An Introduction to Mathematics and Statistics

Solutions

Warm-Up Tasks

First sequence: 34 comes next
(this is the Fibonacci Sequence, where each term is found by adding the two previous terms)

Second sequence: 45 comes next
(this is the sequence of 'triangular numbers', where the difference between terms increases by 1 each time)

Third sequence: 1113213211 comes next
(this is a bit of a trick question! It is the 'say what you see' sequence; if you say the number out loud, one digit at a time, you are describing the previous term)

Bonus Puzzle: $6 \div \left(1 - \frac{5}{7}\right) = 21$

Task 1a (Geometric series convergence)

As we add more terms, the value of the sum moves closer to 1.5 .

Formal Proof: Let the infinite sum be equal to x .

$$x = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$$

$$3x = 3\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\right) = 3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$$

$$3x - 3 = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots \qquad \Rightarrow \qquad 3x - 3 = x$$

$$\Rightarrow \qquad 2x = 3 \qquad \Rightarrow \qquad x = \frac{3}{2}$$

Task 1c (Harmonic series divergence)

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \cdots$$

After the numbers 1 and $\frac{1}{2}$, we can separate the terms into a group of 2 , then a group of 4 , then a group of 8 , then a group of 16 , and so on ...

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right) + \cdots$$

We can make this series smaller by changing each number into another copy of the last fraction in its bracket:

$$1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}\right) + \cdots$$

Then we notice that each bracket will be equal to $\frac{1}{2}$, and there will be infinitely many of these brackets, so this series diverges. And since we made the harmonic series *smaller* to reach this one, the harmonic series diverges also.

Bonus Tasks (Analysis)

The sum of reciprocals of primes *diverges* .
(another fact proven by Euler in the 18th Century)

The series with n^{th}-term $\frac{n^2-1}{n-n^2}$ must diverge because the terms themselves do not tend to zero. As n grows, the terms tend towards -1 according to the given formula.

Task 2b (Hausdorff Dimension of the Sierpinski Triangle)

By trial and error, the power which satisfies $2^x = 3$ is $x \approx 1.585 \ldots$

For those who have knowledge of logarithms, a calculator tells us that:

$$x = \log_2 3 = 1.584962501 \ldots$$

Bonus Tasks (Fractal Geometry)

We need 8 copies of the Sierpinski Carpet to form a larger version, and the larger version will have side-length 3 times more. Therefore the Hausdorff Dimension can be found by solving $3^x = 8 \quad \Rightarrow \quad x \approx 1.893 \ldots$

We need 20 copies of the Menger Sponge to form a larger version, and the larger version will have side-length 3 times more. Therefore the Hausdorff Dimension can be found by solving $3^x = 20 \quad \Rightarrow \quad x = 2.727 \ldots$

Task 3a (Introduction to counting with and without repetition)

There are $999 - 99 = 900$ three-digit numbers.
(all the numbers up to 999 with the numbers below 100 removed)

There are $99999 - 9999 = 90000$ five-digit numbers.
(all the numbers up to 99999 with the numbers below 10000 removed)

Following the pattern shown above, there are $9 \times 10^{n-1}$ n-digit numbers.

If we can only choose from the set $1, 2, 3, 4, 5$ then we have 5 options for the first digit, and 5 options again for the second digit, and so on...

So the total amount of possible numbers we can make is:

$$5 \times 5 \times 5 \times 5 \times 5 = 3125$$

If each digit can only be used once, then the number of options decreases with each digit (5 for the first, then only 4 for the second, then 3 for the third ...) :

$$5 \times 4 \times 3 \times 2 \times 1 = 120$$

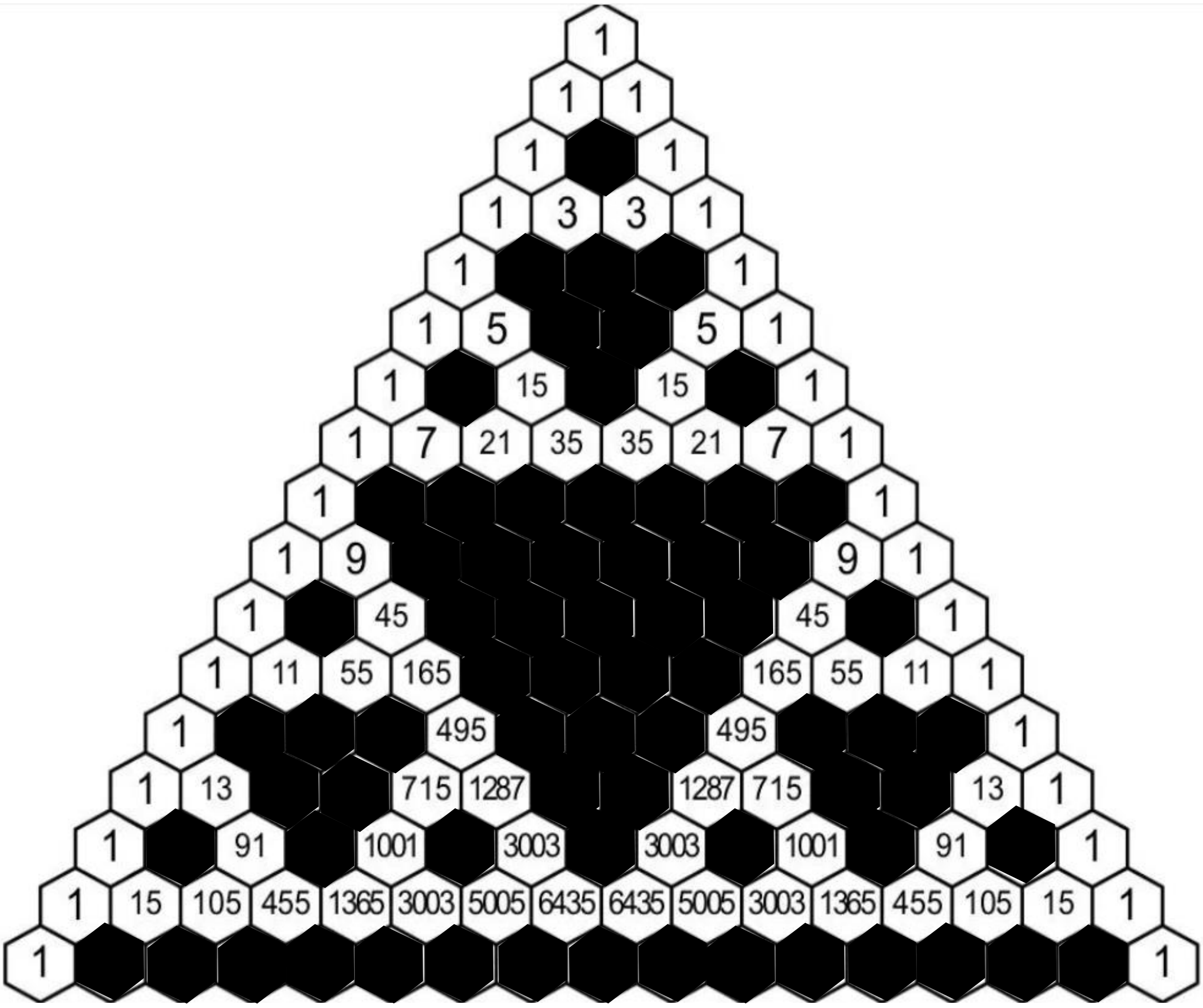
Task 3b (Permutation vs Combination)

There are 9 options for the first digit, then 8 remaining options for the second digit, then 7 remaining options for the third digit.

$$9 \times 8 \times 7 = 504$$

Task 3c (Pascal's Triangle)

Shading the even numbers of Pascal's Triangle gives a shape that looks similar to the Sierpinski Triangle! In fact, as the number of rows increases towards infinity (here we have only used 17 rows), the limit of the resulting shape made by shading even numbers, is exactly the Sierpinski Triangle.



Bonus Tasks (Combinatorics)

If there are 2 people at the party, there is 1 handshake.

If a 3rd person joins the party, they must shake hands with each of the first 2 people. $1 + 2 = 3$ handshakes in total.

If a 4th person joins, they now have 3 people to shake hands with, so there are 3 'new' handshakes, plus the ones from before. $1 + 2 + 3 = 6$.

We can see that the number of 'new' handshakes increases by 1 with each additional person. Therefore we find the sequence of Triangular Numbers from the second warm-up task.

2 people: 1 handshake
3 people: $1 + 2$
4 people: $1 + 2 + 3$
5 people: $1 + 2 + 3 + 4$
…
11 people: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$

So if there are 55 handshakes in total, there must be 11 people at the party.

To be certain that the group you choose contains a married couple, you need to choose 1 more than half of the total number of people (if you choose half the total number of people, there is a small chance you chose one person from each couple hence there could be no couple in your group).

n married couples $\Rightarrow$ $2n$ people in total

Answer: $\frac{2n}{2} + 1 = n + 1$

Task 4 (Co-prime probability experiment)

The limitations of the experiment include having a small sample size, potentially the method of generating the 'random' integers, and restricting the integers to a maximum of 100.

Proof of the theoretical probability:

Using the notation $P(\ )$ to denote the probability of something occurring;

Suppose we choose two random positive integers and call them m and n .

Then $P(m \text{ and } n \text{ are coprime})$

$= P(\text{the highest common factor of } m \text{ and } n \text{ is equal to } 1)$

Let's simply call the probability x , and write: $x = P(\ \text{hcf}(m,n) = 1\ )$

If we choose a random integer $k > 0$ (which is smaller than m and n) then for k to be the highest common factor of m and n , we need three things to be true:

(i) k must be a factor of m
(ii) k must be a factor of n
(iii) If k is a factor of both, no larger number is also a factor of both

The probability of (i) and (ii) are each equal to $\frac{1}{k}$

(e.g. 2 is a factor of $\frac{1}{2}$ of all integers, 3 is a factor of $\frac{1}{3}$ of all integers, ...)

To find the probability of the third statement (iii) we can use a fact that if you divide two integers by their highest common factor, the resulting two numbers must be co-prime (you can test this to convince yourself that it must be true!).

$$\Rightarrow \quad P(\ (\text{iii})\ ) = P\left(\ \text{hcf}\left(\frac{m}{k}, \frac{n}{k}\right) = 1\ \right) = x$$

This is equal to x because that is the name we gave to the probability that two random integers are co-prime.

The probability that all of (i) and (ii) and (iii) are true can be found by multiplying: $P(\ \text{hcf}(m,n) = k\ ) = \frac{1}{k} \times \frac{1}{k} \times x = \frac{x}{k^2}$

Two integers must have a highest common factor, so there is a probability of 1 that the highest common factor is somewhere in the set $\{1\ ,\ 2\ ,\ 3\ ,\ 4\ ,\ \ldots\ \}$:

$$\Rightarrow \quad P(\ \mathrm{hcf}(m,n) = 1\ ) \ + \ P(\ \mathrm{hcf}(m,n) = 2\ ) \ + \ P(\ \mathrm{hcf}(m,n) = 3\ ) \\ + \ P(\ \mathrm{hcf}(m,n) = 4\ ) \ + \ \ldots \quad = \quad 1$$

$$\Rightarrow \quad x + \frac{x}{2^2} + \frac{x}{3^2} + \frac{x}{4^2} + \ \ldots \ = \ 1$$

$$\Rightarrow \quad x\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ \ldots\right) \ = \ 1$$

$$\Rightarrow \quad x\left(\frac{\pi^2}{6}\right) = 1$$ *Using the solution to the Basel Problem in Task 1b !*

$$\Rightarrow \quad x \ = \ \frac{6}{\pi^2}$$

Bonus Tasks (Probability)

The solution to the Monty Hall Problem is that your chances of winning the big prize *doubles* if you change from your original choice.

This is because you only *lose* by changing if you originally picked the correct door, which has a probability of $\frac{1}{3}$.

You win by switching if you originally picked one of the incorrect doors (probability $\frac{2}{3}$) because then the host reveals the *other* incorrect door.

Suppose the expected (average) time for the miner to escape is x .

Then we can make a table of probabilities:

Door	First	Second	Third
Expected escape time:	3	$5 + x$	$7 + x$
Probability	1/3	1/3	1/3

Therefore the overall expected (average) time to escape is:

$$x \ = \ \frac{1}{3}(3) + \frac{1}{3}(5 + x) + \frac{1}{3}(7 + x) \qquad \Rightarrow \qquad x = 15 \ \text{ hours}$$