Introduction to Game Theory Seminar

University of Warwick

Pre-University Summer School

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What we will do

- Go through some essential game-theory concepts and definitions.
- Play and analyse the *Beauty Contest game* (Question 1).
- Try to solve the *Hawk–Dove game* (Question 2).
- Explore some advanced material if time permits.

Essential game-theory concepts

What is game theory?

Game theory studies situations in which the outcome for each decision maker depends on **their own choice** and on the **choices of others**.

A game normally specifies:

1. **Players**: who makes decisions?
2. **Actions**: what can each player choose?
3. **Information**: what does each player know when choosing?
4. **Payoffs**: how much does each player value every possible outcome?

Payoffs and utility

A player's **payoff**, or utility, is a number representing how much that player values an outcome. Players prefer higher payoffs.

- Payoffs need not be money: they can represent profit, points, probability of winning, or general wellbeing.
- Utility is an *ordinal measure*: only the ordering matters. A payoff of 10 is preferred to a payoff of 5, but it need not be "twice as good".
- We assume the payoff number incorporates everything relevant to the player. We can therefore compare the payoffs rather than separately listing every benefit and cost.
- Each player maximises *their own* payoff, not necessarily the total payoff of the group.

How to read a payoff matrix

Consider the Prisoner's Dilemma game, and we will use it only to recall the notation.

		Ben	
		Confess	Silent
Alex	Confess	$(-10, -10)$	$(0, -12)$
	Silent	$(-12, 0)$	$(-1, -1)$

- Alex chooses a row; Ben chooses a column.
- The first number in a cell is Alex's payoff; the second is Ben's payoff.
- An outcome lists one strategy for each player. We also call it a **strategy profile**.
- For example, $(\text{Confess}, \text{Confess})$ is both an outcome and a strategy profile; the resulting payoffs are $(-10, -10)$.

Best responses

A player's **best response** is the strategy that gives that player the highest payoff, given the opponent's strategy.

To find a best response:

1. Fix the opponent's strategy.
2. Compare your available payoffs.
3. Choose the strategy giving the highest payoff.

A best response is therefore a **function** of the opponent's strategy: your best response is conditional on what your opponent chooses.

Strict dominance and strict domination

- A strategy is **strictly dominant** if it gives a higher payoff than any other strategy for every possible opponent action.
- A strategy is **strictly dominated** if it gives a lower payoff than a strategy for every possible opponent action.

		Player 2		
		Left	Centre	Right
Player 1	Up	$(4, 3)$	$(3, 2)$	$(5, 1)$
	Down	$(1, 1)$	$(2, 0)$	$(0, 4)$

- For Player 1, **Up strictly dominates Down**:

$$4 > 1, \qquad 3 > 2, \qquad 5 > 0.$$

- For Player 2, **Centre is strictly dominated by Left**:

$$3 > 2 \quad \text{and} \quad 1 > 0.$$

Nash equilibrium

- A **Nash equilibrium** is a strategy profile in which every player is choosing a best response to the strategies chosen by the other players.
- In other words, no player wants to *unilaterally* change their strategy. No player can obtain a higher payoff by changing *only* their own strategy.

		Player 2	
		A	B
Player 1	A	$(2, 2)$	$(0, 0)$
	B	$(0, 0)$	$(1, 1)$

- At (A, A), either player who changes alone falls from 2 to 0. Hence (A, A) is a Nash equilibrium.
- At (B, B), either player who changes alone falls from 1 to 0. Hence (B, B) is also a Nash equilibrium.
- A game can therefore have more than one Nash equilibrium.

Pure and mixed strategies

Suppose a player has two actions, A and B.

- A **pure strategy** chooses one action with certainty. For example, the pure strategy is to choose A.
- A **mixed strategy** randomises between the actions. The strategy is now the probability:

$$pA + (1 - p)B, \qquad p \in [0, 1].$$

- The player chooses A with probability p and B with probability $1 - p$.

Question 1: The Beauty Contest game

Question 1: The rules

Everyone chooses one integer from 1 to 100.

Players are trying to choose a number as close as possible to

$$\frac{2}{3} \times \text{the average.}$$

You are not trying to guess the average itself. You are trying to get close to $\frac{2}{3}$ of the average.

The formal payoff formula

The seminar question writes player i's payoff as

$$U_i(s_i, s_{-i}) = 100 - \left| s_i - \frac{2\sum_{j=1}^{n} s_j}{3n} \right|.$$

In words:

Start from 100 and subtract the distance between player i's choice and $\frac{2}{3}$ of the average.

A smaller distance implies a larger payoff.

In other words, players are trying to get as close as possible to $\frac{2}{3}$ of the average.

Understanding the notation

$$U_i(s_i, s_{-i}) = 100 - \left| s_i - \frac{2\sum_{j=1}^{n} s_j}{3n} \right|.$$

- i is referring to the player i; the player that we are focusing on
- s_i is player i's selected number.
- U_i is player i's payoff.
- s_{-i} means the choices of *all players other than player* i. Read "$-i$" as "everyone except i".
- n is the total number of players, so $\frac{\sum_{j=1}^{n} s_j}{n}$. is the average.
- $\left| s_i - \frac{2\sum_{j=1}^{n} s_j}{3n} \right|$ is the distance between i's number and the target.

Understanding the payoff formula

$$U_i(s_i, s_{-i}) = 100 - \underbrace{\left| s_i - \overbrace{\frac{2\sum_{j=1}^{n} s_j}{3n}}^{\text{target}} \right|}_{\text{distance from player } i\text{'s choice to the target}}.$$

- The **target** is two-thirds of the average.
- The absolute value measures the **distance** between player i's selected number and the target.
- A smaller distance gives a higher payoff.

Players want to get as close as possible to $\frac{2}{3}$ of the average.

Numerical example

Beauty Contest: Round 1

Choose one integer from 1 to 100

Write your choice on the piece of paper.

Do not discuss your choice with anyone else.

Open Excel calculator

Beauty Contest: Round 2

Choose again, choose one integer from 1 to 100

Write your choice on the piece of paper.

You may revise your reasoning after seeing the Round 1 result.

Open Excel calculator

Compare Rounds 1 and 2

Compare the two sets of choices:

- Did the average fall?
- Did the number with the highest payoff fall?
- How did seeing Round 1 change beliefs about how other students would reason?

A lower Round 2 average suggests that people responded strategically after learning from Round 1.

Question 1(a)

(a). How might you expect players to play this game? Which players are more successful - players picking high numbers, medium numbers or small numbers?

Solution (a): A natural first thought is:

- Numbers range from 1 to 100, so perhaps the average will be around 50.
- Then $\frac{2}{3}$ of the average would be about

$$\frac{2}{3} \times 50 \approx 33.$$

- A player following this first step may therefore choose a number near 33.

Very high choices should perform poorly because the target can never exceed 66.67. Whether a medium or low choice performs best depends on the class's actual choices.

Question 1(a): Reasoning about other players

The choice 33 is based on expecting an average near 50. But other players may also reason strategically.

If many players choose near 33, then

$$\frac{2}{3} \times 33 = 22.$$

If players anticipate this, they may choose near 22.

Reasoning one step further gives

$$\frac{2}{3} \times 22 \approx 15.$$

The game therefore depends on beliefs about *how other people reason* and how many steps of reasoning they will use.

Question 1(b)

(b). Can we apply the idea of dominated strategies to this game? Iteratively eliminate strictly dominated strategies and see what remains.

Solution (b): The highest possible average is 100. Therefore the highest possible target is

$$\frac{2}{3} \times 100 = \frac{200}{3} \approx 66.67.$$

No target can ever be above 66.67.

First round of elimination

Because the target is never above 66.67, choosing 67 is always closer to the target than choosing 68, 69, . . . , or 100.

Therefore 68, 69, . . . , or 100 are strictly dominated by 67, so we eliminate

$$68, 69, \ldots, 100.$$

The remaining choices are $1, 2, \ldots, 67$.

Second round of elimination

After choices above 67 are removed, the highest possible average is 67. The highest possible target becomes

$$\frac{2}{3} \times 67 \approx 44.67.$$

Therefore 45 is always closer to the target than any choice between 46, 47, . . . , 67. Eliminate

$$46, 47, \ldots, 67.$$

Third elimination

After choices above 45 are removed, the highest possible average is 45. The highest possible target becomes

$$\frac{2}{3} \times 45 = 30.$$

Using the same reasoning, eliminate

$$31, 32, \ldots, 45.$$

Continue the same reasoning

Repeating the argument gives

$$100 \longrightarrow 67 \longrightarrow 45 \longrightarrow 30 \longrightarrow 20 \longrightarrow 14 \longrightarrow 10 \longrightarrow 7 \longrightarrow 5 \longrightarrow 4 \longrightarrow 3 \longrightarrow 2.$$

At every step, removing high choices lowers the highest possible average, and therefore lowers the highest possible target.

This is *iterated elimination of strictly dominated strategies*.

The final elimination

In the final round, only 1 and 2 remain. The highest possible average is 2, so the highest possible target is

$$\frac{2}{3} \times 2 = \frac{4}{3} \approx 1.33.$$

1 is closer to the target than 2. Therefore 2 is strictly dominated by 1.

Only the strategy 1 remains.

Question 1(c)

(c). What are the Nash Equilibria of this game?

Solution (c): Iterated elimination leaves only one possible strategy for every player:

$$(s_1, s_2, \ldots, s_n) = (1, 1, \ldots, 1).$$

If all other players choose 1, choosing a larger number moves a player further away from the target. Hence choosing 1 is a best response.

The unique Nash equilibrium is that every player chooses 1.

Equilibrium and our result

The equilibrium prediction does not mean that everybody must choose 1 in our two rounds.

- People may use different numbers of reasoning steps.
- They may hold different beliefs about their opponents.
- Actual play may not immediately reach the equilibrium prediction.
- If the game is played repeatedly, it may converge to the equilibrium.

The experiment lets us compare a formal game-theory prediction with observed behaviour.

Question 2: The Hawk–Dove game

Question 2: The situation

Two players compete for a resource worth 10 units. Each chooses:

- **Dove (D)**: act gently and share if the other player also chooses Dove.
- **Hawk (H)**: act aggressively and try to take the whole resource.
- Both choose Dove: each receives 5.
- One chooses Hawk and the other Dove: Hawk receives 10 and Dove receives 0.
- Both choose Hawk: each receives 5 but suffers injury cost c, giving $5 - c$.

Question 2(a)

(a). Draw a payoff matrix representing this game.

Solution (a):

		Player 2	
		H	D
Player 1	H	$(5-c, 5-c)$	$(10, 0)$
	D	$(0, 10)$	$(5, 5)$

The first number in each cell is Player 1's payoff; the second is Player 2's payoff.

Only the payoff from the outcome (H, H) depends on the injury cost:

$$(H, H) = (5-c, 5-c).$$

A larger c makes fighting less attractive.

Question 2(b): Set $c = 10$

When $c = 10$, we have $5 - c = -5$:

		Player 2	
		H	D
Player 1	H	$(-5, -5)$	$(10, 0)$
	D	$(0, 10)$	$(5, 5)$

Question 2(b)(i)

(b)(i). Suppose Player 1 selects Hawk with probability $p \in [0, 1]$ and so Dove with probability $1 - p$. Calculate the expected utility to Player 2 of choosing Hawk and the expected utility of choosing Dove. (Answers should be functions of p).

Solution (b)(i): Player 1's mixed strategy is

$$pH + (1 - p)D, \qquad p \in [0, 1].$$

- $p = 1$: always H.
- $p = 0$: always D.
- $0 < p < 1$: randomise between H and D.

Expected payoff: the basic rule

When an outcome is uncertain, expected payoff is a weighted average:

$$\mathbb{E}[U] = (\text{probability of outcome 1})(\text{payoff 1}) + (\text{probability of outcome 2})(\text{payoff 2}).$$

For Player 2, the uncertainty comes from Player 1 choosing H with probability p and D with probability $1 - p$.

Player 2's expected payoff from H

Payoff matrix when $c = 10$

		Player 2	
		H	D
Player 1	H	$(-5, -5)$	$(10, 0)$
	D	$(0, 10)$	$(5, 5)$

The second number in each cell is Player 2's payoff.

Suppose Player 2 chooses H.

- If Player 1 chooses H, Player 2 receives -5.
- If Player 1 chooses D, Player 2 receives 10.
- Player 1 chooses H with probability p and D with probability $1 - p$.

Player 2's expected payoff is the probability-weighted average:

$$U_2(H) = p(-5) + (1 - p)10.$$

$$U_2(H) = 10 - 15p.$$

Player 2's expected payoff from D

Payoff matrix when $c = 10$

		Player 2	
		H	D
Player 1	H	$(-5, -5)$	$(10, 0)$
	D	$(0, 10)$	$(5, 5)$

The second number in each cell is Player 2's payoff.

Suppose Player 2 chooses D.

- If Player 1 chooses H, Player 2 receives 0.
- If Player 1 chooses D, Player 2 receives 5.
- Player 1 chooses H with probability p and D with probability $1 - p$.

Player 2's expected payoff is the probability-weighted average:

$$U_2(D) = p(0) + (1 - p)5.$$

$$U_2(D) = 5 - 5p.$$

Question 2(b)(ii)

(b)(ii). Find the best response of Player 2 as a function of p.

Solution (b)(ii): Compare the expected payoffs:

$$\begin{aligned} U_2(H) &\geq U_2(D) \\ 10 - 15p &\geq 5 - 5p \\ 5 &\geq 10p \\ p &\leq \frac{1}{2}. \end{aligned}$$

Thus H is better when $p < \frac{1}{2}$, D is better when $p > \frac{1}{2}$, and Player 2 is indifferent when $p = \frac{1}{2}$.

Why is best response is a function of p?

The Player 2's best response depends on the probability of Player 1 chooses H, which is p.

A best response is therefore a function of the opponent's strategy.

In other words, it specifies the best action of Player 2, given every possible action of Player 1.

$$\mathrm{BR}_2(p) = \begin{cases} H, & p < \frac{1}{2}, \\ \text{any mixture of } H \text{ and } D, & p = \frac{1}{2}, \\ D, & p > \frac{1}{2}. \end{cases}$$

At $p = \frac{1}{2}$, the two pure actions give the same expected payoff, so any mixture also gives that payoff.

Question 2(b)(iii)

(b)(iii) Player 2 chooses H with probability q and D with probability $1-q$. Find Player 1's best response.

Solution (b)(iii):
Player 2's mixed strategy is

$$qH + (1-q)D.$$

		Player 2	
		H	D
Player 1	H	$(-5, -5)$	$(10, 0)$
	D	$(0, 10)$	$(5, 5)$

The first number in each cell is Player 1's payoff.

If Player 1 chooses H,

$$U_1(H) = q(-5) + (1-q)10 = 10 - 15q.$$

If Player 1 chooses D,

$$U_1(D) = q(0) + (1-q)5 = 5 - 5q.$$

Question 2(b)(iii): Best response

From the previous slide:

$$U_1(H) = 10 - 15q, \qquad U_1(D) = 5 - 5q.$$

Compare the expected payoffs:

$$\begin{aligned} U_1(H) \geq U_1(D) &\iff 10 - 15q \geq 5 - 5q \\ &\iff 5 \geq 10q \\ &\iff q \leq \frac{1}{2}. \end{aligned}$$

Therefore,

$$\mathrm{BR}_1(q) = \begin{cases} H, & q < \frac{1}{2}, \\ \text{any mixture of } H \text{ and } D, & q = \frac{1}{2}, \\ D, & q > \frac{1}{2}. \end{cases}$$

Question 2(b)(iv)

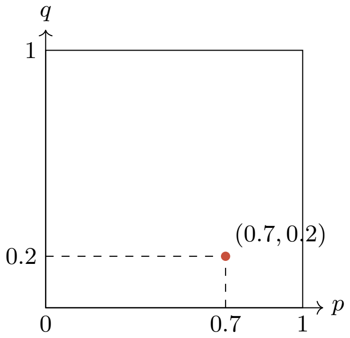
(b)(iv) Draw a diagram with $p \in [0,1]$ on the horizontal axis and $q \in [0,1]$ on the vertical axis. Discuss the relevance of the $[0,1]^2$ unit square. Draw the best responses of both players and find the Nash equilibria.

Solution (b)(iv):

- p is Player 1's probability of choosing H.
- q is Player 2's probability of choosing H.
- Every point (p,q) in the unit square is one mixed-strategy profile.

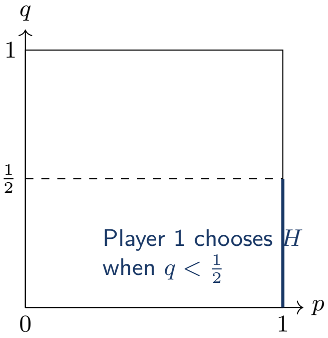
For example, consider the point:
$(p,q) = (0.7, 0.2)$

- Player 1 chooses H with probability 0.7 and D with probability 0.3.
- Player 2 chooses H with probability 0.2 and D with probability 0.8.



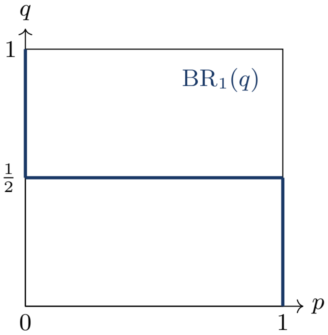
Drawing Player 1's best response: first region

If $q < \frac{1}{2}$, Player 1's best response is pure H, so $p = 1$.



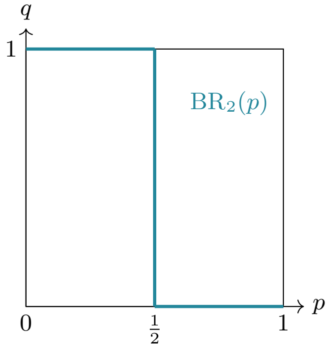
Drawing Player 1's complete best response

- If $q < \frac{1}{2}$, then $p = 1$.
- If $q = \frac{1}{2}$, any $p \in [0, 1]$ is a best response.
- If $q > \frac{1}{2}$, then $p = 0$.



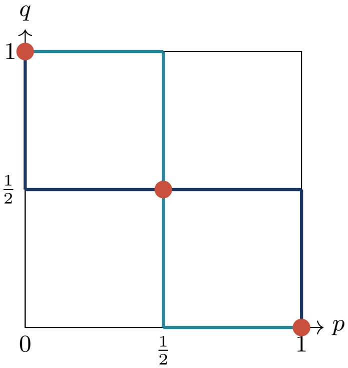
Drawing Player 2's best response

- If $p < \frac{1}{2}$, then $q = 1$.
- If $p = \frac{1}{2}$, any $q \in [0, 1]$ is a best response.
- If $p > \frac{1}{2}$, then $q = 0$.



Put both best responses together

A Nash equilibrium occurs where the two best-response functions intersect.



Question 2(b)(iv): Nash equilibria

The three intersections give three Nash equilibria:

1. $(p, q) = (1, 0)$: Player 1 chooses H and Player 2 chooses D.
2. $(p, q) = (0, 1)$: Player 1 chooses D and Player 2 chooses H.
3. $(p, q) = \left(\frac{1}{2}, \frac{1}{2}\right)$: both choose H half the time and D half the time.

Question 2(b)(v)

(b)(v). What type of game have we got?

Solution (b)(v): This is a **game of Chicken**.

- If the opponent chooses H, a player wants to choose D.
- If the opponent chooses D, a player wants to choose H.
- In a game of Chicken, the players want to choose **opposite actions**.

A common analogy is two countries in a military standoff: each wants the other country to back down, but if both escalate, the outcome can be very costly for both.

Question 2(c): Set $c = 3$

When $c = 3$, we have $5 - c = 2$:

		Player 2: H	Player 2: D
Player 1	H	$(2, 2)$	$(10, 0)$
	D	$(0, 10)$	$(5, 5)$

Question 2(c)(i)

(c)(i). Let players mix as before: Player 1 mix $pH + (1-p)D$ and Player 2 mix $qH + (1-q)D$ where $p, q \in [0, 1]$. Find Player 1's best response as a function of q and Player 2's best response as a function of p.

Solution (c)(i):

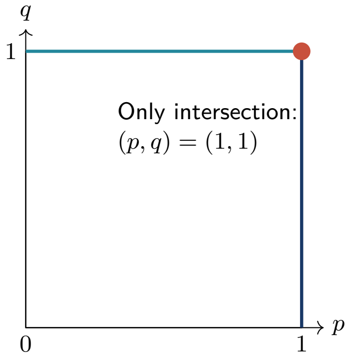
		Player 2	
		H	D
Player 1	H	$(\mathbf{2}, 2)$	$(\mathbf{10}, 0)$
	D	$(\mathbf{0}, 10)$	$(\mathbf{5}, 5)$

For Player 1, $2 > 0$ against H and $10 > 5$ against D. Thus H is better in both columns: H strictly dominates D.
By symmetry, H also strictly dominates D for Player 2.
Therefore, H is the best response for both players for any strategy their opponent chooses.

Question 2(c)(ii)

(c)(ii). Draw best responses on a diagram and find the NE.

Solution (c)(ii):
Player 1 always chooses H, so $p = 1$. Player 2 always chooses H, so $q = 1$.



Both players have the strictly dominant strategy H. Therefore the unique Nash equilibrium is

$$(H, H), \qquad (p, q) = (1, 1), \qquad \text{with payoff } (2, 2).$$

Question 2(c)(iii)

(c)(iii). What type of game have we got?

Solution (c)(iii): This has the structure of a **Prisoner's Dilemma**:

- Each player has a strictly dominant strategy: H.
- Dominant-strategy reasoning leads to (H, H) with payoffs $(2, 2)$.
- Both would be better off at (D, D) with payoffs $(5, 5)$.

Individual incentives produce an outcome that is worse for both players.

Question 2(d)

(d). Discuss what happens to our diagram and our set of NE as c changes between 3 and 10. What would happen for $c > 10$?

Solution (d): Suppose Player 2 chooses H with probability q.

$$U_1(H) = q(5-c) + (1-q)10,$$
$$U_1(D) = q(0) + (1-q)5.$$

The difference is

$$U_1(H) - U_1(D) = 5 - cq.$$

Thus Player 1 is indifferent when

$$q = \frac{5}{c}.$$

By symmetry, Player 2 is indifferent when $p = \frac{5}{c}$.

Low and high injury costs

- **Low injury cost, $c < 5$:** the threshold $5/c$ is above 1, so H is always better. Both always choose H, and the game has a Prisoner's Dilemma structure.
- **High injury cost, $c > 5$:** the threshold $5/c$ lies between 0 and 1. A player chooses H when the opponent is unlikely to choose H, but chooses D when the opponent is sufficiently likely to choose H. The game has a game of Chicken structure.

Economic interpretation of changing c

As the injury cost increases, fighting becomes less attractive.

- Players switch from **always fighting** at low c to **sometimes fighting** at high c.
- In the symmetric mixed equilibrium, the probability of choosing H is $5/c$, so it decreases as c increases.
- When $c = 10$,
$$\frac{5}{c} = \frac{5}{10} = \frac{1}{2},$$
so each player chooses H half the time.
- When $c > 10$, $5/c < 1/2$, so each player fights less than half the time in the mixed equilibrium.

Hawk–Dove: main lessons

- Changing one payoff can change the strategic structure of a game.
- For low c, the game is a Prisoner's Dilemma; for high c, it is a game of Chicken.
- Best responses can depend on the opponent's probability of an action.
- A Nash equilibrium is an intersection of the players' best responses.

Advanced material: Sequential games

Simultaneous and sequential games

In the games so far, players choose without observing the other player's action.

In a **sequential game**:

- one player moves first;
- another player observes that move;
- the second player then chooses a response.

The order of moves and the information available can change the outcome.

Game trees

A sequential game is often represented by a **game tree**.

- A node shows where a player makes a decision.
- A branch shows an available action.
- The tree moves from left to right or top to bottom.
- Terminal points show the final payoffs.

Strategy

Strategy in a sequential game. A strategy is a **complete plan of action**: it specifies what a player would do at every decision point where that player might be called upon to act.

This includes decision points that are **not** reached during actual play.

Backward induction

Backward induction. Solve a sequential game by starting at the final decision points, finding the optimal choices there, and then working backwards to earlier decisions.

Nash equilibrium and SPE are solution concepts

A **solution concept** is a rule used by game theorists to predict or select strategic outcomes.

- **Nash equilibrium**: no player wants to change their complete strategy alone.
- **Subgame-perfect equilibrium (SPE)**: strategies must also be optimal after every possible history of the game.

  SPE is a refinement of Nash equilibrium.

It requires the players to play optimally even when a decision node is not reached. (We will see this more clearly in the question).

Advanced material: Ultimatum Game

The bargaining problem

Player 1 proposes how to divide £1, or 100 pence.
Player 1 chooses:

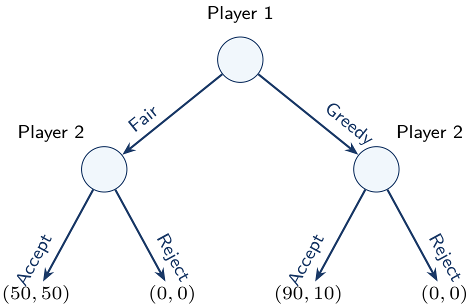
- **Fair**: 50 pence for each player.
- **Greedy**: 90 pence for Player 1 and 10 pence for Player 2.

Player 2 observes the offer and chooses Accept or Reject.
If Player 2 rejects, both players receive 0.

Question 1(a)

(a). Draw a tree diagram (game theorists call this extensive form) representing this game.

Solution (a):



Player 2 needs a complete strategy

Player 2 may have to respond after either offer.

Therefore Player 2's strategy must answer two questions:

1. What would Player 2 do after a Fair offer?
2. What would Player 2 do after a Greedy offer?

A strategy need to specify what the player would do after every possible scenario.

Player 2's four pure strategies

Use the first letter for the response after Fair and the second for the response after Greedy.

Strategy	After Fair	After Greedy
AA	Accept	Accept
AR	Accept	Reject
RA	Reject	Accept
RR	Reject	Reject

These are complete plans.

Question 1(b)

(b) Suppose Player 2 accepts regardless of what Player 1 offers. What should Player 1 offer? Do these strategies form a Nash equilibrium?

Solution (b):
If Player 2's strategy is AA:

Player 1's offer	Player 1's payoff
Fair	50
Greedy	90

Since $90 > 50$, Player 1 chooses Greedy.
The resulting strategy profile is

$$(G, AA).$$

To check whether this is a Nash equilibrium:

- Given AA, Greedy is Player 1's best response.
- Given Greedy, Player 2 prefers Accept to Reject because $10 > 0$.

Neither player can improve by changing strategy alone. Therefore,

$$\boxed{(G, AA) \text{ is a Nash equilibrium.}}$$

Question 1(c)

(c) Suppose Player 2 accepts if the offer is Fair, but rejects if the offer is Greedy. What should Player 1 offer? Do these strategies form a Nash equilibrium?

Solution (c):
If Player 2's strategy is AR:

Player 1's offer	Player 1's payoff
Fair	50
Greedy	0

Since $50 > 0$, Player 1 chooses Fair.
The resulting strategy profile is

$$(F, AR).$$

To check whether this is a Nash equilibrium:

- Given AR, Fair is Player 1's best response.
- Given Fair, Player 2 prefers Accept to Reject because $50 > 0$.

Neither player can improve by changing strategy alone. Therefore,

$$\boxed{(F, AR) \text{ is a Nash equilibrium.}}$$

Question 1(d)

(d). Which of b) and c) is more likely? Your seminar tutor might discuss backwards induction and SPE with you.

Solution (d):
After Player 1 chooses Fair:

- Accept gives Player 2: 50.
- Reject gives Player 2: 0.

Therefore Player 2 **accepts** after Fair.

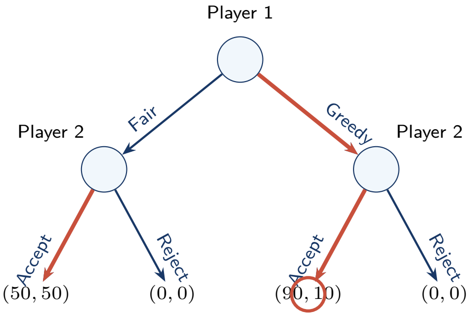
After Player 1 chooses Greedy:

- Accept gives Player 2: 10.
- Reject gives Player 2: 0.

Therefore Player 2 **accepts** after Greedy.

So Player 2's optimal choice is to accept after either action of Player 1.

Player 1 anticipates Player 2's responses



Backward induction tells Player 1 that Player 2 will accept both offers.

Player 1 therefore compares:

- Fair and accepted: Player 1 gets 50.
- Greedy and accepted: Player 1 gets 90.

Because $90 > 50$, Player 1 chooses Greedy.

Backward induction therefore predicts:

Greedy $\longrightarrow$ Accept.

The subgame-perfect equilibrium

The backward-induction solution is

$$(G, AA).$$

This strategy profile is optimal at every decision point:

- Player 2 accepts after Fair.
- Player 2 accepts after Greedy.
- Player 1 anticipates these responses and chooses Greedy.

(G, AA) **is the unique subgame-perfect equilibrium.**

Why is (F, AR) a Nash equilibrium but not an SPE?

- (F, AR) requires Player 2 to reject Greedy, but at that decision point Player 2 would prefer 10 to 0.

This behaviour is not optimal for Player 2.

Therefore it is a Nash equilibrium but not subgame-perfect equilibrium.

For sequential games, SPE is often the more persuasive solution concept because it checks what players would rationally do at every point in the game.

Advanced material: Sequential Hawk–Dove

Question 2

2. (Independent study) Modify the Hawk–Dove game by having Player 1 move before Player 2. Analyse this when $c = 3$. Analyse this when $c = 10$.

Solution: Player 1 moves first and chooses Hawk or Dove.
Player 2 observes Player 1's action and then chooses Hawk or Dove.
Player 2's complete strategy must specify:

- the response after Player 1 chooses Hawk;
- the response after Player 1 chooses Dove.

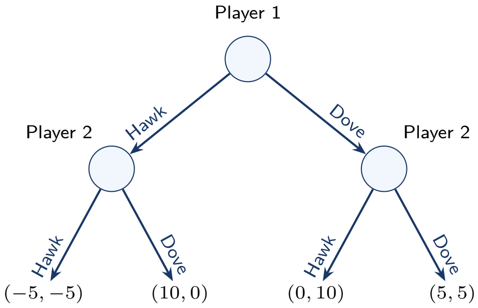
Player 2's strategy notation

Write Player 2's strategy using two letters:

Strategy	Response after Hawk	Response after Dove
HH	Hawk	Hawk
HD	Hawk	Dove
DH	Dove	Hawk
DD	Dove	Dove

The first letter is the response after Hawk; the second is the response after Dove.

Sequential Hawk–Dove when $c = 10$



Each terminal node: (Player 1's payoff, Player 2's payoff).

Step 1: Find Player 2's responses

- After Player 1 chooses Hawk:

$$0 > -5 \quad \Rightarrow \quad \text{Player 2 chooses Dove.}$$

- After Player 1 chooses Dove:

$$10 > 5 \quad \Rightarrow \quad \text{Player 2 chooses Hawk.}$$

Hence Player 2's strategy is DH.

Step 2: Player 1 anticipates these responses

Player 1 chooses	Player 1 receives
H	10
D	0

Therefore, Player 1 chooses Hawk.

Subgame-perfect equilibrium:

$$(H, DH).$$

The resulting outcome is (H, D), with payoffs

$$(10, 0).$$

First-mover advantage when $c = 10$

In the simultaneous game, the two asymmetric pure equilibria favour different players.

In the sequential game, Player 1 commits first to Hawk.

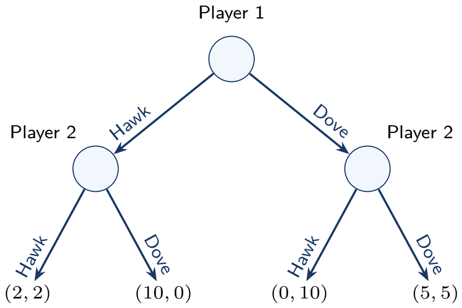
Player 2 then prefers to avoid the costly fight and chooses Dove.

Player 1 obtains the entire resource:

$$(10, 0).$$

This is a first-mover advantage.

Sequential Hawk–Dove when $c = 3$



Each terminal node: (Player 1's payoff, Player 2's payoff).

Step 1: Find Player 2's responses

- After Player 1 chooses Hawk:

$$2 > 0 \quad \Rightarrow \quad \text{Player 2 chooses Hawk.}$$

- After Player 1 chooses Dove:

$$10 > 5 \quad \Rightarrow \quad \text{Player 2 chooses Hawk.}$$

Hence Player 2's strategy is HH.

Step 2: Player 1 anticipates these responses

Player 1 chooses	Player 1 receives
H	2
D	0

Therefore, Player 1 chooses Hawk.

Subgame-perfect equilibrium:

$$(H, HH).$$

The resulting outcome is (H, H), with payoffs

$$(2, 2).$$

Advanced material: Modelling a penalty kick

Question 3

3. Discussion - modelling a game yourself: Consider a penalty kick in a football match. Who are the players? Discuss the various options available to each of the players. How might you look to model this situation using Game Theory?

Solution: First of all, the game must specify

- Who are the players?
- What actions can each player take?
- What do they know when choosing?
- What are the payoffs?

We must also consider which real-world details the model should keep or omit.

There is no single correct model. Different models answer different questions.

Some possible models

Different models emphasise different parts of the penalty kick.

1. Left or Right Open

Both players simultaneously choose one of two directions.

2. Left, Centre or Right Open

Add a third possible direction for both players.

3. Goalkeeper dives early or waits Open

Model the trade-off between moving early and revealing information.

4. Sequential run-up Open

Allow one player to observe an earlier decision before responding.

5. Other features we might include Open

Add more detail about the players, the kick and the available information.

Model 1: Left or Right

The players are: the *striker* and the *goalkeeper*
Both simultaneously choose Left or Right.

- If they choose the same direction, the goalkeeper saves the penalty.
- If they choose different directions, the striker scores.

	Goalkeeper	
	Left	Right
Striker: Left	$(0,1)$	$(1,0)$
Striker: Right	$(1,0)$	$(0,1)$

Model 1: Is there a pure-strategy Nash equilibrium?

Check each possible outcome.

- If both choose the same direction, the striker wants to switch direction.
- If they choose different directions, the goalkeeper wants to switch direction.

When players play pure strategies, one of them wants to deviate.

So there is no pure-strategy Nash equilibrium.
Intuition:

- This is expected. Because if a striker always kicks left, then the goalkeeper will always save the penalty.
- So the striker wants to kick left and right sometimes to keep the goalkeeper uncertain.

Back to models

Model 2: Left, Centre or Right

We can extend the player's possible action set:

$$\{\text{Left}, \text{Centre}, \text{Right}\}.$$

	Goalkeeper		
Striker	Left	Centre	Right
Left	$(0, 1)$	$(1, 0)$	$(1, 0)$
Centre	$(1, 0)$	$(0, 1)$	$(1, 0)$
Right	$(1, 0)$	$(1, 0)$	$(0, 1)$

This model allows more possible behaviour, but similarly one can expect that there is no pure-strategy Nash equilibrium, and players prefer mixing between these actions.

Back to models

Model 3: Goalkeeper dives early or waits

Suppose the choices are:

- Striker: watch the goalkeeper or focus on the kick;
- Goalkeeper: dive early or wait.

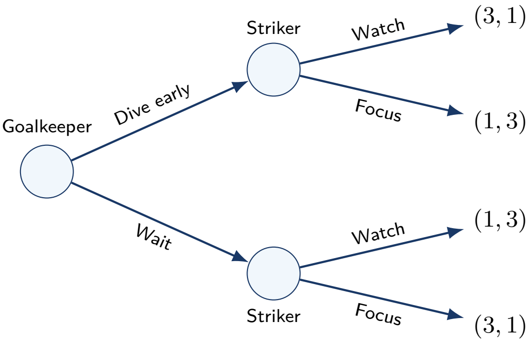
	Goalkeeper	
Striker	Dive early	Wait
Watch	$(3, 1)$	$(1, 3)$
Focus	$(1, 3)$	$(3, 1)$

- Watching is useful when the goalkeeper moves early.
- Focusing on the kick is useful when the goalkeeper waits.

Back to models

Model 4: A sequential run-up

Suppose the goalkeeper acts first. The striker observes that choice before responding.



Unlike the previous model, the players do not choose simultaneously. The striker can respond to the goalkeeper's observed action.

Back to models

Other features we might include in the model

A more detailed model could also include:

- the exact location of the shot;
- shot speed, height and spin;
- the striker's preferred foot;
- the goalkeeper's reaction time;
- information from previous penalties;
- pressure, fatigue and match importance.

Adding more detail may make the model more realistic, but it also makes the model harder to analyse.

What makes a useful model?

A useful game-theory model:

- should simplify the real situation enough while keeping the important details
- does not need to include every detail of reality. It needs to capture the features that matter for the question.

Economic models are like maps: they are simplifications and deliberate distortions of the underlying reality.

Further reading

The following books are listed in an increasing order of difficulty.

1. **Martin J. Osborne, *An Introduction to Game Theory*.**
   A good place to start if you would like a clear and systematic introduction.

2. **Robert Gibbons, *Game Theory for Applied Economists*.**
   A natural next step if you are comfortable with Osborne and would like more compact and mathematical models.

3. **Steven Tadelis, *Game Theory: An Introduction*.**
   A more advanced and comprehensive choice, moving towards the level used in master's study.

Final summary

- Game theory studies choices whose consequences depend on other people's choices.
- Best responses describe optimal action in response to opponent's strategy.
- Nash equilibrium implies players are best responding to each other.
- Mixed strategies involves randomising between actions using probabilities.
- In sequential games, strategies are complete plans and backward induction identifies credible behaviour.
- Subgame-perfect equilibrium refines Nash equilibrium by requiring optimal play in every subgame.

Thank you!!

Appendix

Numerical example

Suppose five players choose

$$12, \quad 18, \quad 21, \quad 27, \quad 42.$$

The total is

$$12 + 18 + 21 + 27 + 42 = 120.$$

The average is

$$\frac{120}{5} = 24,$$

so the target is

$$\frac{2}{3} \times 24 = 16.$$

Numerical example: Payoffs

The target is 16, so each player's payoff is 100 minus the distance from 16.

Choice s_i	12	18	21	27	42
Distance $\lvert s_i - 16 \rvert$	4	2	5	11	26
Payoff U_i	96	**98**	95	89	74

The player who chose 18 has the highest payoff because 18 is the closest number to the target 16.

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