An Introduction to Mathematics and Statistics

Student Workbook

Warm-Up Tasks

What comes next in each sequence?

- 1 , 1 , 2 , 3 , 5 , 8 , 13 , 21 , ...
- 1 , 3 , 6 , 10 , 15 , 21 , 28 , 36 , ...
- 1 , 11 , 21 , 1211 , 111221 , 312211 , 13112221 , ...

Bonus Puzzle: Make the answer 21 using 1 , 5 , 6 , 7 once each.
(You can only use standard operations $+$, $-$, $\times$, $\div$)

Task 1a

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots + \frac{1}{3^{n-1}} + \cdots$$

Use a calculator to complete the table below (use decimal values!)

$\boldsymbol{n}$	Sum of the first $\boldsymbol{n}$ terms
1	1
2	
3	
4	
5	
6	
…	…

Task 1b

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots+\frac{1}{n^2}+\cdots$$

Use a calculator to complete the table below (use decimal values!)

$\boldsymbol{n}$	Sum of the first $\boldsymbol{n}$ terms
1	1
2	
3	
4	
5	
6	
…	…

Task 1c

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \cdots$$

Use a calculator to complete the table below (use decimal values!)

$\boldsymbol{n}$	Sum of the first $\boldsymbol{n}$ terms
1	1
2	
3	
4	
5	
6	
…	…

Bonus Tasks (Analysis):

Consider the following infinite series (sum of reciprocals of prime numbers):

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\cdots$$

Investigate whether or not you think this series converges, by adding more and more terms together in the same way as we have been doing.

Investigate the infinite series whose terms are given by the n^{th}-term formula:

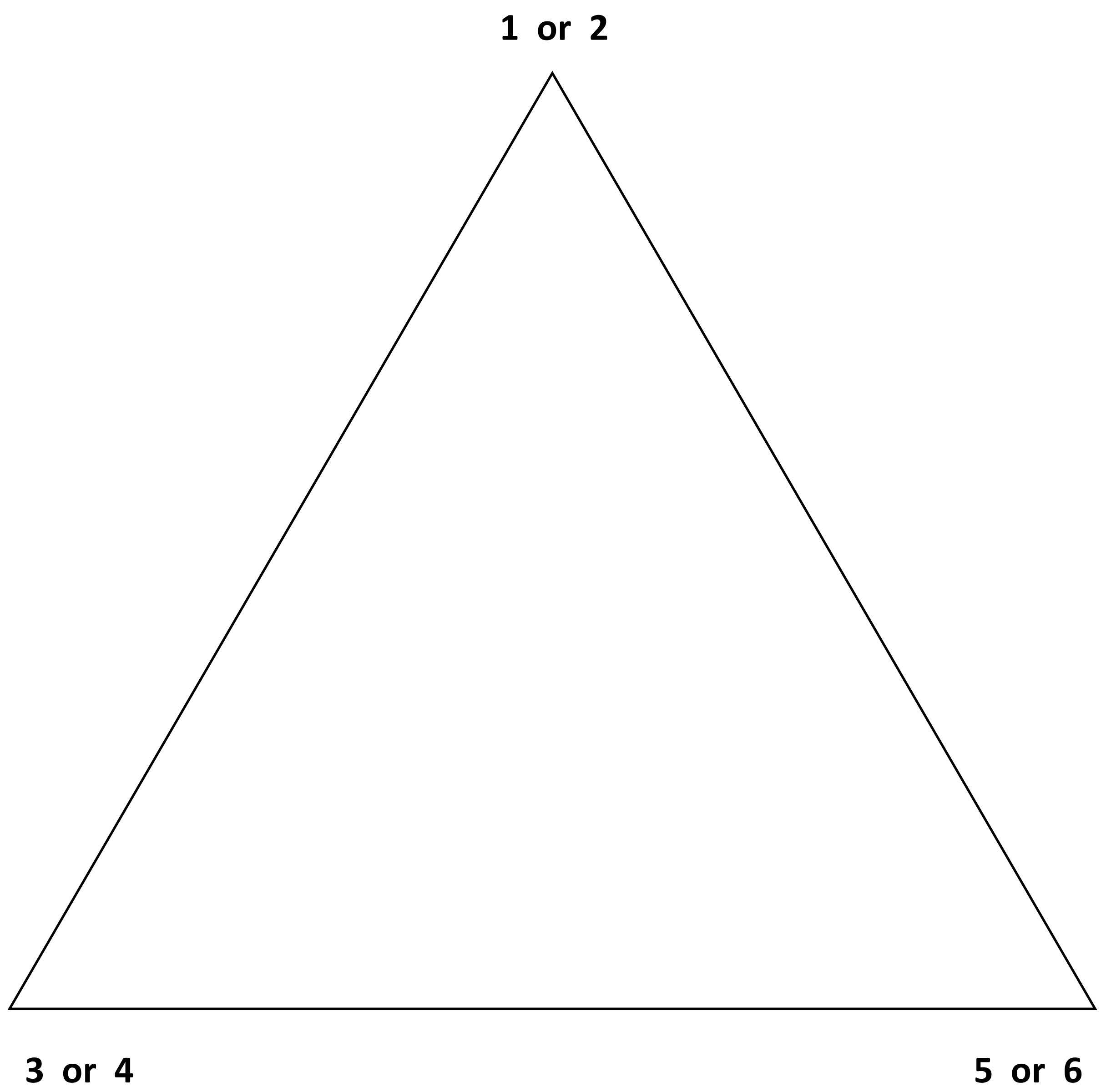
$$\frac{n^2-1}{n-n^2}$$

Can you think of a way to decide whether this series converges or diverges, *without adding terms together* ?

Research online a function called the *Riemann Zeta Function*.

Task 2a

Roll a dice to determine which corner to start at.
Then roll again and move half-way towards that number.
Repeat this process and draw a point each time you move.



Task 2b

Use this table to find an approximate value for the value of x such that:

$$2^x = 3$$

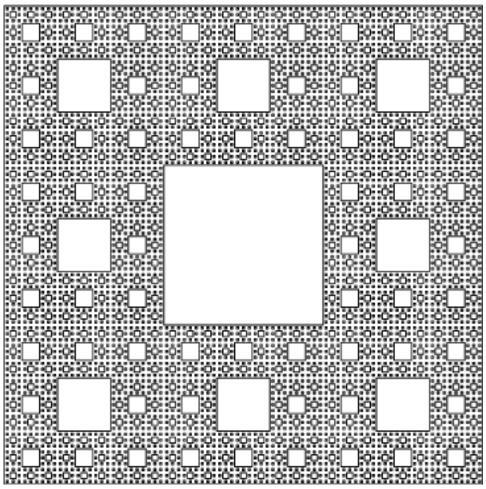
and therefore estimate the Hausdorff Dimension for the Sierpinski Triangle.

Guess	Result	Too large or too small?
2^1	2	Too small
2^2	4	Too large
$2^{1.5}$	2.828 …	Too small

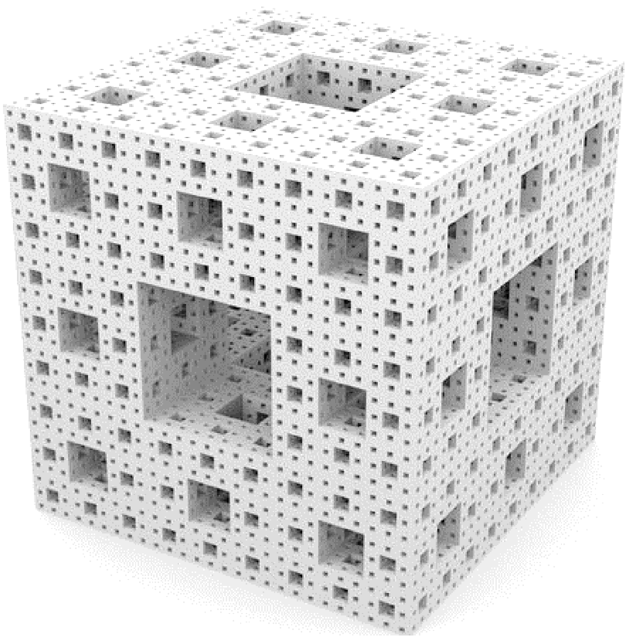
Bonus Tasks (Fractal Geometry):

Can you calculate the Hausdorff Dimension of the Sierpinski Carpet?

(Hint: 3 copies of the Sierpinski Triangle produces a larger version of itself with side-length 2 times larger, which is why we solved $2^x = 3$) .



Now find the Hausdorff Dimension of the Menger Sponge:



Task 3a:

How many 3-digit numbers are there?

How many 5-digit numbers are there?

(Challenge question: How many n-digit numbers are there?)

How many 5-digit numbers can be made using only digits from the set $1, 2, 3, 4, 5$? What if each digit can only be used once?

Task 3b

How many 3-digit numbers can be made using $1, 2, 3, 4, 5, 6, 7, 8, 9$ *without repetition* (i.e. maximum once each) ?

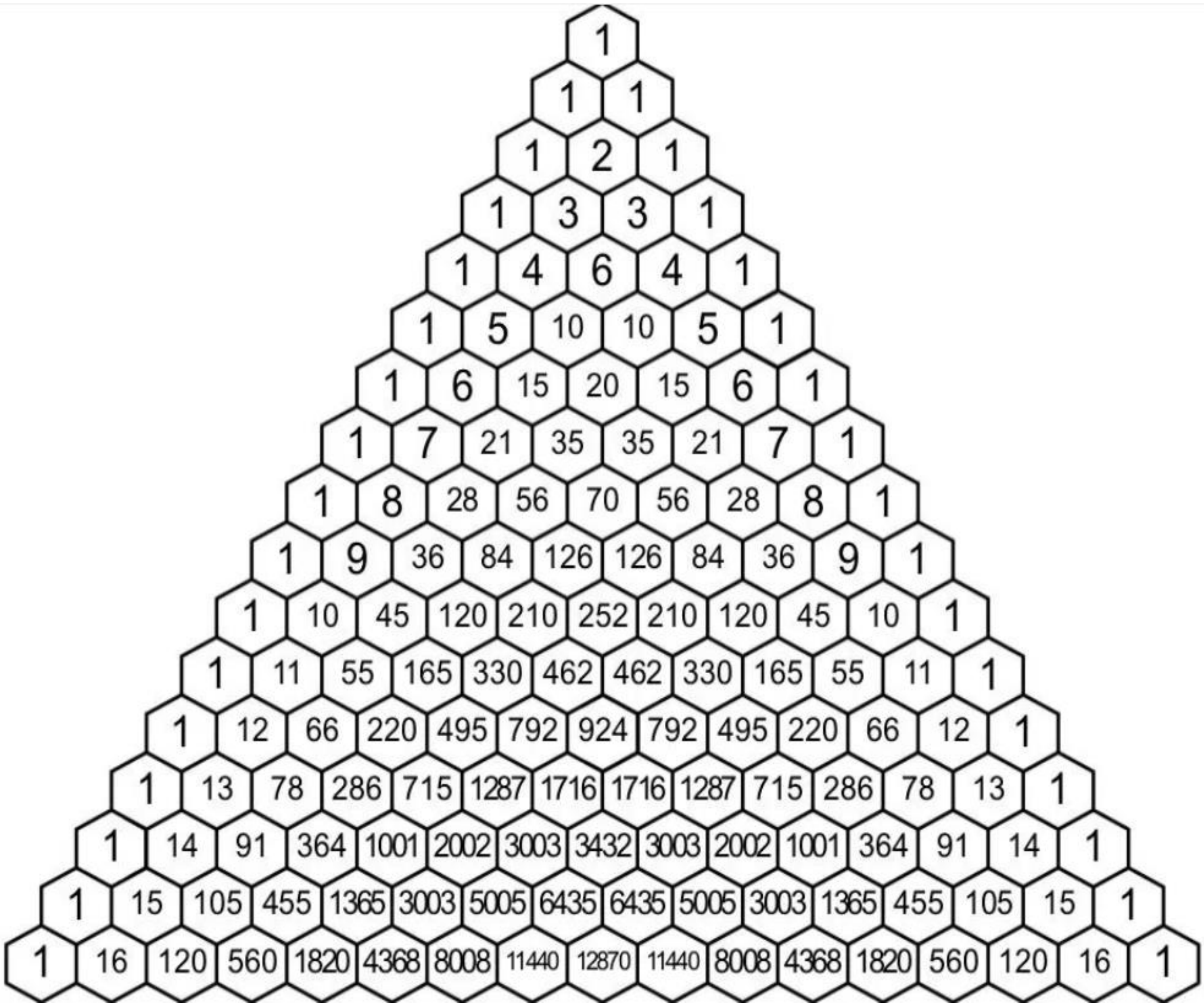
Does the following question have the same answer as the one above?
Why, or why not?

‘Students at a university must choose 3 *modules to study from a list of* 9 *options. How many total choices are possible?’*

Task 3c

Shade all of the even numbers in Pascal's Triangle.

Do you notice any familiar pattern?



Bonus Tasks (Combinatorics)

There are some people at a party, and every person at the party shakes hands with everyone else. If there are 55 handshakes that take place in total, how many people are there at the party?

Suppose there are n married couples in a room, standing in a mixed up order. You do not know which people are married to each other. You are going to enter the room and choose a group of people. How many people do you need to choose, in order to be absolutely certain that your selection includes at least one married couple?

(This is an example of something called the *pigeonhole principle*).

Task 4

- Use a random number generator (e.g. on Google) to produce two random integers (whole numbers) from 1 to 100 .
- Decide whether the two numbers are co-prime (i.e. they share *no factors in common* apart from 1) or not.
- Record the result in the table below (e.g. as tally marks)
- Repeat this process as many times as you can.

Co-Prime (1 is the only common factor)	Not Co-Prime (there is some common factor apart from 1)

How can you use your data to estimate the probability that any two randomly selected integers are co-prime?

What are the limitations of your estimation?
(i.e. what are the reasons why your estimate might not be perfect, and what could you do to improve it?)

Bonus Tasks (Probability)

There is a long-running American TV game show called *Let's Make A Deal* , which was formerly hosted by Monty Hall. At one point in the show, the winning contestant was offered a choice of three doors. Behind one random door was an expensive prize (e.g. a car), and behind the other two doors was a low-value prize. The contestant was invited to choose any door, and then Monty would reveal the location of a low-value prize behind one of the *other* two doors. The contestant could then either keep their original choice of door, or change to the remaining un-opened door.

If you were a contestant on the show, would you keep your original choice when the options are reduced down to two doors, or would you change to the other door? Does it make any difference to your chances of winning?

(Warning: this problem is difficult!) A miner is trapped in a mine containing three doors. The first door leads to a tunnel that will bring him outside after 3 hours of walking. The second door leads to a tunnel that will return him to the mine after 5 hours of walking. The third door leads to a tunnel that will return him to the mine after 7 hours of walking. If we assume that the miner is *at all times* equally likely to choose any of the 3 doors, what is the expected (average) length of time until he reaches the air?

Research online the *Sleeping Beauty Paradox* .
(Julia Galef has a great introduction video on YouTube)