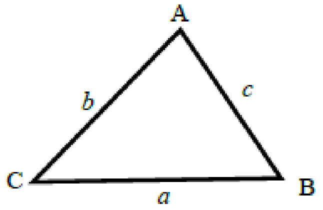
b



The cosine rule

Key points

- *a* is the side opposite angle A.
  b is the side opposite angle B.
  c is the side opposite angle C.

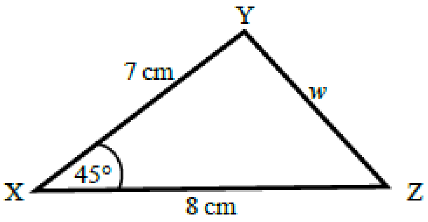


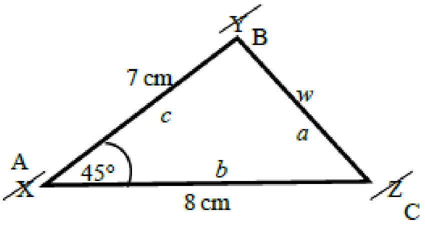
- You can use the cosine rule to find the length of a side when two sides and the included angle are given.
- To calculate an unknown side use the formula $a^2 = b^2 + c^2 - 2bc\cos A$.

- Alternatively, you can use the cosine rule to find an unknown angle if the lengths of all three sides are given.
- To calculate an unknown angle use the formula $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

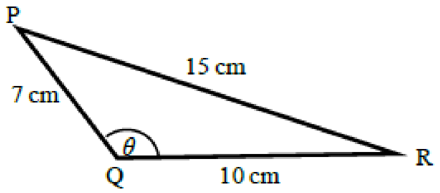
Examples

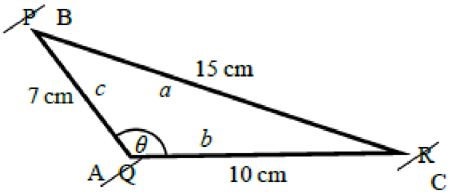
Example 1 Work out the length of side *w*.
Give your answer correct to 3 significant figures.



	**1** Always start by labelling the angles and sides.
$a^2 = b^2 + c^2 - 2bc\cos A$	**2** Write the cosine rule to find the side.
$w^2 = 8^2 + 7^2 - 2 \times 8 \times 7 \times \cos 45°$	**3** Substitute the values *a*, *b* and *A* into the formula.
$w^2 = 33.804\,040\,51...$ $w = \sqrt{33.80404051}$ $w = 5.81$ cm	**4** Use a calculator to find w^2 and then *w*. **5** Round your final answer to 3 significant figures and write the units in your answer.

Example 2 Work out the size of angle θ.
Give your answer correct to 1 decimal place.

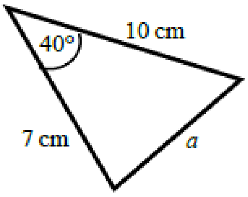


	**1** Always start by labelling the angles and sides.
$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$	**2** Write the cosine rule to find the angle.
$\cos\theta = \dfrac{10^2 + 7^2 - 15^2}{2 \times 10 \times 7}$	**3** Substitute the values *a*, *b* and *c* into the formula.
$\cos\theta = \dfrac{-76}{140}$	**4** Use $\cos^{-1}$ to find the angle.
$\theta = 122.878\ 349...$	**5** Use your calculator to work out $\cos^{-1}(-76 \div 140)$.
$\theta = 122.9°$	**6** Round your answer to 1 decimal place and write the units in your answer.

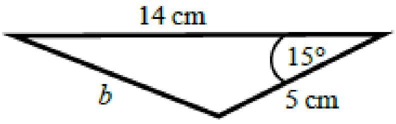
Practice

1 Work out the length of the unknown side in each triangle.
Give your answers correct to 3 significant figures.

a

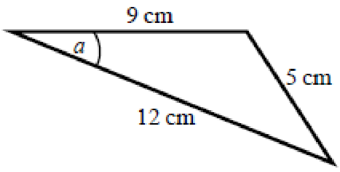


b

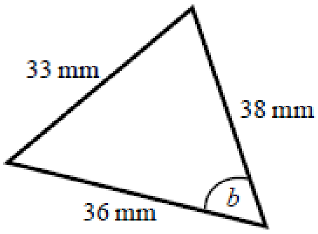


2 Calculate the angles labelled θ in each triangle.
Give your answer correct to 1 decimal place.

a

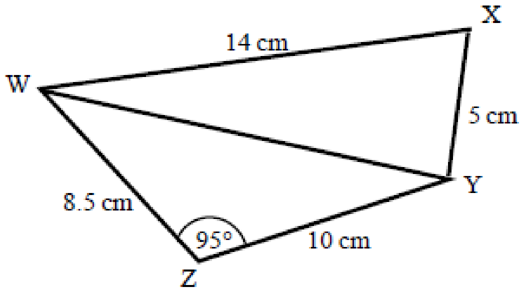


b



3 **a** Work out the length of WY.
Give your answer correct to
3 significant figures.

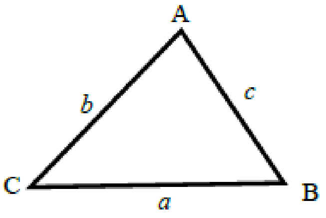
b Work out the size of angle WXY.
Give your answer correct to
1 decimal place.



The sine rule

Key points

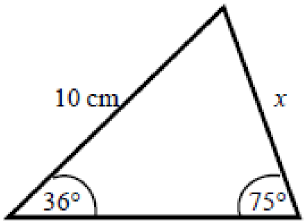
- *a* is the side opposite angle A.
  b is the side opposite angle B.
  c is the side opposite angle C.

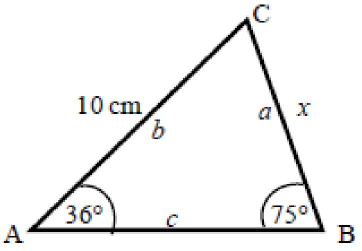


- You can use the sine rule to find the length of a side when its opposite angle and another opposite side and angle are given.
- To calculate an unknown side use the formula $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
- Alternatively, you can use the sine rule to find an unknown angle if the opposite side and another opposite side and angle are given.
- To calculate an unknown angle use the formula $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

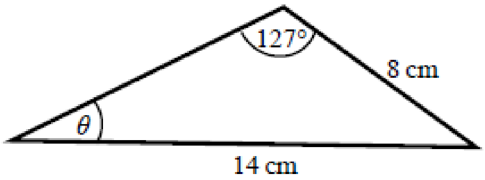
Examples

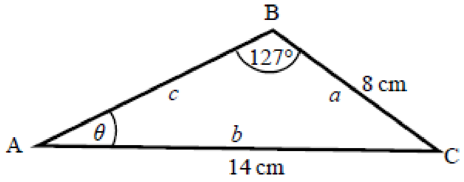
Example 1 Work out the length of side *x*.
Give your answer correct to 3 significant figures.



	**1** Always start by labelling the angles and sides.
$\frac{a}{\sin A} = \frac{b}{\sin B}$	**2** Write the sine rule to find the side.
$\frac{x}{\sin 36°} = \frac{10}{\sin 75°}$	**3** Substitute the values *a*, *b*, *A* and *B* into the formula.
$x = \frac{10 \times \sin 36°}{\sin 75°}$	**4** Rearrange to make *x* the subject.
$x = 6.09$ cm	**5** Round your answer to 3 significant figures and write the units in your answer.

Example 2 Work out the size of angle θ.
Give your answer correct to 1 decimal place.

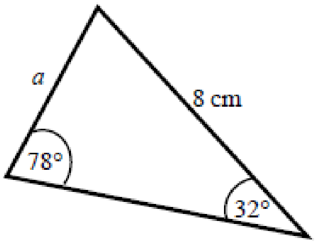


	**1** Always start by labelling the angles and sides.
$\frac{\sin A}{a} = \frac{\sin B}{b}$	**2** Write the sine rule to find the angle.
$\frac{\sin\theta}{8} = \frac{\sin 127°}{14}$	**3** Substitute the values *a*, *b*, *A* and *B* into the formula.
$\sin\theta = \frac{8 \times \sin 127°}{14}$	**4** Rearrange to make sin θ the subject.
θ = 27.2°	**5** Use $\sin^{-1}$ to find the angle. Round your answer to 1 decimal place and write the units in your answer.

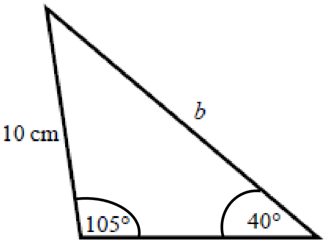
Practice

1 Find the length of the unknown side in each triangle.
Give your answers correct to 3 significant figures.

a

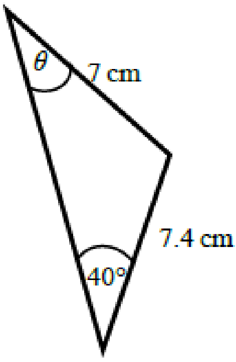


b

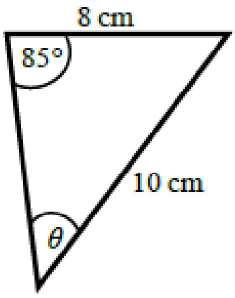


2 Calculate the angles labelled θ in each triangle.
Give your answer correct to 1 decimal place.

a

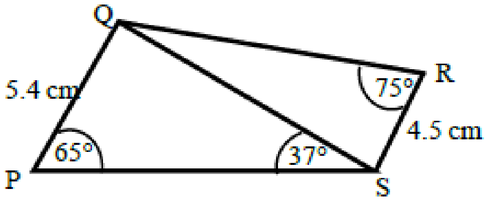


b



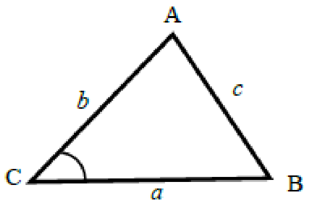
3 **a** Work out the length of QS.
Give your answer correct to 3 significant figures.

b Work out the size of angle RQS.
Give your answer correct to 1 decimal place.



Areas of triangles

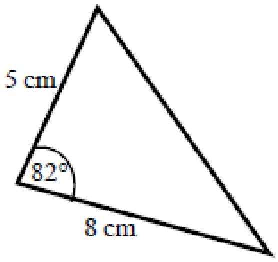
Key points

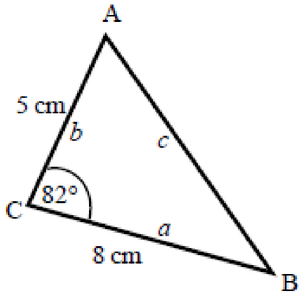


- a is the side opposite angle A.
  b is the side opposite angle B.
  c is the side opposite angle C.
- The area of the triangle is $\frac{1}{2}ab\sin C$.

Examples

Example 1 Find the area of the triangle.

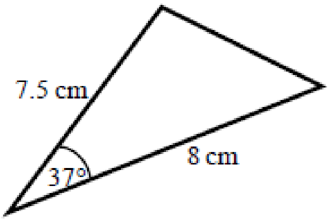


	**1** Always start by labelling the sides and angles of the triangle.
Area = $\frac{1}{2}ab\sin C$	**2** State the formula for the area of a triangle.
Area = $\frac{1}{2}\times 8\times 5\times \sin 82°$	**3** Substitute the values of a, b and C into the formula for the area of a triangle.
Area = 19.805 361...	**4** Use a calculator to find the area.
Area = 19.8 cm^2	**5** Round your answer to 3 significant figures and write the units in your answer.

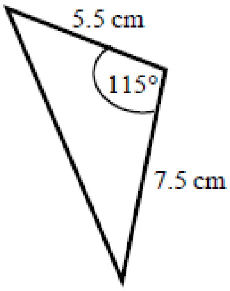
Practice

1 Work out the area of each triangle.
Give your answers correct to 3 significant figures.

a



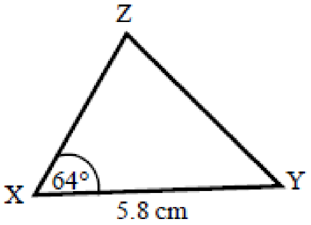
b



2 The area of triangle XYZ is 13.3 cm^2.
Work out the length of XZ.

Hint:

Rearrange the formula to make a side the subject.

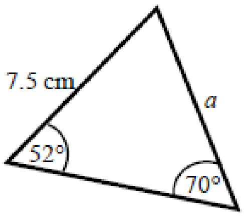


3 Find the size of each lettered angle or side.
Give your answers correct to 3 significant figures.

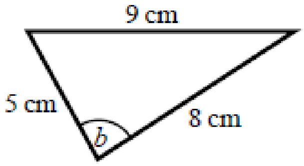
Hint:

For each one, decide whether to use the cosine or sine rule.

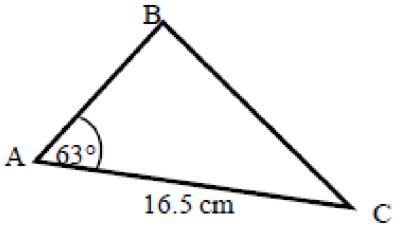
a



b



4 The area of triangle ABC is 86.7 cm^2.
Work out the length of BC.
Give your answer correct to 3 significant figures.



Rearranging equations

Key points

- To change the subject of a formula, get the terms containing the subject on one side and everything else on the other side.
- You may need to factorise the terms containing the new subject.

Examples

Example 1 Make *t* the subject of the formula $v = u + at$.

$v = u + at$ $v - u = at$ $t = \frac{v - u}{a}$	**1** Get the terms containing *t* on one side and everything else on the other side. **2** Divide throughout by *a*.

Example 2 Make *t* the subject of the formula $r = 2t - \pi t$.

$r = 2t - \pi t$ $r = t(2 - \pi)$ $t = \frac{r}{2 - \pi}$	**1** All the terms containing *t* are already on one side and everything else is on the other side. **2** Factorise as *t* is a common factor. **3** Divide throughout by $2 - \pi$.

Example 3 Make *t* the subject of the formula $\frac{t + r}{5} = \frac{3t}{2}$.

$\frac{t + r}{5} = \frac{3t}{2}$ $2t + 2r = 15t$ $2r = 13t$ $t = \frac{2r}{13}$	**1** Remove the fractions first by multiplying throughout by 10. **2** Get the terms containing *t* on one side and everything else on the other side and simplify. **3** Divide throughout by 13.

Example 4 Make t the subject of the formula $r = \frac{3t+5}{t-1}$.

$r = \frac{3t+5}{t-1}$ $r(t-1) = 3t + 5$ $rt - r = 3t + 5$ $rt - 3t = 5 + r$ $t(r-3) = 5 + r$ $t = \frac{5+r}{r-3}$	**1** Remove the fraction first by multiplying throughout by $t - 1$. **2** Expand the brackets. **3** Get the terms containing t on one side and everything else on the other side. **4** Factorise the LHS as t is a common factor. **5** Divide throughout by $r - 3$.

Practice

Change the subject of each formula to the letter given in the brackets.

1 $C = \pi d$ $[d]$

2 $P = 2l + 2w$ $[w]$

3 $D = \frac{S}{T}$ $[T]$

4 $\frac{y-7x}{2} = \frac{7-2y}{3}$ $[y]$

5 $x = \frac{2a-1}{3-a}$ $[a]$

6 $x = \frac{b-c}{d}$ $[d]$

7 Make r the subject of the following formulae.

a $A = \pi r^2$

b $V = \frac{4}{3}\pi r^3$

8 Make x the subject of the following formulae.

a $\frac{xy}{z} = \frac{ab}{cd}$

b $\frac{4\pi cx}{d} = \frac{3z}{py^2}$

9 Make $\sin B$ the subject of the formula $\frac{a}{\sin A} = \frac{b}{\sin B}$

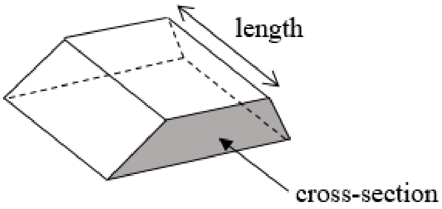
10 Make x the subject of the following equations.

a $\frac{p}{q}(sx+t) = x - 1$

b $\frac{p}{q}(ax+2y) = \frac{3p}{q^2}(x-y)$

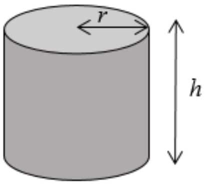
Volume and surface area of 3D shapes

Key points

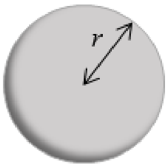


- Volume of a prism = cross-sectional area × length.
- The surface area of a 3D shape is the total area of all its faces.

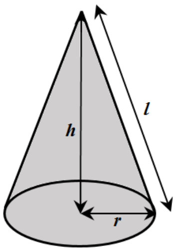
- Volume of a pyramid = $\frac{1}{3}$ × area of base × vertical height.



- Volume of a cylinder = $\pi r^2 h$
- Total surface area of a cylinder = $2\pi r^2 + 2\pi rh$



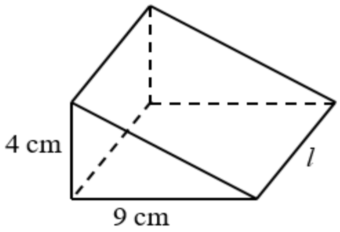
- Volume of a sphere = $\frac{4}{3}\pi r^3$
- Surface area of a sphere = $4\pi r^2$



- Volume of a cone = $\frac{1}{3}\pi r^2 h$
- Total surface area of a cone = $\pi rl + \pi r^2$

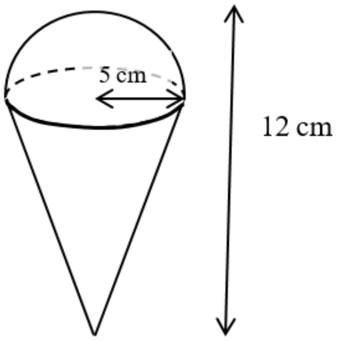
Examples

Example 1 The triangular prism has volume 504 cm^3.
Work out its length.



$V = \frac{1}{2}bhl$	**1** Write out the formula for the volume of a triangular prism.
$504 = \frac{1}{2} \times 9 \times 4 \times l$	**2** Substitute known values into the formula.
$504 = 18 \times l$	**3** Simplify
$l = 504 \div 18$ = 28 cm	**4** Rearrange to work out *l*. **5** Remember the units.

Example 2 Calculate the volume of the 3D solid.
Give your answer in terms of π.

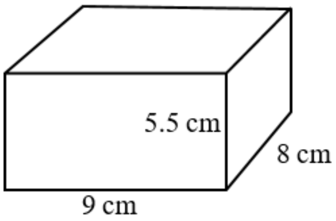


Total volume = volume of hemisphere + Volume of cone $= \frac{1}{2}$ of $\frac{4}{3}\pi r^3 + \frac{1}{3}\pi r^2 h$	**1** The solid is made up of a hemisphere radius 5 cm and a cone with radius 5 cm and height 12 − 5 = 7 cm.
Total volume = $\frac{1}{2} \times \frac{4}{3} \times \pi \times 5^3$ $+ \frac{1}{3} \times \pi \times 5^2 \times 7$	**2** Substitute the measurements into the formula for the total volume.
$= \frac{425}{3}\pi \text{ cm}^3$	**3** Remember the units.

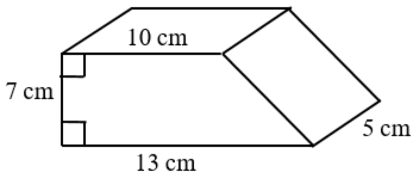
Practice

1 Work out the volume of each solid.
Leave your answers in terms of π where appropriate.

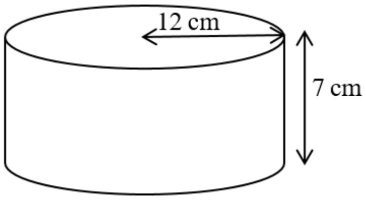
a



b

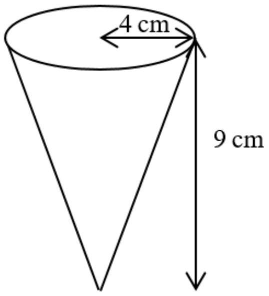


c



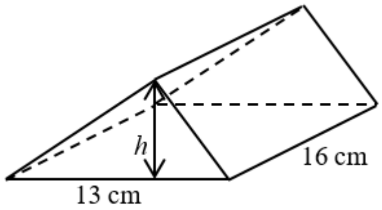
d a sphere with diameter 9 cm

e



2 A cuboid has width 9.5 cm, height 8 cm and volume 1292 cm^3.
Work out its length.

3 The triangular prism has volume 1768 cm^3.
Work out its height.



4 The diagram shows a large catering size tin of beans in the shape of a cylinder.

The tin has a radius of 8 cm and a height of 15 cm.

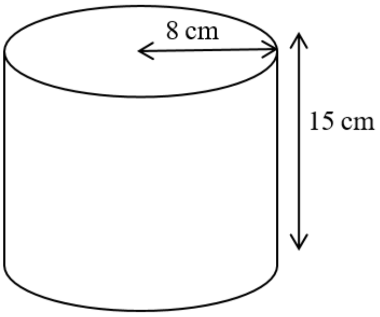
A company wants to make a new size of tin.

The new tin will have a radius of 6.7 cm.

It will have the same volume as the large tin.

Calculate the height of the new tin.

Give your answer correct to one decimal place.

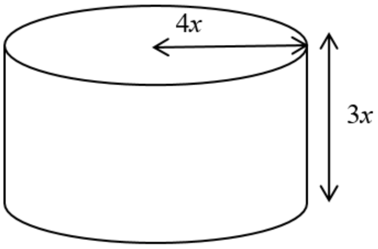


5 The diagram shows a solid metal cylinder.

The cylinder has base radius $4x$ and height $3x$.

The cylinder is melted down and made into a sphere of radius r.

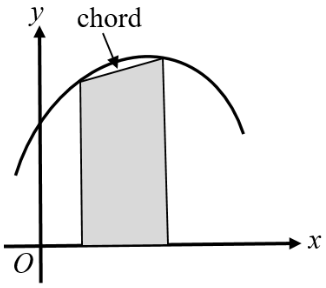
Find an expression for r in terms of x.



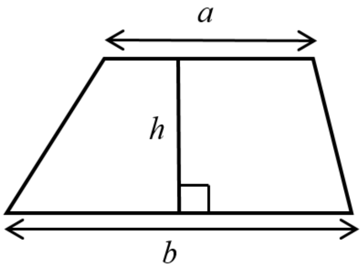
Area under a graph

Key points

- To estimate the area under a curve, draw a chord between the two points you are finding the area between and straight lines down to the horizontal axis to create a trapezium.
The area of the trapezium is an approximation for the area under a curve.

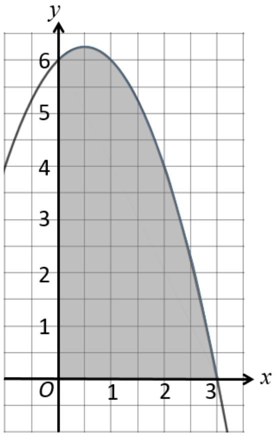


- The area of a trapezium = $\frac{1}{2}h(a+b)$



Examples

Example 1 Estimate the area of the region between the curve $y = (3 - x)(2 + x)$ and the x-axis from $x = 0$ to $x = 3$. Use three strips of width 1 unit.



x	0	1	2	3
$y = (3 - x)(2 + x)$	6	6	4	0

Trapezium 1:
$a_1 = 6 - 0 = 6$, $b_1 = 6 - 0 = 6$
Trapezium 2:
$a_2 = 6 - 0 = 6$, $b_2 = 4 - 0 = 4$
Trapezium 3:
$a_3 = 4 - 0 = 4$, $a_3 = 0 - 0 = 0$

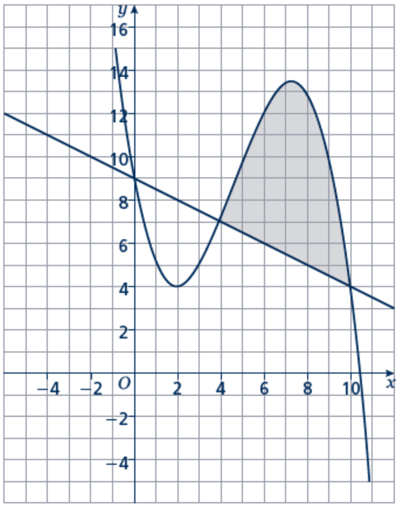
1 Use a table to record the value of y on the curve for each value of x.

2 Work out the dimensions of each trapezium. The distances between the y-values on the curve and the x-axis give the values for a.

(continued on next page)

$\frac{1}{2}h(a_1+b_1)=\frac{1}{2}\times1(6+6)=6$ $\frac{1}{2}h(a_2+b_2)=\frac{1}{2}\times1(6+4)=5$ $\frac{1}{2}h(a_3+b_3)=\frac{1}{2}\times1(4+0)=2$	**3** Work out the area of each trapezium. h = 1 since the width of each trapezium is 1 unit.
Area = 6 + 5 + 2 = 13 units^2	**4** Work out the total area. Remember to give units with your answer.

Example 2 Estimate the shaded area.
Use three strips of width 2 units.



<table>
<tr><td>
<table>
<tr><td><i>x</i></td><td>4</td><td>6</td><td>8</td><td>10</td></tr>
<tr><td><i>y</i></td><td>7</td><td>12</td><td>13</td><td>4</td></tr>
</table>
</td><td>1 Use a table to record <i>y</i> on the curve for each value of <i>x</i>.</td></tr>
<tr><td>
<table>
<tr><td><i>x</i></td><td>4</td><td>6</td><td>8</td><td>10</td></tr>
<tr><td><i>y</i></td><td>7</td><td>6</td><td>5</td><td>4</td></tr>
</table>
</td><td>2 Use a table to record <i>y</i> on the straight line for each value of <i>x</i>.</td></tr>
<tr><td>Trapezium 1:
$a_1=7-7=0$, $b_1=12-6=6$
Trapezium 2:
$a_2=12-6=6$, $b_2=13-5=8$
Trapezium 3:
$a_3=13-5=8$, $a_3=4-4=0$</td><td>3 Work out the dimensions of each trapezium. The distances between the <i>y</i>-values on the curve and the <i>y</i>-values on the straight line give the values for <i>a</i>.</td></tr>
<tr><td>$\frac{1}{2}h(a_1+b_1)=\frac{1}{2}\times2(0+6)=6$
$\frac{1}{2}h(a_2+b_2)=\frac{1}{2}\times2(6+8)=14$
$\frac{1}{2}h(a_3+b_3)=\frac{1}{2}\times2(8+0)=8$</td><td>4 Work out the area of each trapezium. <i>h</i> = 2 since the width of each trapezium is 2 units.</td></tr>
<tr><td>Area = 6 + 14 + 8 = 28 units^2</td><td>5 Work out the total area. Remember to give units with your answer.</td></tr>
</table>

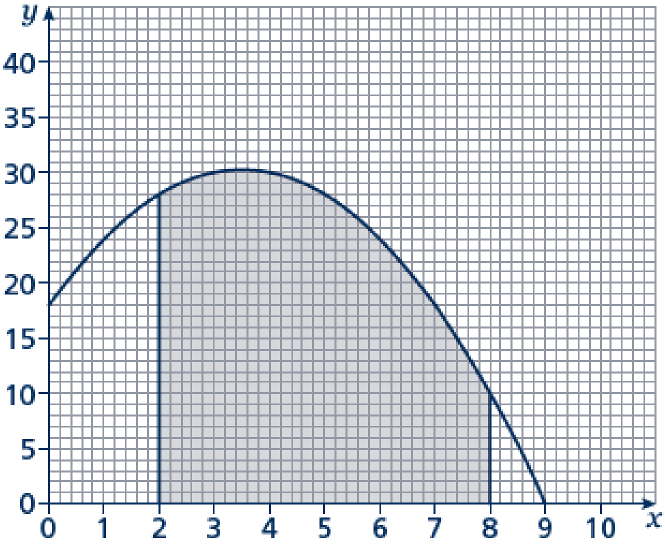
Practice

Hint:

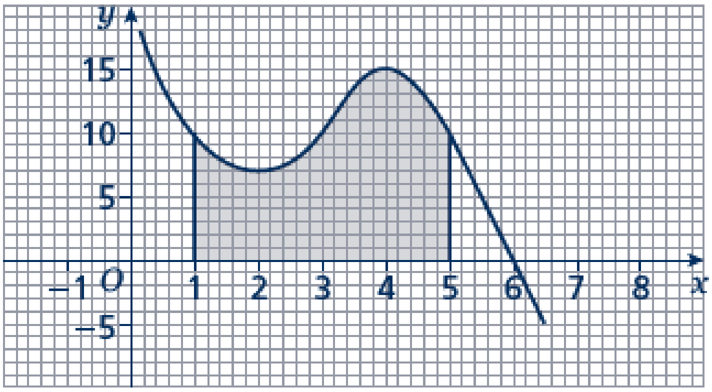
For a full answer, remember to include ‘units^2’.

1 Estimate the area of the region between the curve $y = (5 - x)(x + 2)$ and the x-axis from $x = 1$ to $x = 5$.
Use four strips of width 1 unit.

2 Estimate the shaded area shown on the axes.
Use six strips of width 1 unit.



3 Estimate the shaded area.
Use four strips of equal width.



4 Estimate the area of the region between the curve $y = -x^2 + 2x + 15$ and the x-axis from $x = 2$ to $x = 5$.
Use six strips of equal width.

5 The curve $y = 8x - 5 - x^2$ and the line $y = 2$ are shown in the sketch.
Estimate the shaded area using six strips of equal width.