Introduction to Probability

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Summer 2019

Introduction

Discrete mathematics: study of discrete structures, e.g. integer numbers, finite or countable sets, finite or countable relations.

Less concerned with continuous structures, e.g. real numbers, continuous functions, limits, derivatives etc.

The distinction between discrete and continuous mathematics is informal and often hazy. The are many concepts and ideas common to both kinds of maths.

Discrete mathematics is intimately related to *computation* and therefore *computer science*

CS130 Maths for Computer Scientists 1 (this term): mostly discrete structures

CS131 Maths for Computer Scientists 2 (Spring term): mostly continuous structures

Introduction

30 lectures in Autumn Term

Seminars *from week 3*, please check the module website!

Four sets of assessed exercises: issued by Thursday of week 2, 4, 6, 8; work to be submitted by noon on Thursday of week 3, 5, 7, 9; to be discussed during the Seminars of Week 4, 6, 8, 10 (submissions through Tabula!)

Four sets of unassessed exercises: issued by Tuesday of week 2 and Thursday of week 3, 5, 7; to be discussed during the Seminars of Week 3, 5, 7, 9.

Submitted work must have a cover sheet! Download at `http://www2.warwick.ac.uk/fac/sci/dcs/intranet/coversheet`

Exam in Summer Term (usually week 1), all exam results at the end of Summer Term

Introduction

Module website: `http://www2.warwick.ac.uk/fac/sci/dcs/teaching/material/cs130`

Lecture notes are available (subject to minor updates)

Problem/Exercise sets and solutions will be posted incrementally

Module forum: `http://www2.warwick.ac.uk/fac/sci/dcs/teaching/material/cs130/forum/`

Slides slightly adapted from Steve Russ and Rajagopal Nagarajan

Introduction

Recommended books:

Rosen, *Discrete Mathematics and its Applications*

Ross and Wright, *Discrete Mathematics*

Truss, *Discrete Mathematics for Computer Scientists*

Which is the best? In my personal opinion:

- Rosen: the most *interesting*
- Ross and Wright: the most *helpful*
- Truss: the most *advanced*

Introduction

Another useful book:

Bloch, *Proofs and Fundamentals: a First Course in Abstract Mathematics*

Does not cover whole course, but helps with proofs (which is worth a lot!)

Hundreds more books on Discrete Maths, the choice is yours...

Any questions?

A Brief Tour of the Syllabus

Mathematics: the science of abstraction

Natural numbers: $0, 1, 2, 3, 4, 5, 6, 7, \ldots$

What does "5" mean? Five apples, five hats, five lottery numbers...

Abstraction of all five-element sets

What set does 0 represent? The *empty set*

A Brief Tour of the Syllabus

A *set* is any collection of *elements*

{Peter Pan, Gingerbread Man, Wrestling Fan}

$\{\spadesuit, \heartsuit, \clubsuit, \diamondsuit\}$

$\{0, 1, 2, 3, 4, 5, 6, 7, \ldots\} = \mathbb{N}$: natural numbers

$\{0, 2, 4, 6, 8, 10, 12, 14, \ldots\} = \mathbb{N}_{\text{even}}$: even natural numbers

$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} = \mathbb{Z}$: integers

First two examples are *finite sets*, last three *infinite sets*

The *empty set*: $\{\} = \emptyset$

Plays a role for sets similar to 0 for numbers

A Brief Tour of the Syllabus

Some "tricky" questions:

Do infinite sets have "sizes"?

Yes, but these are beyond natural numbers

Can infinite sets have different sizes?

Yes, a huge variety of possible different sizes

Is there a set of all sets?

No! Not even a set of all (infinite) set sizes

Why? We cannot answer right now, but it can be *proved*!

A Brief Tour of the Syllabus

Natural sciences are based on *evidence*

Mathematics is based on *proof* (but evidence helps to understand proofs)

To write proofs, we need a special language:

- precise (unambiguous)
- concise (clear and relatively brief)

The "grammar" of this language is *logic*

A Brief Tour of the Syllabus

1. *All eagles can fly*
2. *Some pigs cannot fly*

Therefore, *some pigs are not eagles*

Proof. Consider all creatures.

By (1), if a creature is an eagle, it can fly. Hence, if it cannot fly, it is not an eagle.

By (2), there is a creature that is a pig and cannot fly. We already know that, since it cannot fly, it cannot be an eagle. □

A Brief Tour of the Syllabus

The same in mathematical notation:

If (1) for all x, $eagle(x) \Rightarrow canfly(x)$,
and (2) for some y, $pig(y) \wedge \neg canfly(y)$,
then for some z, $pig(z) \wedge \neg eagle(z)$

Proof. Consider all $x \in Creatures$.

By (1), $\neg canfly(x) \Rightarrow \neg eagle(x)$.

By (2), there exists $y \in Creatures$, such that $pig(y) \wedge \neg canfly(y)$.

By the above, we have $pig(y) \wedge \neg eagle(y)$.

Substituting y for z, we reach the required conclusion. □

A Brief Tour of the Syllabus

Some concepts are *basic*, i.e. need no definition.

E.g.: "set", "natural number"

All other concepts must be *defined*.

E.g.: "finite set", "even number"

Some facts are *axioms*, i.e. need no proof.

E.g.: "equal sets have the same elements"

All other facts must be *proved*.

E.g.: "the set of all even numbers is infinite"

This is *the axiomatic method*

A Brief Tour of the Syllabus

Module structure:

- Logic
- Sets
- Relations
- Functions
- Induction
- Graphs
- Probability

Any questions?

1 Introduction

2 A Brief Tour of the Syllabus

3 Probability

Probability

Consider an experiment (e.g. throwing a six-sided die)

The *sample space* is the set of all possible outcomes (e.g. $\{1, 2, 3, 4, 5, 6\}$). Its elements are sometimes called *points*.

An *event* is some subset of outcomes (e.g. $\{1, 3, 5\}$)

An *elementary event* is an event with a single outcome (e.g. $\{3\}$)

The *impossible event* is an event with no outcomes ($\emptyset$)

The *certain event* is the event containing all possible outcomes, i.e. the complete sample space (in the above example, $\{1, 2, 3, 4, 5, 6\}$)

A pair of events are *mutually exclusive*, if they have no outcomes in common (e.g. $\{1, 3, 5\}$ and $\{2, 4\}$)

Probability

Examples:

Two coins tossed
Sample space: $\{(H, H), (H, T), (T, H), (T, T)\}$
An elementary event: $\{(H, T)\}$

Three dice thrown
Sample space: $\{(1, 1, 1), (1, 1, 2), \ldots, (6, 6, 6)\}$
An elementary event: $\{(4, 3, 4)\}$

Hand of five cards
Sample space: $(S_\clubsuit \cup S_\diamondsuit \cup S_\spadesuit \cup S_\heartsuit)^5$

$S_\clubsuit = \{\clubsuit 2, \clubsuit 3, \ldots, \clubsuit 10, \clubsuit J, \clubsuit Q, \clubsuit K, \clubsuit A\}$, etc.

An elementary event: $\{(\clubsuit 3, \diamondsuit 4, \diamondsuit 5, \heartsuit A, \spadesuit A)\}$

Probability

Example: die throw

Event "odd number": $A = \{1, 3, 5\}$

Event "greater than 4": $B = \{5, 6\}$

Event "at most 3": $C = \{1, 2, 3\}$

$A \cup B = \{1, 3, 5, 6\}$

$A \cap B = \{5\}$ — an elementary event

$B \cap C = \emptyset$ — the impossible event (B and C are mutually exclusive)

$\bar{C} = \{4, 5, 6\}$ — the complement of event C

Probability

Multiplication Principle. If two experiments are performed in succession, where the first has m possible outcomes, and following this the second has n possible outcomes, then the total number of ways of possible combined outcomes is mn

Examples:

Number of outcomes for a coin toss followed by a die throw: $2 \cdot 6 = 12$

Number of outcomes for ten coin tosses: $2^{10} = 1024$

Probability

Recall: a *permutation* is a bijective function from a set to itself

Can be thought of as an experiment with n outcomes, followed by an experiment with $n-1$ outcomes, etc. down to 1 outcome

The number of different permutations of a finite set of n objects is called *n factorial*

$n! = n(n-1)(n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$

Example: the number of ways to order a pack of 52 cards
$52! = 80658175170943878571660636856403766975289505440883277824000000000000$

Probability

Permutations of n objects taken r at a time

Can be thought of as an experiment with n outcomes, followed by an experiment with $n-1$ outcomes, etc. down to $n-r+1$ outcomes

$P(n,r) = n(n-1)(n-2) \cdot \ldots \cdot (n-r+1) = \frac{n!}{(n-r)!}$

Example: the number of ways of assigning first, second and third prizes to 10 contestants is $10 \times 9 \times 8 = 10!/7! = 720$

Probability

A *combination of n objects taken r at a time* is any subset containing r of the objects (the order is not taken into account)

The number of combinations of n objects taken r at a time:
$C(n,r) = P(n,r)/r! = \frac{n!}{r!(n-r)!} = \binom{n}{r}$

Example: the number of "hands" of 5 cards from a pack of 52 cards
$C(52,5) = \binom{52}{5} = \frac{52!}{5! \cdot 47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2598960$

Example: the number of ways of choosing a committee of three from 10 people $C(10,3) = \binom{10}{3} = \frac{10!}{3! \cdot 7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$

Probability

Let $\mathcal{E}$ be the set of all events of some sample space S: $E = \mathcal{P}(S)$

Each event A in $\mathcal{E}$ is to be assigned a real number $p(A)$ called the *probability* of A

Let p be this function from $\mathcal{E}$ to the set of real numbers. Function p is called a *probability measure*, if

- $0 \leq p(A) \leq 1$ for every event A;
- $p(S) = 1$;
- if A_1 and A_2 are mutually exclusive events ($A_1 \cap A_2 = \emptyset$), then $p(A_1 \cup A_2) = p(A_1) + p(A_2)$;
- if $A_1, A_2, A_3, \ldots$ is a sequence of pairwise mutually exclusive events ($\forall i, j : A_i \cap A_j = \emptyset$), then $p(A_1 \cup A_2 \cup A_3 \cup \cdots) = p(A_1) + p(A_2) + p(A_3) + \cdots$

Probability

Using this definition, we can prove some additional properties of p:

- $p(\emptyset) = 0$;
- for any event A, $p(\bar{A}) = 1 - p(A)$;
- if $A \subseteq B$, then $p(A) \leq p(B)$;
- for any two (not necessarily mutually exclusive) events A and B, $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

Alternative notation: $\Pr(A)$

Probability

Let S be a *finite* sample space: $S = \{a_1, a_2, \ldots, a_n\}$

We can get a *finite probability space* by assigning a probability $p_i \geq 0$ to each elementary event $\{a_i\}$, $a_i \in S$, where $p_1 + p_2 + \cdots + p_n = \sum_{1 \leq i \leq n} p_i = 1$

The probability $p(E)$ of an event E is then the sum of the probabilities assigned to the points in E: $p(E) = \sum_{e \in E} p(\{e\})$

The situation is trickier for *infinite* sample spaces. In particular, in an infinite sample space, an event may not have a well-defined probability.

Probability

An *equiprobable* (or *uniform*) finite probability space is one in which each elementary event is assigned the same probability

In a finite equiprobable space containing n elementary events, the probability assigned to each elementary event is $1/n$

If event E contains r points, then $p(E) = r/n$.

Probability

Examples:

A *fair coin* has equal probability of landing heads or tails

An experiment consisting of a single toss can be described by the sample space $\{H, T\}$, with probabilities are $p(\{H\}) = p(\{T\}) = 1/2$.

A *fair die* has equal probability of landing on any face

The sample space of a single throw is $\{1, 2, 3, 4, 5, 6\}$ where each elementary event has uniform probability $1/6$

For a *loaded die*, the probabilities assigned to the individual outcomes are not necessarily identical

For example, if a certain die yields a 6 half of the time, while the other sides are equally likely, then we have $p_6 = 1/2$, $p_1 = p_2 = p_3 = p_4 = p_5 = 1/10$.

Probability

Example: Find the probability of exactly two heads, when a fair coin is tossed three times

Solution. We take the sample space $S = \{HHH, THH, HTH, HHT, HTT, THT, TTH, TTT\}$

Since each outcome is equally likely, we assign each element of S probability $1/8$

The event "exactly two heads are thrown" corresponds to the subset $\{THH, HTH, HHT\}$, and hence has probability $3/8$

Probability

Example: Find the probability of being dealt a flush at poker, i.e., of five cards, selected at random from a standard pack of 52, all belonging to the same suit.

Solution. The sample space consists of all possible sets of five distinct cards. The number of elementary events in this sample space is $\binom{52}{5}$.

The number of ways in which five diamonds can be selected is $\binom{13}{5}$, since there are a total of 13 diamonds. Hence, the probability of getting five diamonds is $\binom{13}{5}/\binom{52}{5}$.

Considering the other suits as well, we see that the probability of a flush is $4 \cdot \binom{13}{5}/\binom{52}{5} = 4 \cdot \frac{13!}{5! \cdot 8!} \cdot \frac{5! \cdot 47!}{52!} = \frac{4 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13}{48 \cdot 49 \cdot 50 \cdot 51 \cdot 52} = \frac{33}{16660} \approx 0.0020$

Probability

Let A, B be events

The *joint probability* of A and B is $p(A \cap B)$

Let $p(A) \neq 0$

The probability of B occurring, when it is known that A has already occurred, is called the *conditional probability of B given A*

$p(B \mid A) = \frac{p(B \cap A)}{p(A)}$

Probability

Example: A card is dealt from a standard pack of 52. Consider the probability that it is a king.

$p(\text{king}) = 4/52 = 1/13$

If we know in advance that it is one of 12 "face cards" (J,Q,K), then the probability that it is a king becomes $4/12 = 1/3$

This is known as the *conditional probability* of obtaining a king, *given that* a face card is dealt

$p(\text{king} \mid \text{face card}) = 1/3$

Probability

Example: A fair coin is tossed three times. The sample space is $S = \{HHH, THH, HTH, HHT, HTT, THT, TTH, TTT\}$

The probability of getting three tails is $1/8$

If we know in advance that the last toss was a tail, then the sample space is reduced to $A = \{HHT, HTT, THT, TTT\}$

The conditional probability of three tails, given that the last toss was a tail, is $1/4$

Probability

Example: Two dice are tossed. Find the probability that the first is a 6, given that the sum is 8.

Solution. An outcome is a pair of digits, giving the values on the first and the second die respectively

There is a total of $6 \cdot 6 = 36$ possible equally likely outcomes

$p(\text{first is } 6 \mid \text{sum is } 8) = p(\{61, 62, 63, 64, 65, 66\} \mid \{26, 35, 44, 53, 62\}) = \frac{p(\{62\})}{p(\{26,35,44,53,62\})} = \frac{1/36}{5/36} = \frac{1}{5}$

Probability

Let A, B be events

Events A, B are *independent*, if the probability of either event is not affected by whether or not the other event occurs

$p(A \mid B) = p(A)$ and $p(B \mid A) = p(B)$

Theorem. Events A, B are independent, iff $p(A \cap B) = p(A) \cdot p(B)$

Proof. We have $p(A \mid B) = p(A)$ and $p(B \mid A) = p(B)$

Equivalently, $\frac{p(A \cap B)}{p(B)} = p(A)$ and $\frac{p(B \cap A)}{p(A)} = p(B)$

Each of these is equivalent to $p(A \cap B) = p(A) \cdot p(B)$ □

Examples:

Let $P_2 =$ "divisible by 2" and $P_3 =$ "divisible by 3"

Roll a 6-sided die. The events defined by P_2 and P_3 are independent.

$p(P_2) = 3/6 = 1/2 \quad p(P_3) = 2/6 = 1/3$

$p(P_2 \cap P_3) = 1/6 = (1/2) \cdot (1/3)$

Roll a 10-sided die. The events defined by P_2 and P_3 are *not* independent.

$p(P_2) = 5/10 = 1/2 \quad p(P_3) = 3/10$

$p(P_2 \cap P_3) = 1/10 \neq (1/2) \cdot (3/10)$

A *random variable* can be thought of as some quantity which is measured in connection with a random experiment

Examples:

The number of heads when a coin is tossed ten times

The total obtained when two dice are thrown

The winnings from a single play of a fruit machine

If S is a sample space, then a *random variable* on S is a function from S to the set of real numbers

Let $S = \{a_1, a_2, \ldots, a_n\}$ be a finite sample space

Let X be a random variable on S

The *expectation* (or *mean*) of X is $E[X] = \sum_{1 \leq i \leq n} p(a_i) \cdot X(a_i)$

The expectation can be thought of as a measure of the average value of X

Problem. A roulette wheel contains the 37 numbers $0, 1, 2, \ldots, 36$. 18 numbers are black, 18 numbers are red, number 0 is green.

You can bet on black of red. If you are correct, you get back twice your stake. If 0 comes up, you get half your stake. If you are incorrect, you lose your stake.

What are your expected returns for a £1 bet?

Solution: Returns are a random variable R defined by

$R(\text{win}) = 2 \quad R(\text{lose}) = 0 \quad R(0) = 0.5$

$E[R] = p(\text{win}) \cdot R(\text{win}) + p(\text{lose}) \cdot R(\text{lose}) + p(0) \cdot R(0) =$
$\frac{18}{37} \cdot 2 + \frac{18}{37} \cdot 0 + \frac{1}{37} \cdot 0.5 = \frac{36.5}{37} \approx 0.9865$

So in the long run, you can expect to get back about 99p for every £1 bet. The "house" takes about 1.35% of all stakes. □

Probability

A charity advert:

> *The average donation is 10 pounds, but most people give more.*

Can this be right?

Yes! E. g.

- 100 people giving £6 each
- 200 people giving £12 each

Probability

Let X be a random variable on sample space $\{a_1, \ldots, a_n\}$

Let g be a function from real numbers to real numbers

Then $g(X)$ is also a random variable with expectation
$E[g(X)] = \sum_{1 \le i \le n} p(a_i) \cdot g(X(a_i)) = \sum_{1 \le i \le n} p(a_i) \cdot (X \circ g)(a_i)$

In particular, consider $g(x) = x^k$ for some $k \in \mathbb{N}$

Value $E[X^k]$ is the *k-th moment* of X

The expectation of X is the first moment of X

The *variance* of X is the second moment of $X - E[X]$:
$\mathrm{Var}(X) = E[(X - E[X])^2]$

The *standard deviation* of X is the square root of the variance:
$\sigma_X = \sqrt{\mathrm{Var}(X)} = (E[(X - E[X])^2])^{1/2}$

Probability

The variance and standard deviation are measures of the random variable's dispersion about the mean

If the variance of X is large, then there is a high probability that the value of X will be far from its mean, and vice versa

Theorem. The random variable's variance is the difference between the mean of its square and the square of its mean

$\mathrm{Var}(X) = E[X^2] - E[X]^2$

Proof: Let $p(a_i) = p_i$ for all i. We have

$\mathrm{Var}(X) = E[(X - E[X])^2] = \sum_i (X(a_i) - E[X])^2 p_i =$
$\sum_i X(a_i)^2 p_i - 2E[X] \sum_i X(a_i) p_i + E[X]^2 \sum_i p_i =$
$E[X^2] - 2E[x] \cdot E[X] + E[X]^2 \cdot 1 = E[X^2] - E[X]^2$ □

Probability

In many applications, we do not know the probability distribution of a random variable in advance. However, it is possible to estimate probabilities from only the mean (and maybe variance).

Let X be a *non-negative* random variable over $S = \{a_1, \ldots, a_n\}$. Suppose we know $E[X]$, but not $\mathrm{Var}(X)$ or $p(a_i)$.

Theorem (Markov's Inequality). Let k be any positive real number. Then $p(X \ge k) \le \frac{E[X]}{k}$

Proof: Let $p(a_i) = p_i$ for all i. We have

$E[X] = \sum_i p_i X(a_i) = \sum_{i:X(a_i) \ge k} p_i X(a_i) + \sum_{i:X(a_i) < k} p_i X(a_i) \ge$
(since X non-negative) $\sum_{i:X(a_i) \ge k} p_i X(a_i) \ge$
(by the condition on i) $\sum_{i:X(a_i) \ge k} p_i k = k \cdot \sum_{i:X(a_i) \ge k} p_i = k \cdot p(X \ge k)$

Hence $p(X \ge k) \le \frac{E[X]}{k}$ □

Probability

A sequence of n *Bernoulli trials* is a sequence of n independent observations or experiments, each of which has only two possible outcomes, often called *success* and *failure*.

The probability of success p and of failure $1 - p$ are the same for each observation

Example:

Toss a coin independently n times: heads or tails.

Examine components produced on an assembly line: acceptable or defective

Transmit bits through a communication channel: received correctly or incorrectly

Probability

A sequence of n Bernoulli trials can be described by the sample space $\{success, failure\}^n$, consisting of all finite sequences of length n

The probability of getting k successes followed by $n - k$ failures is $p^k \cdot (1-p)^{n-k}$

This is also the probability of any other *single* outcome containing k successes and $n - k$ failures

The total number of *all* outcomes consisting of k successes and $n - k$ failures is just the number of ways we can select positions for the successes. Hence, the probability of k successes in n Bernoulli trials is $\binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$.

Probability

Problem. Find the probability that a decimal string with seven digits, chosen randomly and uniformly, contains

- exactly two 4's;
- exactly one digit greater than 5.

Solution: We think of each digit as the outcome of a trial

Let success be the digit 4: $p(\text{success}) = 1/10$

The probability of two successes in seven trials: $\binom{7}{2} \cdot \left(\frac{1}{10}\right)^2 \cdot \left(\frac{9}{10}\right)^5 \approx 0.124$

Now let success be all digits greater than 5: $p(\text{success}) = 4/10$

The probability of exactly one success in seven trials: $\binom{7}{1} \cdot \frac{2}{5} \cdot \left(\frac{3}{5}\right)^6 \approx 0.131$ □

Probability

Let A, B be events, and p a probability measure

Let $p(A) \neq 0$ and $p(B) \neq 0$

Recall $p(A \mid B) \cdot p(B) = p(A \cap B) = p(B \mid A) \cdot p(A)$

Therefore, $p(A \mid B) = \frac{p(B|A) \cdot p(A)}{p(B)}$

If events $A_1, A_2, \ldots, A_n$ *partition* a sample space S (i.e., are mutually exclusive and have union S), then any event E is the union of the disjoint sets $E \cap A_1, E \cap A_2, \ldots, E \cap A_n$

Total Probability Theorem. If events $A_1, A_2, \ldots, A_n$ partition the sample space S, then for any event E, $p(E) = \sum_{1 \leq i \leq n} p(E \cap A_i) = \sum_{1 \leq i \leq n} p(A_i) \cdot p(E \mid A_i)$

Probability

Example: Only two factories manufacture zoggles. 20% of zoggles from Factory 1 are defective, and 5% from Factory 2 are defective. Factory 1 produces twice as many zoggles as Factory 2.

- What is the probability that a zoggle chosen randomly is satisfactory?
- If a chosen zoggle is defective, what is the probability that it came from Factory 1?

Solution. Let
$G =$ "chosen zoggle is good"
$F_1 =$ "chosen zoggle was made in Factory 1"
$F_2 =$ "chosen zoggle was made in Factory 2"

We are asked $p(G)$ and $p(F_1 \mid \bar{G})$

Probability

The information we are given:

$p(\bar{G} \mid F_1) = 1/5 \quad p(G \mid F_1) = 4/5 \quad p(F_1) = 2/3$

$p(\bar{G} \mid F_2) = 1/20 \quad p(G \mid F_2) = 19/20 \quad p(F_2) = 1/3$

Part 1. From the Total Probability Theorem,

$p(G) = p(G \mid F_1) \cdot p(F_1) + p(G \mid F_2) \cdot p(F_2) =$
$(4/5) \cdot (2/3) + (19/20) \cdot (1/3) = 8/15 + 19/60 = 17/20$

Part 2. We have

$p(F_1 \mid \bar{G}) = \frac{p(F_1 \cap \bar{G})}{p(\bar{G})} = \frac{p(\bar{G} \mid F_1) \cdot p(F_1)}{p(\bar{G})}$

From Part 1, we have $p(\bar{G}) = 1 - p(G) = 3/20$, so

$p(F_1 \mid \bar{G}) = \frac{(1/5)\cdot(2/3)}{3/20} = \frac{2}{15} \cdot \frac{20}{3} = 8/9$ □

Probability

Let S be a sample space, and p a probability measure

Bayes's Theorem. For any events A, B, we have $p(A \mid B) = \frac{p(B|A)p(A)}{p(B)}$

If events $A_1, A_2, \ldots, A_n$ partition S, then for any event B,
$p(A_k \mid B) = \frac{p(B|A_k)p(A_k)}{p(B)} = \frac{p(B|A_k)p(A_k)}{\sum_{1 \le i \le n} p(B|A_i)p(A_i)}$

Possible interpretation: events $A_1, A_2, \ldots, A_n$ take place in a "black box", event B is observed. What is the likelihood of each of A_i *a posteriori* (i.e. knowing that B has happened)?

Probability

Example: A message is coded as a sequence of 0's and 1's. The probabilities of transmission of the two symbols are 0.45 and 0.55 respectively.

The symbols 0 are distorted into 1's with a probability of 0.2; the 1's are distorted into 0's with probability 0.1.

Find the probability that a received 0 has not been distorted.

Solution. Let

$T_0 =$ "a 0 has been transmitted"
$T_1 =$ "a 1 has been transmitted"
$R_0 =$ "a 0 has been received"
$R_1 =$ "a 1 has been received"

Probability

Solution (contd.) We have

$p(T_0) = 0.45 \qquad p(R_1 \mid T_0) = 0.2 \qquad p(R_0 \mid T_0) = 0.8$

$p(T_1) = 0.55 \qquad p(R_0 \mid T_1) = 0.1 \qquad p(R_1 \mid T_1) = 0.9$

We want to find $p(T_0 \mid R_0)$

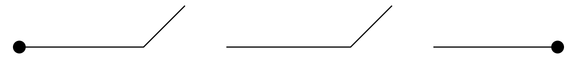
Take $\{T_0, T_1\}$ as the partition of the sample space

Bayes's theorem gives:

$$p(T_0 \mid R_0) = \frac{p(R_0|T_0)p(T_0)}{p(R_0|T_0)p(T_0)+p(R_0|T_1)p(T_1)} = \frac{0.8 \cdot 0.45}{(0.8 \cdot 0.45)+(0.1 \cdot 0.55)} \approx 0.87$$

Probability

Example: Switches 1 and 2 in the diagram below work independently. The probability that Switch 1 is closed (allowing a signal to get through) is 0.7. The probability that Switch 2 is closed is 0.2.



If a signal from the input fails to arrive at the output, find the probability that Switch 2 was open.

Solution: Let

S = "a signal is received"
C_1 = "Switch 1 is closed"
C_2 = "Switch 2 is closed"

Probability

We are given:

$p(C_1) = 0.7 \qquad p(C_2) = 0.2 \qquad p(\bar{C}_2) = 1 - p(C_2) = 0.8$

We want to find $p(\bar{C}_2 \mid \bar{S}) = \frac{p(\bar{C}_2 \cap \bar{S})}{p(\bar{S})}$

A signal will be transmitted iff both switches are closed:

$p(S) = p(C_1) \cdot p(C_2) = 0.7 \cdot 0.2 = 0.14$

$p(\bar{S}) = 1 - p(S) = 0.86$

If Switch 2 is open, then no signal, so $\bar{C}_2 \subseteq \bar{S}$

$$p(\bar{C}_2 \mid \bar{S}) = \frac{p(\bar{C}_2 \cap \bar{S})}{p(\bar{S})} = \frac{p(\bar{C}_2)}{p(\bar{S})} = \frac{0.8}{0.86} \approx 0.93$$